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Pointwise approximation by Bernstein type operators in mobile interval

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ABSTRACT

In the present paper, we study pointwise approximation by Bernstein–Durrmeyer type operators in the mobile interval $x \in \left[0, 1 - \frac{1}{n+1}\right]$, with use of Peetre's K-functional and $\omega^2_{\varphi^\lambda}(f, t)\,(0 \leqslant \lambda \leqslant 1)$, we give its properties and obtain the direct and inverse theorems for these operators.

1. Introduction and axillary results

In the year 2008, the operators $\tilde{B}_n$, were introduced and studied by Deo et al. [6] and defined as:

$$\tilde{B}_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$p_{n,k}(x) = \left(1 + \frac{1}{n}\right)^n \binom{n}{k} x^k \left(1 - \frac{1}{n+1} - x\right)^{n-k} \tag{1.2}$$

and $x \in \left[0, 1 - \frac{1}{n+1}\right]$. If n is sufficient large then operators (1.1) convert in the classical Bernstein operators:

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{1.3}$$

Now we consider Durrmeyer type operators

$$V_n(f, x) = \frac{(n+1)^2}{n} \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)\, dt. \tag{1.4}$$

In 2003, a very interesting general sequence of linear positive operators was introduced by Srivastava and Gupta [18] and investigated as well as estimated the rate of convergence. Then the faster rate of convergence was studied by Deo [2] for

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these operators and similar type modification have given and studied simultaneous approximation for these operators (1.1) in [4].

Ditzian [8] used $\omega^2_{\varphi^\lambda}(f,t)$ and gave an interesting direct estimate for the Bernstein polynomials. Felten [10] studied local and global approximation theorems for positive linear operators, later on similar type results studied for Durrmeyer operators by Guo et al. [15] and gave the direct and inverse theorems for pointwise approximation by Bernstein–Durrmeyer operators via Ditzian–Totik moduli $\omega^2_{\varphi^\lambda}(f,t)$. In the year 1998, Guo et al. [14] studied pointwise estimate for Szász–Durrmeyer operators with the help of Ditzian–Totik modulus of smoothness $\omega^r_{\varphi^\lambda}(f,t)$ in the interval $[0,\infty)$. Deo [3] studied pointwise estimate for modified Baskakov type operator. Recently Abel et al. [1] used properties of the Jacobi polynomials in order to give a new proof of geometric series of Bernstein operators. Very recently, Gupta and Agarwal [12] have given last two decades, literature on positive linear operators in their book and some interesting results were given by researchers [5,11,13,17] on approximation operators.

In a similar manner, in this research work, we give the direct and the inverse theorem for pointwise approximation by Bernstein type operators by Ditzian–Totik modulus of smoothness $\omega^2_{\varphi^\lambda}(f,t)$ in the mobile interval $\left[0,1-\frac{1}{n+1}\right]$.

First we give some notations. Let $C\left[0,1-\frac{1}{n+1}\right]$ be the set of continuous and bounded functions on $\left[0,1-\frac{1}{n+1}\right]$ and

$$\omega^2_{\varphi^\lambda}(f,t)=\sup_{0<h\leqslant t}\sup_{x\pm h\varphi^\lambda\in\left[0,1-\frac{1}{n+1}\right]}\left|\Delta^2_{h\varphi^\lambda}f(x)\right|, \tag{1.5}$$

$$D^2_\lambda=\left\{f\in C\left[0,1-\frac{1}{n+1}\right], f'\in A.C_{loc}, \|\varphi^{2\lambda}f''\|<+\infty\right\},$$

$$K_{\varphi^\lambda}(f,t^2)=\inf_{g\in D^2_\lambda}\{\|f-g\|+t^2\|\varphi^{2\lambda}g''\|\}, \tag{1.6}$$

$$\tilde{D}^2_\lambda=\left\{f\in D^2_\lambda, \|f''\|<+\infty\right\},$$

$$\tilde{K}_{\varphi^\lambda}(f,t^2)=\inf_{g\in\tilde{D}^2_\lambda}\{\|f-g\|+t^2\|\varphi^{2\lambda}g''\|+t^{4/(2-\lambda)}\|g''\|\}, \tag{1.7}$$

and $\varphi(x)=\sqrt{x\left(1-\frac{1}{n+1}-x\right)}$, $0\leqslant\lambda\leqslant 1$. It is well known (see [9, Theorem 3.1.2]) that

$$\omega^2_{\varphi^\lambda}(f,t)\sim K_{\varphi^\lambda}(f,t^2)\sim\tilde{K}_{\varphi^\lambda}(f,t^2) \tag{1.8}$$

($x\sim y$ means that there exists $c>0$ such that $c^{-1}y\leqslant x\leqslant cy$). Now we give some basic properties of the operators (1.1) as follow:

Lemma 1.1. *Let $e_i(t)=t^i$, $i=0,1,2$, then for $x\in\left[0,1-\frac{1}{n+1}\right]$ and $n\in N$. The operators $\tilde{B}_n$ verify the following:*

$$\tilde{B}_n(e_0;x)=1, \tag{1.9}$$

$$\tilde{B}_n(e_1;x)=x+\frac{1}{n}x, \tag{1.10}$$

$$\tilde{B}_n(e_2;x)=\frac{(n+1)^2(n-1)}{n^3}x^2+\frac{n+1}{n^2}x. \tag{1.11}$$

We give next Lemma along the line of proposition 1.2 p. 326 of Derriennic [7].

