VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0047-9
Math. Slovaca **61** (2011), No. 5, 799–806

A NOTE ON IMPROVED ESTIMATIONS FOR INTEGRATED SZÁSZ-MIRAKYAN OPERATORS

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(*Communicated by Wladyslaw Wilczynski*)

ABSTRACT. In the present paper, we propose a modification of the integrated Szász-Mirakyan operators having the weight function of general Beta basis function. We study some direct results on the modified operators. It is also observed that our modified operators have better estimates over the original operators.

1. Introduction

For $f \in C[0,\infty)$, the Szász-Mirakyan operator (spelled also Szász-Mirakjan) was introduced independently by several authors, i.e. G. Mirakyan [9], J. Favard [5] and O. Szász [10] as

$$B_n(f,x) = \sum_{v=0}^{\infty} s_{n,v}(x) f(v/n), \qquad x \in [0,\infty), \tag{1.1}$$

where the Szász-Mirakyan basis function is defined by

$$s_{n,v}(x) = \mathrm{e}^{-nx} \frac{(nx)^v}{v!}.$$

The first integral modification of Szász-Kantorovich operators is given as

$$K_n(f,x) = \sum_{v=0}^{\infty} s_{n,v}(x) \int_{k/n}^{(k+1)/n} f(t)\,\mathrm{d}t, \qquad x \in [0,\infty).$$

2010 Mathematics Subject Classification: Primary 41A25, 41A30.
Keywords: Szász-Mirakyan operators, Beta basis function, asymptotic formula, Lipschitz condition, bounded variation.

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The Durrmeyer type integral modification of the Szász-Mirakyan operator was first introduced by Mazhar and Totik [8] and also studied in [6] and [1] is defined as

$$D_n(f,x) = n\sum_{v=0}^{\infty} s_{n,v}(x)\int_0^{\infty} s_{n,v}(t)f(t)\,\mathrm{d}t, \qquad x\in[0,\infty).$$

In Gupta et al. [7] considered another integral modification of the Szász-Mirakyan operators by considering the weight functions of Beta basis functions. Very recently Dubey and Jain [3] considered a parameter c in the definition of [7] and for $c>0$, they proposed the modified operators as

$$S_{n,c}(f,x) = \sum_{v=0}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v,c}(t)f(t)\,\mathrm{d}t, \qquad x\in[0,\infty), \tag{1.2}$$

where $s_{n,v}(x)$ is as defined in (1.1) above and

$$b_{n,v,c}(t) = c\frac{\Gamma\big(\frac{n}{c}+v+1\big)}{\Gamma(v+1)\Gamma\big(\frac{n}{c}\big)}\frac{(ct)^v}{(1+ct)^{(\frac{n}{c}+v+1)}}.$$

In case $c=1$, the above operators reduce to the Szász-Beta operators [7]. It is observed that these operators are linear positive operators and reproduce only constant function. The approximation properties of operators $S_{n,c}(f,x)$ are different from the operators studied by Duman et al. [4]. The operators studied in [4] reproduce constant as well as linear functions.

Lemma 1.1. ([3]) *Let the function $\mu_{n,m,r,c}(x)$, $m\in c^0$ and $c>0$ be defined as*

$$\mu_{n,m,c}(x) = \sum_{v=0}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v,c}(t)(t-x)^m\,\mathrm{d}t.$$

For $n>c(m+1)$, we have the recurrence relation:

$$\begin{aligned}&[n-c(m+1)]\mu_{n,m+1,c}(x)\\ &= x\mu^{(1)}_{n,m,c}(x) + [(m+1)(1+2cx)-cx]\mu_{n,m,c}(x) + mx(cx+2)\mu_{n,m-1c}(x)\end{aligned}$$

Also by easy computation, we have $\mu_{n,0,c}(x)=1$, $\mu_{n,1,c}(x)=\frac{(1+cx)}{n-c}$ and

$$\mu_{n,2,c}(x) = \frac{(4cx+2c^2x^2+2)+nx(cx+2)}{(n-c)(n-2c)}.$$

Consequently for each $x\in[0,\infty)$, it follows from the recurrence relation that

$$\mu_{n,m,c}(x) = O\big(n^{-[(m+1)/2]}\big).$$

Remark 1. It can be easily verified from Lemma 1.1, with $e_i = t^i$, $i = 0, 1, 2,$ that

$$S_{n,c}(e_0, x) = 1, \qquad S_{n,c}(e_1, x) = \frac{(1 + nx)}{n - c}$$

and

$$S_{n,c}(e_2, x) = \frac{n^2x^2 + 4nx + 2}{(n - c)(n - 2c)}.$$

The present paper deals with some direct theorems in simultaneous approximation, for the operators $S_{n,c}(f, x)$.

