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Jain–Durrmeyer operators associated with the inverse Pólya–Eggenberger distribution

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ABSTRACT

The present paper deals with the Jain–Durrmeyer operators based on inverse Pólya–Eggenberger distribution. First, we give the moments with the help of Vandermonde convolution formula and then study approximation properties of these operators which include uniform convergence and degree of approximation.

1. Introduction

The Pólya–Eggenberger (P-E) distribution was introduced by Eggenberger and Pólya [11] in the year 1923. The P-E distribution with parameters (n, A, B, S) is defined as:

$$P(X=k)=\binom{n}{k}\frac{\prod_{i=0}^{k-1}(A+iS)\prod_{i=0}^{n-k-1}(B+iS)}{\prod_{i=0}^{n-1}(A+B+iS)},\quad k=0,1,\ldots,n. \tag{1.1}$$

This gives the probability of getting k white balls out of n drawings from an urn *contains* A white and B black balls, if each time one ball is drawn at random and then replaced together with S balls of the same color.

In literatures, the inverse Pólya–Eggenberger(I-P-E) distribution is defined as:

$$P(X=k)=\binom{n+k-1}{k}\frac{\prod_{i=0}^{k-1}(A+iS)\prod_{i=0}^{n-1}(B+iS)}{\prod_{i=0}^{n+k-1}(A+B+iS)},\quad k=0,1,\ldots,n, \tag{1.2}$$

gives the probability that k white balls are drawn preceding the n-th black ball. The details have been given about these two distributions (1.1) and (1.2) in [21].

Based on I-P-E (1.2), Stancu [28] studied a generalization of the Baskakov operators for a real-valued function bounded on $[0,\infty)$, defined as:

$$V_n^{(\alpha)}(f;x)=\sum_{k=0}^{\infty}v_{n.k}^{(\alpha)}(x)f\left(\frac{k}{n}\right)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}f\left(\frac{k}{n}\right) \tag{1.3}$$

where α is a non-negative parameter which may depend only on the natural number n and $t^{[n,h]}=t(t-h)(t-2h)\cdot\ldots\cdot(t-\overline{n-1}h)$, $t^{[0,h]}=1$ represents the factorial power of t with increment h. In the case when $\alpha=0$, operators (1.3) reduce

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to the following classical Baskakov operators [6].

$$V_n(f,x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right). \tag{1.4}$$

In the year 1972, Jain [20] introduced a new class of linear operators (known as Jain operators) as:

$$J_n^{(\beta)}(f;x)=\sum_{k=0}^{\infty}b_{n,k}^{(\beta)}(x)f\left(\frac{k}{n}\right),\quad x\geq 0, \tag{1.5}$$

where $0\leq\beta<1$ and

$$b_{n,k}^{(\beta)}(x)=\frac{nx(nx+k\beta)^{k-1}e^{-(nx+k\beta)}}{k!}.$$

These operators reduce to Szász–Mirakyan operators for special case $\beta=0$. Several mathematicians studied these Jain operators and their integral variant (See [3,4,12,15,25,29]). Very recently, Gupta and Greubel [16] introduced the Durrmeyer variant of these operators (1.5) in a very different manner as:

$$\begin{aligned} D_n^{(\beta)}(f;x)&=\sum_{k=0}^{\infty}\left(\int_0^{\infty}b_{n,k}^{(\beta)}(t)dt\right)^{-1}b_{n,k}^{(\beta)}(x)\int_0^{\infty}b_{n,k}^{(\beta)}(t)f(t)dt \\ &=\sum_{k=0}^{\infty}\frac{\left\langle b_{n,k}^{(\beta)}(t),f(t)\right\rangle}{\left\langle b_{n,k}^{(\beta)}(t),1\right\rangle}b_{n,k}^{(\beta)}(x), \end{aligned} \tag{1.6}$$

where $\langle f,g\rangle=\int_0^{\infty}f(t)g(t)dt$.