Lemma 1.2 [7]. *Let $e_s(t)=t^s$, $s=0,1,2,\ldots$ with the properties $s\leqslant n$, then for $x\in\left[0,1-\frac{1}{n+1}\right]$ and $n\in N$ we have*

$$(V_ne_s)(x)=\frac{(n!)^2}{(n+s+1)!}\sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\frac{n^{s-r}}{(n-r)!(n+1)^{s-r-1}}x^r. \tag{1.12}$$

Proof. In the account of (1.4), we obtain

$$\int_0^{\frac{n}{n+1}}p_{n,k}(t)t^s dt=\left(1+\frac{1}{n}\right)^n\binom{n}{k}\int_0^{\frac{n}{n+1}}t^{k+s}\left(1-\frac{1}{n+1}-t\right)^{n-k}dt=\left(1+\frac{1}{n}\right)^n\binom{n}{k}\left(\frac{n}{n+1}\right)^{n+s+1}\beta(k+s+1,n-k+1)$$
$$=\left(\frac{n}{n+1}\right)^{s+1}\frac{n!}{(n+s+1)!}\frac{(k+s)!}{k!}.$$

Therefore from (1.4), we obtain

$$(V_n e_s)(x) = \frac{n^s}{(n+1)^{s-1}} \frac{n!}{(n+s+1)!} \sum_{k=0}^{n} p_{n,k}(x) \frac{(k+s)!}{k!}. \tag{1.13}$$

For any $x, y \in R$ and any $s, n \in N$ satisfying the inequality $s \leqslant n$ we have

$$\frac{\partial^s}{\partial x^s}\{x^s(x+y)^n\} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \frac{(k+s)!}{k!}. \tag{1.14}$$

By Leibnitz formula, the left side of (1.14), we have

$$\frac{\partial^s}{\partial x^s}\{x^s(x+y)^n\} = \sum_{r=0}^{s} \binom{s}{r} \frac{\partial^{s-r}}{\partial x^{s-r}} x^s \frac{\partial^r}{\partial x^r}(x+y)^n = \sum_{r=0}^{s} \binom{s}{r} \frac{s!}{r!} \frac{n!}{(n-r)!} x^r (x+y)^{n-r}. \tag{1.15}$$

From (1.14) and (1.15), with $y = \left(1 - \frac{1}{n+1} - x\right)$

$$\sum_{r=0}^{s} \binom{s}{r} \frac{s!}{r!} \frac{n!}{(n-r)!} x^r \left(\frac{n}{n+1}\right)^{n-r} = \left(\frac{n}{n+1}\right)^n \sum_{k=0}^{n} p_{n,k}(x) \frac{(k+s)!}{k!}. \tag{1.16}$$

Now from (1.13), with the help of (1.16), we obtain

$$(V_n e_s)(x) = \frac{(n!)^2}{(n+s+1)!} \sum_{r=0}^{s} \binom{s}{r} \frac{s!}{r!} \frac{n^{s-r}}{(n-r)!(n+1)^{s-r-1}} x^r. \qquad \square$$

Lemma 1.3. *Let $e_i(t) = t^i$, $i = 0, 1, 2$, then for $x \in \left[0, 1 - \frac{1}{n+1}\right]$ and $n \in N$. The operators V_n verify the following:*

$$V_n(e_0; x) = 1, \tag{1.17}$$

$$V_n(e_1; x) = \frac{n}{(n+1)(n+2)} + \frac{n}{(n+2)} x, \tag{1.18}$$

$$V_n(e_2; x) = \frac{2n^2}{(n+1)^2(n+2)(n+3)} + \frac{4n^2}{(n+1)(n+2)(n+3)} x + \frac{n(n-1)}{(n+2)(n+3)} x^2. \tag{1.19}$$

Proof. From (1.16), we obtain these assertions for $s = 0, 1, 2$. $\square$

Lemma 1.4. *For $x \in \left[0, 1 - \frac{1}{n+1}\right]$, we have*

$$V_n((e_1 - x); x) = \frac{n - 2(n+1)x}{(n+1)(n+2)}, \tag{1.20}$$

$$V_n((e_1 - x)^2; x) = \frac{2n^2}{(n+1)^2(n+2)(n+3)} + \frac{2n(n-3)}{(n+1)(n+2)(n+3)} x - \frac{2(n-3)}{(n+2)(n+3)} x^2 = \frac{2\left[n^2 + (n+1)^2(n-3)\varphi^2(x)\right]}{(n+1)^2(n+2)(n+3)}. \tag{1.21}$$

Proof. From Lemma 1.3, we obtain these assertions. $\square$

2. Direct estimates

In this section we give rate of convergence.

Let $f \in C\left[0, 1 - \frac{1}{n+1}\right]$, the modulus of continuity of f, defined as:

$$\omega(f, \delta) = \sup_{x,t \in \left[0, 1 - \frac{1}{n+1}\right], |t-x| \leqslant \delta} |f(t) - f(x)|. \tag{2.1}$$

In the year 1990, Lenze [16] introduced the Lipschitz type maximal functions of order α as follows:

$$\hat{\omega}_\alpha(f, \delta) = \lim_{t \neq x; t \in \left[0, 1 - \frac{1}{n+1}\right]} \frac{|f(t) - f(x)|}{|t - x|^\alpha}, \quad x \in \left[0, 1 - \frac{1}{n+1}\right], \; 0 < \alpha \leqslant 1.$$

The remarkable is that the roundedness of $\hat{\omega}_\alpha(f,\delta)$ is equivalent to $f \in Lip_M(\alpha)$.

Theorem 2.1. *Let $f \in C\left[0, 1-\frac{1}{n+1}\right], V_n(f,x)$ be given by (1.4), then we have*

$$\|V_n(f;.) - f\| \leqslant \left(1+\sqrt{\beta}\right)\omega\left(f, \sqrt{\lambda_n}\right), \tag{2.2}$$

where $\beta = \sqrt{\frac{n}{n+1}}$.