2. Construction of the operators

In this section, we modify the operators given by (1.2) such that the linear functions are preserved.

For $n > c$, we start by defining

$$r_n(x) = \frac{(n - c)x - 1}{n}, \tag{2.1}$$

we replace x in the definition of $S_{n,c}(f, x)$ by $r_n(x)$. So, to get $r_n(x) \in [0, \infty)$ for any $n \in \mathbb{N}$. Then we give the following modification of the operators $S_{n,c}(f, x)$ defined by (1.2):

$$V_{n,c}(f, x) = \sum_{v=0}^{\infty} s_{n,v}(r_n(x)) \int_0^{\infty} b_{n,v,c}(t) f(t)\, \mathrm{d}t, \tag{2.2}$$

where

$$s_{n,v}(r_n(x)) = \mathrm{e}^{-nr_n(x)} \frac{(nr_n(x))^v}{v!}$$

and $x \in [0, \infty)$, $n \in \mathbb{N}$, the term $b_{n,k,c}(t)$ is given in (1.2).

Lemma 2.1. *For $x \in [0, \infty)$ and $n \in \mathbb{N}$, we have*

(i) $V_{n,c}(e_0; x) = 1,$

(ii) $V_{n,c}(e_1; x) = x,$

(iii) $V_{n,c}(e_2; x) = \dfrac{(n - c)^2x^2 + 2(n - c)x - 1}{(n - c)(n - 2c)}.$

Lemma 2.2. *For $x \in [0, \infty)$, $n \in \mathbb{N}$ and with $\varphi_x = t - x$, we have*

(i) $V_{n,c}(\varphi_x; x) = 0,$

(ii) $V_{n,c}(\varphi_x^2; x) = \dfrac{c(n - c)x^2 + 2(n - c)x - 1}{(n - c)(n - 2c)}.$

The operators $V_{n,c}(f,x)$ preserve the linear functions, that is, for $h(t)=at+b$, where a,b any real constants, we obtain $V_{n,c}(h,x)=h(x)$.

We first establish a direct local approximation theorem for the modified operators $V_{n,c}(f,x)$ in ordinary approximation.

Let $C_B[0,\infty)$ be the space of all real valued continuous bounded functions f on $[0,\infty)$, equipped with the norm $\|f\|=\sup\limits_{x\in[0,\infty)}|f(t)|$. The K-functional is defined by

$$K_2(f,\delta)=\inf\big\{\|f-g\|+\delta\|g''\|:\ g\in W^2_\infty\big\},$$

where $W^2_\infty=\big\{g\in C_B[0,\infty):\ g',g''\in C_B[0,\infty)\big\}$. By [2], there exists a positive constant C such that

$$K_2(f,\delta)\le C\omega_2\big(f,\sqrt{\delta}\,\big),\qquad \delta>0, \tag{2.3}$$

and

$$\omega_2\big(f,\sqrt{\delta}\,\big)=\sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|.$$

Theorem 2.1. *Let $f\in C_B[0,\infty)$. Then for every $x\in[0,\infty)$ and for $n>2c$, $C>0$, we have*

$$|V_{n,c}(f,x)-f(x)|\le C\omega_2\left(f,\sqrt{\frac{(n-c)^2x^2+2(n-c)x-1}{(n-c)(n-2c)}}\,\right).$$

Proof. Let $g\in W^2_\infty$ and $x\in[0,\infty)$. Using Taylor's expansion

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)\,\mathrm{d}u,\qquad t\in[0,\infty),$$

and Lemma 2.2, we have

$$V_{n,c}(g,x)-g(x)=V_{n,c}\left(\int_x^t(t-u)g''(u)\,\mathrm{d}u,x\right).$$

Also $\big|\int_x^t(t-u)g''(u)\,\mathrm{d}u\big|\le(t-x)^2\|g''\|$. Thus

$$|V_{n,c}(g,x)-g(x)|\le V_{n,c}((t-x)^2,x)\|g''\|=\frac{c(n-c)x^2+2(n-c)x-1}{(n-c)(n-2c)}\|g''\|.$$