Proof. By Popoviciu's technique we can write

$$|f(t) - f(x)| \leqslant \omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right), \quad \text{for any} \quad \delta > 0. \tag{2.3}$$

Using positivity and linearity properties of the operators $V_n(f;x)$ and for $n \in N$ and $x \in \left[0, 1-\frac{1}{n+1}\right]$, we have

$$|V_n(f;x) - f(x)| \leqslant V_n(|f(t) - f(x)|;x), \tag{2.4}$$

From (2.3) and (2.4), we get

$$|V_n(f;x) - f(x)| \leqslant \omega(f,\delta)\left(\frac{V_n(|t-x|;x)}{\delta}+1\right), \tag{2.5}$$

Now using Cauchy–Schwartz inequality and from Lemma 1.4, we get

$$\begin{aligned} |V_n(f;x) - f(x)| &\leqslant \omega(f,\delta)\left(\frac{\left(V_n\left((t-x)^2;x\right)\right)^{1/2}}{\delta}+1\right), \\ &\leqslant \omega(f,\delta)\left(\frac{\sqrt{\lambda_n x}}{\delta}+1\right). \end{aligned} \tag{2.6}$$

If we choose $\delta = \sqrt{\lambda_n}$ in (2.6) and take maximum over $x \in \left[0, 1-\frac{1}{n+1}\right]$, we get the required result (2.2). □

Motivated from Guo et al. [15], now we give direct theorem.

Theorem 2.2. *For $f \in C\left[0, 1-\frac{1}{n+1}\right], 0 \leqslant \lambda \leqslant 1$, then we have*

$$|V_n(f,x) - f(x)| \leqslant C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right) + \omega^*\left(f, 1/n^2\right). \tag{2.7}$$

Proof. From Lemma 1.4, we have

$$V_n((t-x)^2;x) = \frac{2\left[n^2+(n+1)^2(n-3)\varphi^2(x)\right]}{(n+1)^2(n+2)(n+3)} \leqslant Cn^{-1}\delta_n^2(x).$$

Let

$$M_n(f,x) = f(x) - f\left(x + \frac{n-2(n+1)x}{(n+1)(n+2)}\right)$$

and

$$L_n(f,x) = V_n(f,x) + M_n(f,x)$$

then we obtain

$$|M_n(f,x)| \leqslant \omega^*\left(f, \frac{|n+2(n+1)x|}{(n+1)(n+2)}\right) \leqslant \omega^*(f, 1/n).$$

$$L_n(1,x) = 1, L_n((t-x),x) = 0, L_n\left((t-x)^2,x\right) \leqslant Cn^{-1}\delta_n^2(x) \text{ and } \|L_n\| \leqslant 3. \tag{2.8}$$

From (1.7) and (1.8), we choose $g = g_{n,x,\lambda} \in \bar{D}^2_\lambda$, for a fixed x and λ, we get

$$\|f-g\| \leqslant C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right),$$

$$\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^2\left\|\varphi^{2\lambda}g''\right\| \leqslant C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right),$$

$$\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{4/(2-\lambda)}\|g''\| \leqslant C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right).$$

From ([9], p. 141), for $t < u < x$, we have

$$\frac{|t-u|}{\varphi^{2\lambda}(u)} \leqslant \frac{|t-x|}{\varphi^{2\lambda}(x)} \quad \text{and} \quad \frac{|t-u|}{\delta_n^{2\lambda}(u)} \leqslant \frac{|t-x|}{\delta_n^{2\lambda}(x)}. \tag{2.9}$$

So that

$$|L_n(f,x) - f(x)| \leqslant |L_n((f-g),x)| + |f(x) - g(x)| + |L_n(g,x) - g(x)| \leqslant 4\|f-g\| + |L_n(g,x) - g(x)|$$

Using Taylor expansion with integral reminder

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du \tag{2.10}$$

(see [9], p. 134) and by (2.8) and (2.9), we have

$$\begin{aligned}
|L_n(g,x) - g(x)| &= \left|L_n\left(\int_x^t (t-u)g''(u)du, x\right)\right| \leqslant \left|V_n\left(\int_x^t (t-u)g(u)du, x\right)\right| \\
&\quad + \left|\int_x^{x+\frac{n+2(n+1)x}{(n+1)(n+2)}} \left(x + \frac{n+2(n+1)x}{(n+1)(n+2)} - u\right)g(u)du\right| \leqslant C\|\delta_n^{2\lambda}g''\|V_n\left(\frac{(t-x)^2}{\delta_n^{2\lambda}(x)}, x\right) \\
&\quad + \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda}g''\|\left(\frac{n+2(n+1)x}{(n+1)(n+2)}\right)^2 \leqslant C\left(\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^2\|\delta_n^{2\lambda}g''\| + n^{-2}\delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda}g''\|\right) \\
&\leqslant C\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^2\|\delta_n^{2\lambda}g''\|.
\end{aligned} \tag{2.11}$$

For $x \in \left[0, 1 - \frac{1}{n+1}\right]$, $\delta_n^{2\lambda} = 1/n^{\lambda}$ and from (1.8) we get

$$\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^2\|\varphi^{2\lambda}g''\| \leqslant CK_{\varphi^\lambda}\left(f, \left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^2\right) \sim \omega_{\varphi^\lambda}^2\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right).$$

From (2.11), we have

$$|L_n(f,x) - f(x)| \leqslant 4\|f-g\| + Cn^{-1}\delta_n^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\| \leqslant C\left(\|f-g\| + n^{-1}\delta_n^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\|\right) \leqslant C\omega_{\varphi^\lambda}^2\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right).$$

Hence, for $f \in C\left[0, 1 - \frac{1}{n+1}\right]$, we obtain

$$|V_n(f,x) - f(x)| \leqslant |L_n(f,x) - f(x)| + |L_n(f,x)| \leqslant C\omega_{\varphi^\lambda}^2\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right) + \omega^*\left(f, 1/n^2\right). \tag{2.12}$$

This completes the proof. □

3. Inverse theorem

In this section we give inverse theorem.