Also by Lemma 2.1, we have

$$|V_{n,c}(f,x)|\le\sum_{v=0}^{\infty}s_{n,v}(r_n(x))\int_0^\infty b_{n,v,c}(t)|f(t)|\,\mathrm{d}t\le\|f\|.$$

Therefore

$$\begin{aligned}|V_{n,c}(f,x)-f(x)| &\leq |V_{n,c}(f-g,x)-(f-g)(x)|+|V_{n,c}(g,x)-g(x)| \\ &\leq 2\|f-g\|+\frac{c(n-c)x^2+2(n-c)x-1}{(n-c)(n-2c)}\|g''\|.\end{aligned}$$

Finally taking the infimum on the right side over all $g \in W^2_\infty$ and using (2.3), we get the desired result. □

3. Better error estimation

In this section we compute some results on the rates of convergence of the operators $V_{n,c}$ given by (2.2). Then, we will show that our modified operators have better estimation than that of the operators $S_{n,c}$.

Recall that, for $f \in C[0,\infty)$ and $x \in [0,\infty]$, the modulus of continuity of f denoted by $\omega(f,\delta)$ is defined to be

$$\omega(f,\delta)=\sup_{x-\delta\leq t\leq x+\delta,\, t\in[0,\infty)} |f(t)-f(x)|\,.$$

Then we obtain the following result.

Theorem 3.1. *For every $f \in C[0,\infty)$, $x \in [0,\infty)$ and $n \in \mathbb{N}$, we have*

$$|V_{n,c}(f;x)-f(x)| \leq 2\omega(f,\delta_x),$$

where

$$\delta_x := \sqrt{\frac{c(n-c)x^2+2(n-c)x-1}{(n-c)(n-2c)}}.$$

Proof. Now, let $f \in C[0,\infty)$ and $x \in [0,\infty)$. Using linearity and monotonicity of $V_{n,c}$ we easily get, for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$|V_{n,c}(f;x)-f(x)| \leq \omega(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{V_{n,c}\left(\varphi_x^2;x\right)}\right\},$$

Now applying Lemma 2.2 (ii) and choosing $\delta=\delta_x$ the proof is completed. □

Remark 2. For the operator $S_{n,c}$ given by (1.2) we may write that, for every $f \in C[0,\infty)$, $x \in [0,\infty)$ and $n \in \mathbb{N}$,

$$|S_{n,c}(f;x)-f(x)| \leq 2\omega(f,c_x), \tag{3.1}$$

where

$$c_x := \sqrt{\frac{(4cx+2c^2x^2+2)+nx(cx+2)}{(n-c)(n-2c)}}.$$

Now we claim that the error estimation in Theorem 3.1 is better than that of (3.1) provided $f \in C[0,\infty)$ and $x \in [0,\infty)$. Indeed, in order to get this better estimation we must show that $\delta_x \le c_x$. One can obtain that

$$\begin{aligned}\delta_x \le c_x \iff & \frac{c(n-c)x^2 + 2(n-c)x - 1}{(n-c)(n-2c)} \\ \le & \frac{(4cx + 2c^2x^2 + 2) + nx(cx+2)}{(n-c)(n-2c)} \\ \iff & 3c^2x^2 + 6cx + 3 \ge 0 \\ \iff & (cx+1)^2 \ge 0,\end{aligned}$$

which holds true, thus we have $\delta_x \le c_x$, which corrects our claim.

Now we are computing rate of convergence of the operators of $V_{n,c}$ by mean of the elements of the Lipschitz class $\mathrm{Lip}_M(\alpha)$ $(0 < \alpha \le 1)$. To get this, we recall that a function $f \in C_B[0,\infty)$ belongs to $\mathrm{Lip}_M(\alpha)$ if the inequality

$$|f(t) - f(x)| \le M|t-x|^{\alpha} \qquad (x, t \in [0,\infty)) . \tag{3.2}$$

Theorem 3.2. *If $f \in \mathrm{Lip}_M(\alpha)$, $x \ge 0$ and $n > 1$, we have*

$$|V_{n,c}(f;x) - f(x)| \le M \left\{ \frac{nr_n(x)\{1 + (1+cx)\} + (1+cx)}{(n-c)(n-2c)} \right\}^{\alpha/2} .$$