Theorem 3.1. *For $f \in C\left[0, 1 - \frac{1}{n+1}\right], 0 < \alpha < \frac{2}{2-\lambda}, 0 \leqslant \lambda \leqslant 1$, then the following statements are equivalent:*

$$|V_n(f,x) - f(x)| = O\left(\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^\alpha\right), \tag{3.1}$$

$$\omega_{\varphi^\lambda}^2(f,t) = O(t^\alpha) \quad \text{and} \quad \omega^*(f,t) = O\left(t^{\alpha(1-\lambda/2)}\right), \tag{3.2}$$

where $\delta_n(x) = \varphi(x) + 1/\sqrt{n} \sim \max\{\varphi(x), 1/\sqrt{n}\}$.

To prove the inverse theorem we need the following notations. Let us denote

$$\|f\|_0 = \sup_{x\in[0,1-\frac{1}{n+1}]}\left\{\left|\delta_n^{\alpha(\lambda-1)}(x)f(x)\right|\right\};$$

$$C_{\alpha,\lambda} = \left\{f \in C\left[0, 1 - \frac{1}{n+1}\right], \|f\|_0 < \infty\right\};$$

$$\|f\|_1 = \sup_{x\in[0,1-\frac{1}{n+1}]}\left\{\left|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)f'(x)\right|\right\};$$

$$C^1_{\alpha,\lambda} = \{f \in C_{\alpha,\lambda}, \|f\|_1 < \infty\};$$

$$\|f\|_2 = \sup_{x\in[0,1-\frac{1}{n+1}]} \left\{ \left|\delta_n^{2+\alpha(\lambda-1)}(x) f''(x)\right| \right\};$$

$$C^2_{\alpha,\lambda} = \{f \in C_{\alpha,\lambda}, f' \in A.C._{loc}, \|f\|_2 < \infty\};$$

$$K^1_{\alpha,\lambda}(f,t) = \inf_{g\in C^1_{\alpha,\lambda}} \{\|f-g\|_0 + t\|g\|_1\};$$

$$K^2_{\alpha,\lambda}(f,t) = \inf_{g\in C^2_{\alpha,\lambda}} \{\|f-g\|_0 + t\|g\|_2\}.$$

Lemma 3.2. *If* $0 \leqslant \lambda \leqslant 1,\ 0 < \alpha < 2$, *then*

$$\|V_n f\|_1 \leqslant Cn^{1/2}\|f\|_0 (f \in C_{\alpha,\lambda}), \tag{3.3}$$

$$\|V_n f\|_1 \leqslant C\|f\|_1 \left(f \in C^1_{\alpha,\lambda}\right), \tag{3.4}$$

$$\|V_n f\|_2 \leqslant Cn\|f\|_0 (f \in C_{\alpha,\lambda}), \tag{3.5}$$

$$\|V_n f\|_2 \leqslant C\|f\|_2 \left(f \in C^2_{\alpha,\lambda}\right). \tag{3.6}$$

Proof. The proof of Lemma 3.2 will be given in Section 4. □

Lemma 3.3 [15]. *For* $0 < t < \frac{1}{8},\ \frac{t}{2} \leqslant x \leqslant 1 - \frac{1}{n+1} - \frac{t}{2},\ x \in \left[0, 1-\frac{1}{n+1}\right],\ \beta < 2$, *we have*

$$\int_{-t\varphi^\lambda(x)/2}^{t\varphi^\lambda(x)/2} \delta_n^{-\beta}(x+u)du \leqslant C(\beta)t\delta_n^{-\beta}(x).$$

Lemma 3.4 [15]. *For* $0 < t < \frac{1}{4},\ t \leqslant x \leqslant 1 - \frac{1}{n+1} - t,\ x \in \left[0, 1-\frac{1}{n+1}\right],\ 0 \leqslant \beta \leqslant 2$, *we have*

$$\int_{-t\varphi^\lambda(x)/2}^{t\varphi^\lambda(x)/2}\int_{-t\varphi^\lambda(x)/2}^{t\varphi^\lambda(x)/2} \delta_n^{-\beta}(x+u+v)dudv \leqslant Ct^2\delta_n^{-\beta}(x).$$

Proof. First, we let a new K-functional as

$$K^\alpha_{\varphi^\lambda}(f,t^r) = \inf_{g\in C^r_\lambda}\{\|(f(x)-g(x))\|_0 + t^r\|g^{(r)}(x)\|_r\}.$$

By this definition we may choose $g \in C^r_\lambda$ such that

$$\|(f(x)-g(x))\|_0 + n^{-r/2}\|g^{(r)}(x)\|_r \leqslant CK^\alpha_{\varphi^\lambda}\left(f, n^{-r/2}\right).$$

Suppose that for $f \in C[0,1]$,

$$\|M_n(f,x) - f(x)\|_0 \leqslant Cn^{-\alpha/2}.$$

Then we will prove

$$\omega^r_{\varphi^\lambda}(f,t) = O(t^\alpha).$$

$$\begin{aligned} K^\alpha_{\varphi^\lambda}(f,t^r) &\leqslant \|(f(x)-M_n(f;x))\|_0 + t^r\left\|M_n^{(r)}(f;x)\right\|_r \leqslant Cn^{-\alpha/2} + t^r\left(\left\|M_n^{(r)}(f-g;x)\right\|_r + \left\|M_n^{(r)}(g;x)\right\|_r\right) \\ &\leqslant C\left(n^{-\alpha/2} + t^r\left(n^{r/2}\|(f(x)-g(x))\|_0 + \|g(x)\|_r\right)\right) \leqslant C\left(n^{-\alpha/2} + \frac{t^r}{n^{-r/2}}K^\alpha_{\varphi^\lambda}\left(f, n^{-r/2}\right)\right), \end{aligned}$$

which implies

$$K^\alpha_{\varphi^\lambda}(f,t^r) \leqslant Ct^\alpha.$$

On the other hand, notice that for $i = 1, 2, \dots, r, x \pm rt\varphi^{\lambda}(x)/2 \in (0,1)$. Let $0 \leqslant r \leqslant 2$. For $g \in C_{\lambda}^{0}$,

$$\left|\Delta_{t\varphi^{\lambda}}^{r} g(x)\right| \leqslant \|g(x)\|_0 \left(\sum_{j=0}^{r}\binom{r}{j}\delta_n^{\alpha(1-\lambda)}\left(x+\left(j-\frac{r}{2}t\varphi^{\lambda}(x)\right)\right)\right) \leqslant C\|g(x)\|_0\left(\sum_{j=0}^{r}\binom{r}{j}\delta_n^{2r}\left(x+\left(j-\frac{r}{2}t\varphi^{\lambda}(x)\right)\right)\right)^{\alpha(1-\lambda)/2r}$$
$$\leqslant C\|g(x)\|_0\delta_n^{\alpha(1-\lambda)}(x).$$