Proof. Since $f \in \mathrm{Lip}_M(\alpha)$ and $x \ge 0$, using inequality (3.2) and then applying the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned}|V_{n,c}(f;x) - f(x)| &\le V_{n,c}\left(|f(t) - f(x)|\,;x\right) \\ &\le MV_{n,c}\left(|t-x|^{\alpha};x\right) \\ &\le M\big\{V_{n,c}\left(\varphi_x^2;x\right)\big\}^{\alpha/2} \\ &\le M\left\{\frac{nr_n(x)\{1 + (1+cx)\} + (1+cx)}{(n-c)(n-2c)}\right\}^{\alpha/2}\end{aligned}$$

proof is completed. □

Remark 3. In the proof of Theorem 3.1, if using $\mu_{n,2,c}(x)$, then we get following result for the operators (1.2)

$$|S_{n,c}(f;x) - f(x)| \le M\left\{\frac{(4cx + 2c^2x^2 + 2) + nx(cx+2)}{(n-c)(n-2c)}\right\}^{\alpha/2}$$

for every $f \in \mathrm{Lip}_M(\alpha)$, $x \ge 0$ and $n > 1$.

Theorem 3.3. *Let f be bounded and integrable on $[0,\infty)$ and admitting second derivative at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n[V_{n,c}f(f;x) - f(x)] = \frac{x(cx+2)}{2} f^{(2)}(x).$$

Remark 4. We may note here that under the conditions of Theorem 3.3, we have

$$\lim_{n\to\infty} n[S_{n,c}f(f;x)-f(x)] = (1+cx)f^{(1)}(x) + \frac{x(cx+2)}{2}f^{(2)}(x).$$

We conclude the paper with the rate of convergence for functions having derivatives of bounded variation.

By $DB_\gamma(0,\infty)$, $\gamma \geq 0$, we denote the class of absolutely continuous functions f defined on $(0,\infty)$ satisfying the growth condition $f(t) = O(t^\gamma)$, $t\to\infty$ and having a derivative f' on the interval $(0,\infty)$ coinciding a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$.

Using Lemma 2.2 and for any number $\lambda > 1$ and $x\in[0,\infty)$ we have for n sufficiently large

$$V_{n,c}((t-x)^2,x) \leq \frac{\lambda x(2+cx)}{n}$$
$$V_{n,c}(|t-x|,x) \leq \sqrt{\frac{\lambda x(2+cx)}{n}}.$$

Theorem 3.4. *Let $f\in DB_\gamma(0,\infty)$, $\gamma>0$ and $x\in(0,\infty)$. Then for $\lambda>1$ and sufficiently large n, we have*

$$\begin{aligned}|V_{n,c}(f,x)-f(x)| \leq{}& \frac{\lambda(cx+2)}{n}\left(\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-x/v}^{x+x/v}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\right)\\ &+\frac{\lambda(cx+2)}{nx}(|f(2x)-f(x)-xf'(x^+)|+|f(x)|)\\ &+\sqrt{\frac{\lambda x(cx+2)}{n}}\left(C2^\gamma O(n^{\frac{-\gamma}{2}})+|f'(x^+)|\frac{|f'(x^+)-f'(x^-)|}{2}\right)\end{aligned}$$

where $\bigvee_a^b f(x)$ is the total variation of f_x on the interval $[a,b]$ and the auxiliary function f_x is defined as

$$f_x(t) = \begin{cases} f(t)-f(x^-), & 0\leq t<x;\\ 0, & t=x;\\ f(t)-f(x^+), & x<t<\infty;\end{cases}$$

$f(x^-)$ and $f(x^+)$ represents the left and right hand limits at x.

The proof of the above theorem is similar to [3], we omit the details.

Remark 5. If we compare the conclusion of the above Theorem 3.4 with the main result of [3], we observe that it has better estimation than [3], under the same conditions. This is because the operators $V_{n,c}(f,x)$ reproduce linear functions.

Acknowledgement. The authors are thankful to the referees for the critical review leading to the better presentation of the paper.

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Received 13. 1. 2009
Accepted 27. 3. 2009

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