For g with $g^{(r-1)} \in AC_{loc}, \|g^{(r)}(x)\delta_n^{r+\alpha(\lambda-1)}(x)\| < \infty$ and for $0 < t\varphi^{\lambda}(x) < 1/8r$ and $rt\varphi^{\lambda}(x)/2 \leqslant x \leqslant 1 - rt\varphi^{\lambda}(x)/2$

$$\left|\Delta_{h\varphi^{\lambda}}^{r} g(x)\right| \leqslant \left|\int_{-\frac{t\varphi^{\lambda}(x)}{2}}^{\frac{t\varphi^{\lambda}(x)}{2}} \cdots \int_{-\frac{t\varphi^{\lambda}(x)}{2}}^{\frac{t\varphi^{\lambda}(x)}{2}} g^{(r)}(x+u_1+\cdots+u_r)du_1\cdots du_r\right|$$
$$\leqslant \|g\|_r\left|\int_{-\frac{t\varphi^{\lambda}(x)}{2}}^{\frac{t\varphi^{\lambda}(x)}{2}} \cdots \int_{-\frac{t\varphi^{\lambda}(x)}{2}}^{\frac{t\varphi^{\lambda}(x)}{2}} \delta_n^{-r+\alpha(1-\lambda)}(x+u_1+\cdots+u_r)du_1\cdots du_r\right| \leqslant Ct^r\delta_n^{(-r+\alpha)(1-\lambda)}(x)\|g(x)\|_r.$$

From the above, for $0 < t\varphi^{\lambda}(x) < 1/8r$ and $rt\varphi^{\lambda}(x)/2 \leqslant x \leqslant 1 - rt\varphi^{\lambda}(x)/2$ and choosing appropriate g we obtain

$$\left|\Delta_{h\varphi^{\lambda}}^{r} f(x)\right| \leqslant \left|\Delta_{h\varphi^{\lambda}}^{r}(f(x)-g(x))\right| + \left|\Delta_{h\varphi^{\lambda}}^{r} g(x)\right| \leqslant C\left(\|f-g\|_0\delta_n^{\alpha(1-\lambda)}(x) + t^r\delta_n^{(-r+\alpha)(1-\lambda)}(x)\|g\|_r\right)$$
$$\leqslant C\delta_n^{\alpha(1-\lambda)}(x)\left(\|f-g\|_0 + t^r\delta_n^{-r(1-\lambda)}(x)\|g\|_r\right) \leqslant C\delta_n^{\alpha(1-\lambda)}(x)K_{\varphi^{\lambda}}^{\alpha}\left(f, \frac{t^r}{\delta_n^{r(1-\lambda)}(x)}\right) \leqslant Ct^{\alpha}. \qquad \square$$

4. Proof of Lemma 3.2

To prove Lemma 3.2, we let $a = \frac{n}{n+1}$ and we will use the following notations, in this section: Let

$$\tilde{q}_{n,k}(x) := (\,nk\,)x^k(1-x)^{n-k}.$$

Then we know for $p_{n,k}(x)$ defined in (1.2) that

$$p_{n,k}(x) = a^{-n}\binom{n}{k}x^k(a-x)^{n-k} = \tilde{q}_{n,k}(y), \quad y = \frac{x}{a}, \quad 0 \leqslant y \leqslant 1.$$

Let us denote

$$\|f\|_{r,[0,1]} = \sup_{x\in(0,1)}\left\{\left|\delta_n^{r+\alpha(\lambda-1)}(x)f^{(r)}(x)\right|\right\};$$
$$C_{\alpha,\lambda,[0,1]}^{r} = \left\{f \in C^r[0,1], \|f\|_{r,[0,1]} < \infty\right\},$$

and

$$\tilde{\delta}_n(x) = \tilde{\varphi}(x) + 1/\sqrt{n}, \quad \tilde{\varphi}(x) = \sqrt{x(1-x)},$$

Then we know

$$\tilde{\delta}_n(x) \sim \max\{\tilde{\varphi}(x), 1/\sqrt{n}\},$$

and

$$\tilde{Q}_n(f,x) = (n+1)\sum_{k=0}^{n}\tilde{q}_{n,k}(x)\int_0^1\tilde{q}_{n,k}(t)f(t)dt. \tag{4.1}$$

Then we have the following results:

Lemma 4.1. *Let $r \in \mathbb{N}$. If $0 \leqslant \lambda \leqslant 1,\ 0 < \alpha < r$, then we have for $f \in C_{\alpha,\lambda,[0,1]}^{0}$*

$$\left|\tilde{\delta}_n^{r+\alpha(\lambda-1)}(x)\tilde{Q}_n^{(r)}(f,x)\right| \leqslant Cn^{r/2}\|f\|_{0,[0,1]},$$

that is,

$$\left\|\tilde{Q}_n(f,x)\right\|_{r,[0,1]} \leqslant Cn^{r/2}\|f\|_{0,[0,1]},$$

Lemma 4.2. *Let $r \in \mathbb{N}$. If $0 \leqslant \lambda \leqslant 1,\ 0 < \alpha < r$, then we have for $f \in C^r_{\alpha,\lambda,[0,1]}$*

$$\left|\tilde{\delta}_n^{r+\alpha(\lambda-1)}(x)\tilde{Q}_n^{(r)}(f,x)\right| \leqslant C\|f\|_r,$$

that is,

$$\left\|\tilde{Q}_n(f,x)\right\|_r \leqslant C\|f\|_r.$$

Proof of Lemma 3.2. Let $\tilde{f}(x) = f(ax)$. Then we see the followings;

$$\tilde{Q}_n\left(\tilde{f},\frac{x}{a}\right) = (n+1)\sum_{k=0}^{n}\tilde{q}_{n,k}\left(\frac{x}{a}\right)\int_0^1 \tilde{q}_{n,k}(t)\tilde{f}(t)dt = \frac{n+1}{a}\sum_{k=0}^{n}p_{n,k}(x)\int_0^a p_{n,k}(s)f(s)dt = V_n(f,x),$$

$$\frac{d}{dx}V_n(f,x) = \frac{d}{dy}\tilde{Q}_n'\left(\tilde{f},y\right)\frac{1}{a},$$

and

$$\frac{d^r}{dx^r}V_n(f,x) = \frac{d^r}{dy^r}\tilde{Q}_n'\left(\tilde{f},y\right)\left(\frac{1}{a}\right)^r, \quad y = \frac{x}{a}.$$

Moreover, since

$$\varphi(x) = \sqrt{x(a-x)} = a\tilde{\varphi}\left(\frac{x}{a}\right),$$

we have

$$\delta_n(x) = \varphi(x) + 1/\sqrt{n} = a\tilde{\varphi}\left(\frac{x}{a}\right) + 1/\sqrt{n} \sim \tilde{\delta}_n\left(\frac{x}{a}\right) = \tilde{\delta}_n(y).$$

Now, we prove Lemma 3.2. For $0 \leqslant x \leqslant a$ and $y = x/a$,

$$\left|\delta_n^{r+\alpha(\lambda-1)}(x)V_n^{(r)}(f,x)\right| \sim \left|\tilde{\delta}_n^{r+\alpha(\lambda-1)}(y)\frac{d^r}{dy^r}\tilde{Q}_n\left(\tilde{f},y\right)\left(\frac{1}{a}\right)^r\right|.$$

From Lemma 4.1, we have

$$\left|\tilde{\delta}_n^{r+\alpha(\lambda-1)}(y)\frac{d^r}{dy^r}\tilde{Q}_n\left(\tilde{f},y\right)\right| \leqslant Cn^{r/2}\|\tilde{f}(y)\|_{0,[0,1]} = Cn^{r/2}\sup_{y\in(0,1)}\left\{\left|\tilde{\delta}_n^{\alpha(\lambda-1)}(y)\frac{d^r}{dy^r}\tilde{f}(y)\right|\right\} \sim Cn^{r/2}a^r\sup_{y\in(0,1)}\left\{\left|\delta_n^{\alpha(\lambda-1)}(x)\frac{d^r}{dx^r}f(x)\right|\right\}.$$

Therefore, we have for $0 \leqslant x \leqslant a$

$$\left|\delta_n^{r+\alpha(\lambda-1)}(x)V_n^{(r)}(f,x)\right| \leqslant Cn^{r/2}\|f\|_0,$$

that is,

$$\|V_n(f,x)\|_r \leqslant Cn^{r/2}\|f\|_0.$$

Similarly, we can prove using Lemma 4.2

$$\left|\delta_n^{r+\alpha(\lambda-1)}(x)V_n^{(r)}(f,x)\right| \leqslant C\|f\|_r,$$

that is,

$$\left\|V_n^{(r)}(f,x)\right\|_r \leqslant C\|f\|_r. \quad \square$$

In the following we give the proof of Lemma 4.1.

Proof of Lemma 4.1. From (4.1) and differentiation r-times on x, we know

$$\tilde{Q}_n^{(r)}(f,x) = (-1)^r\frac{(n+1)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}^{(r)}(t)f(t)dt. \tag{4.2}$$

Then we obtain

$$\left|\tilde{Q}_n^{(r)}(f,x)\right| \leqslant \frac{(n+1)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}^{(r)}(t)\tilde{\delta}_n^{\alpha(1-\lambda)}(t)dt\|f\|_{0,[0,1]}.$$

Since we easily see

$$\tilde{q}_{n,k}^{(r)}(t) = \frac{n!}{(n-r)!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\tilde{q}_{n-r,k-r+i}(t),$$

we have the following relation:

$$\tilde{q}_{n+r,k+r}^{(r)}(t) = \frac{(n+r)!}{n!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\tilde{q}_{n,k+i}(t) \tag{4.3}$$

Using (4.3), we know

$$\left|\tilde{Q}_n^{(r)}(f,x)\right| \leqslant Cn^r\sum_{i=0}^{r}\binom{r}{i}\sum_{k=0}^{n-r}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n,k+i}(t)\tilde{\delta}_n^{\alpha(1-\lambda)}(t)dt\|f\|_{0,[0,1]}.$$

Let

$$p = \frac{2r}{\alpha(1-\lambda)} \quad \text{and} \quad \frac{1}{q} = \frac{2r-\alpha(1-\lambda)}{2r}. \tag{4.4}$$

Then since $1/p + 1/q = 1$, using Hölder inequality, we have

$$\left|\tilde{Q}_n^{(r)}(f,x)\right| \leqslant Cn^r\|f\|_{0,[0,1]}\sum_{i=0}^{r}\binom{r}{i}\left((n+1)\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n,k+i}(t)dt\right)^{1/q}\left((n+1)\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n,k+i}(t)\tilde{\delta}_n^{2r}(t)dt\right)^{1/p}.$$

Then we first know

$$\left((n+1)\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n,k+i}(t)dt\right)^{1/q} \leqslant \left((n+1)\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\frac{1}{n+1}\right)^{1/q} = 1.$$

Since we see

$$\tilde{\varphi}^{2r}(t)\tilde{q}_{n,k}(t) = \tilde{\varphi}^{2r}(t)\binom{n}{k}t^k(1-t)^{n-k} = \frac{\binom{n}{k}}{\binom{n+2r}{k+r}}(n+2rk+r)t^{k+r}(1-t)^{n+r-k}$$
$$= \frac{(k+1)\cdots(k+r)(n-k+1)\cdots(n-k+r)}{(n+1)(n+2)\cdots(n+2r)}\tilde{q}_{n+2r,k+r}(t),$$

then we know

$$\int_0^1 \tilde{\varphi}^{2r}(t)\tilde{q}_{n,k}(t)dt = \frac{(k+1)\cdots(k+r)(n-k+1)\cdots(n-k+r)}{(n+1)(n+2)\cdots(n+2r)(n+2r+1)} := a(n,k,r).$$

Then secondly we have

$$\left((n+1)\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n,k+i}(t)\tilde{\delta}_n^{2r}(t)dt\right)^{1/p} \leqslant C\left((n+1)\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\left(a(n,k+i,r)+\frac{1}{n^r(n+1)}\right)\right)^{1/p}.$$

For $0 \leqslant k \leqslant 2r$ or $n-3r \leqslant k \leqslant n-r$,

$$\left(\sum_{k=0}^{2r}+\sum_{k=n-3r}^{n-r}\right)(n+1)\tilde{q}_{n-r,k}(x)a(n,k+i,r) \leqslant 2\frac{(4r)^r}{n^r}$$

and for $2r+1 \leqslant k \leqslant n-3r-1$

$$\sum_{k=2r+1}^{n-3r-1}(n+1)\tilde{q}_{n-r,k}(x)a(n,k+i,r) \leqslant x^r(1-x)^r\sum_{k=2r+1}^{n-3r-1}\frac{\binom{n-r}{k}}{\binom{n-3r}{k-r}}\tilde{q}_{n-3r,k-r}(x)(n+1)a(n,k+i,r) \leqslant C(r)x^r(1-x)^r,$$

because

$$\frac{\binom{n-r}{k}}{\binom{n-3r}{k-r}}(n+1)a(n,k+i,r) = \frac{(n-3r+1)\cdots(n-r)}{(k-r+1)\cdots k(n-k-2r+1)\cdots(n-k-r)}$$
$$\times\frac{(k+i+1)\cdots(k+i+r)(n-k-i+1)\cdots(n-k-i+r)}{(n+2)\cdots(n+2r)(n+2r+1)}$$
$$\leqslant\left(1+\frac{r+i}{k-r+1}\right)\left(1+\frac{r+i}{k-r+2}\right)\cdots\left(1+\frac{r+i}{k}\right)$$
$$\times\left(1+\frac{2r-i}{n-k-2r+1}\right)\left(1+\frac{2r-i}{n-k-2r+2}\right)\cdots\left(1+\frac{2r-i}{n-k-r}\right)$$
$$\leqslant\left(1+\frac{r+i}{k}\right)^r\left(1+\frac{2r-i}{n-k-r}\right)^r\leqslant C(r).$$

Therefore, we have for $x\in(0,1)$,

$$\left((n+1)\sum_{k=0}^{n-r}\tilde{q}_{n-r,k}(x)\int_0^1\tilde{q}_{n,k+i}(t)\tilde{\delta}_n^{2r}(t)dt\right)^{1/p}\leqslant C\left(n^{-r}+x^r(1-x)^r\right)\sim\tilde{\delta}_n^{2r}. \tag{4.5}$$

Thus, for $x\in(0,1/n)\cup(1-1/n,1)$

$$\left|\tilde{Q}_n^{(r)}(f,x)\right|\leqslant C\|f\|_{0,[0,1]}n^r\tilde{\delta}_n^{\alpha(1-\lambda)}\sim\|f\|_{0,[0,1]}n^{r/2}\tilde{\delta}_n^{-r+\alpha(1-\lambda)}.$$

This implies that

$$\left|\tilde{\delta}_n^{r+\alpha(\lambda-1)}(x)\tilde{Q}_n^{(r)}(f,x)\right|\leqslant C\|f\|_{0,[0,1]}n^{r/2}.$$

Let $x\in(1/n,1-1/n)$. Then $\tilde{\delta}_n(x)\sim\sqrt{x(1-x)}=\tilde{\varphi}^2(x)$. Differentiating $\tilde{q}_{n,k}(x)$, we have the following expressions:

$$\tilde{q}_{n,k}^{(r)}(x)=(x(1-x))^{-r}\sum_{i=0}^{r}Q_i(r,n,x)n^i\left(\frac{k}{n}-x\right)^i\tilde{q}_{n,k}(x), \tag{4.6}$$

where $Q_i(x,n)$ is a polynomial in $nx(1-x)$ of degree $[(r-i)/2]$ with polynomial coefficients such that

$$\left|(x(1-x))^{-r}Q_i(x,n)n^i\right|\leqslant C\left(\frac{n}{x(1-x)}\right)^{\frac{r+i}{2}}. \tag{4.7}$$

Using the expression (4.6), we know

$$\tilde{Q}_n^{(r)}(f,x)=(n+1)\sum_{k=0}^{n}\tilde{q}_{n,k}^{(r)}(x)\int_0^1\tilde{q}_{n,k}(t)f(t)dt=(x(1-x))^{-r}\sum_{i=0}^{r}Q_i(r,n,x)n^i(n+1)\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^i\tilde{q}_{n,k}(x)\int_0^1\tilde{q}_{n,k}(t)f(t)dt.$$

Then we have by (4.7)

$$\left|\tilde{Q}_n^{(r)}(f,x)\right|\leqslant\sum_{i=0}^{r}\left(\frac{n}{x(1-x)}\right)^{\frac{r+i}{2}}(n+1)\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^i\tilde{q}_{n,k}(x)\int_0^1\tilde{q}_{n,k}(t)\tilde{\delta}_n^{\alpha(1-\lambda)}(t)dt\|f\|_{0,[0,1]}.$$

Using Hölder inequality with p and q defined in (4.4), we obtain

$$(n+1)\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^i\tilde{q}_{n,k}(x)\int_0^1\tilde{q}_{n,k}(t)\tilde{\delta}_n^{\alpha(1-\lambda)}(t)dt\leqslant C\left(\sum_{k=0}^{n}(n+1)\left|\frac{k}{n}-x\right|^{qi}\tilde{q}_{n,k}(x)\int_0^1\tilde{q}_{n,k}(t)dt\right)^{1/q}$$
$$\times\left(\sum_{k=0}^{n}(n+1)\tilde{q}_{n,k}(x)\int_0^1\tilde{q}_{n,k}(t)\tilde{\delta}_n^{2r}(t)dt\right)^{1/p}.$$

For the first term in the above equation, using Hőlder inequality for $2m/iq$ and $2m/(2m-iq)$, we have

$$\left(\sum_{k=0}^{n}(n+1)\left|\frac{k}{n}-x\right|^{qi}\tilde{q}_{n,k}(x)\int_0^1\tilde{q}_{n,k}(t)dt\right)^{1/q}=\left(\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{qi}\tilde{q}_{n,k}(x)\right)^{1/q}\leqslant\left(\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{2m}\tilde{q}_{n,k}(x)\right)^{i/2m}\leqslant\left(n^{-m}\tilde{\delta}_n^{2m}\right)^{i/2m}=n^{-i/2}\tilde{\delta}_n^i.$$

For the second term, we know by the same procedure as the proof of (4.5)

$$\left(\sum_{k=0}^{n}(n+1)\tilde{q}_{n,k}(x)\int_0^1 \tilde{q}_{n,k}(t)\tilde{\delta}_n^{2r}(t)dt\right)^{1/p} \leqslant C\tilde{\delta}_n^{2r/p} \sim \tilde{\delta}_n^{\alpha(1-\lambda)}.$$

Then since we know

$$\left(\frac{n}{x(1-x)}\right)^{\frac{r+i}{2}} \sim \left(\frac{n}{\tilde{\delta}_n^2}\right)^{\frac{r+i}{2}},$$

we have

$$\left|\tilde{Q}_n^{(r)}(f,x)\right| \leqslant C\|f\|_{0,[0,1]}\sum_{i=0}^{r}\left(\frac{n}{\tilde{\delta}_n^2}\right)^{\frac{r+i}{2}} n^{-i/2}\tilde{\delta}_n^i\tilde{\delta}_n^{\alpha(1-\lambda)} \leqslant C\|f\|_{0,[0,1]}n^{r/2}\tilde{\delta}_n^{-r+\alpha(1-\lambda)},$$

that is,

$$\left|\tilde{\delta}_n^{r+\alpha(\lambda-1)}(x)\tilde{Q}_n^{(r)}(f,x)\right| \leqslant Cn^{r/2}\|f\|_{0,[0,1]}. \qquad \square$$

Proof of Lemma 4.2. From (4.2), we know

$$\left|\tilde{Q}_n^{(r)}(f,x)\right| \leqslant C\sum_{k=0}^{n-r}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}(t)\tilde{\delta}_n^{-r+\alpha(1-\lambda)}dt\|f\|_{r,[0,1]}.$$

Then we see

$$\sum_{k=0}^{n-r}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}(t)\tilde{\delta}_n^{-r+\alpha(1-\lambda)}dt \leqslant \left(\sum_{k=0}^{n}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}(t)\tilde{\delta}_n^{-2r}dt\right)^{1/p}$$
$$\leqslant \left(\sum_{k=0}^{n-r}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}(t)\min\{n^r, t^{-r}(1-t)^{-r}\}dt\right)^{1/p},$$

where $p = 2r/(r-\alpha(1-\lambda))$. Here, we first know

$$\sum_{k=0}^{n-r}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}(t)n^r dt = n^r\frac{n+1}{n+r+1}$$

and we secondly have

$$\sum_{k=0}^{n-r}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}(t)t^{-r}(1-t)^{-r}dt = \sum_{k=0}^{n-r}(n+1)\tilde{q}_{n+r,k+r}(x)x^{-r}(1-x)^{-r}\int_0^1 \tilde{q}_{n-r,k}(t)dt = \frac{n+1}{n-r+1}x^{-r}(1-x)^{-r},$$

because

$$\tilde{q}_{n+r,k+r}(t)t^{-r}(1-t)^{-r} = \frac{\binom{n+r}{k+r}}{\binom{n-r}{k}}\tilde{q}_{n-r,k}(t) = \frac{(n+r)!}{(k+r)!(n-k)!}\,\frac{k!(n-k-r)!}{(n-r)!}\tilde{q}_{n-r,k}(t).$$

Thus, we have

$$\sum_{k=0}^{n-r}(n+1)\tilde{q}_{n-r,k}(x)\int_0^1 \tilde{q}_{n+r,k+r}(t)\tilde{\delta}_n^{-r+\alpha(1-\lambda)}dt \leqslant C\tilde{\delta}_n^{-r+\alpha(1-\lambda)}.$$

This implies

$$\left|\tilde{Q}_n^{(r)}(f,x)\right| \leqslant C\tilde{\delta}_n^{-r+\alpha(1-\lambda)}\|f\|_{r,[0,1]},$$

that is,

$$\left\|\tilde{Q}_n(f,x)\right\|_{r,[0,1]} \leqslant C\|f\|_{r,[0,1]}. \qquad \square$$

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