These operators also reduce to Szász–Mirakyan Durrmeyer operators for $\beta=0$. Moreover, the estimation of moments for (1.6) has been calculated by using Stirling numbers of first kind and confluent hypergeometric function. Durrmeyer form of different operators have been studied by many mathematicians and researchers (See [7,8,22]). Motivated from [16], now we propose Hybrid type operators as:

$$\begin{aligned} L_n^{(\alpha)}(f;x)&=\sum_{k=0}^{\infty}\left(\int_0^{\infty}b_{n,k}^{(\alpha)}(t)dt\right)^{-1}v_{n,k}^{(\alpha)}(x)\int_0^{\infty}b_{n,k}^{(\alpha)}(t)f(t)dt \\ &=\sum_{k=0}^{\infty}\frac{\left\langle b_{n,k}^{(\alpha)}(t),f(t)\right\rangle}{\left\langle b_{n,k}^{(\alpha)}(t),1\right\rangle}v_{n,k}^{(\alpha)}(x). \end{aligned} \tag{1.7}$$

For $\alpha=0$, these operators include the hybrid operators (Baskakov–Szász–Durrmeyer operators) presented in [1,18] etc.

These days several mathematicians have worked on P-E based positive linear operators [2,13,17,24,26,27] and few mathematicians have discussed I-P-E based positive linear operators [14,19,28]. Very recently Deo et al. [9] defined I-P-E based Baskakov–Kantorovich operators and studied its properties, asymptotic formula as well as degree of approximation.

Now in the present paper, we obtain the moments of I-P-E distribution based operators (1.3) by using Vandermonde convolution formula, however these moments have already been calculated by Stancu in [28] by using hypergeometric series. Then we consider Jain–Durrmeyer operators associated with the inverse Pólya–Eggenberger distribution (1.7) and study uniform convergence and degree of approximation for these operators.

2. Auxiliary results

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $k\in\mathbb{N}_0$, let $e_k(x)=x^k$ be the test functions. The computation of the images of test functions by Stancu–Baskakov operators (1.3) was done in [28]. Now we find those images with the help of Vandermonde convolution formula and following relation:

$$t^{[i+j,h]}=t^{[i,h]}.(t-ih)^{[j,h]} \tag{2.1}$$

for $i,j\in\mathbb{N}$ and $h\neq 0$.

Lemma 2.1. *For the Stancu–Baskakov operators (1.3), the following results hold:*

$$V_n^{(\alpha)}(e_0;x)=1,\qquad V_n^{(\alpha)}(e_1;x)=\frac{x}{1-\alpha},$$

and

$$V_n^{(\alpha)}(e_2;x)=\frac{1}{(1-\alpha)(1-2\alpha)}\left[x^2+\frac{x(x+1)}{n}+\alpha\left(1-\frac{1}{n}\right)x\right].$$

Proof. To achieve the results of this lemma we use the relation (2.1), we have

$$V_n^{(\alpha)}(e_0;x)=\sum_{k=0}^{\infty}v_{n,k}^{(\alpha)}(x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^{[k,-\alpha]}1^{[n,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}$$
$$=\frac{1^{[n,-\alpha]}}{1^{[n,-\alpha]}}=1.$$

From (1.3) we have

$$V_n^{(\alpha)}(e_1;x)=\sum_{k=0}^{\infty}v_{n,k}^{(\alpha)}(x)\frac{k}{n}=\sum_{k=1}^{\infty}\binom{n+k-1}{n}\frac{x^{[k,-\alpha]}1^{[n,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}$$
$$=\sum_{k=1}^{\infty}\binom{n+k-1}{n}\frac{x^{[1,-\alpha]}(x+\alpha)^{[k-1,-\alpha]}1^{[1,-\alpha]}(1+\alpha)^{[n-1,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}$$
$$=x\frac{(1+\alpha)^{[n-1,-\alpha]}}{(1-\alpha)^{[n+1,-\alpha]}}=\frac{x}{1-\alpha},$$

and

$$V_n^{(\alpha)}(e_2;x)=\sum_{k=0}^{\infty}v_{n,k}^{(\alpha)}(x)\frac{k^2}{n^2}=\sum_{k=0}^{\infty}\frac{k(k-1)+k}{n^2}v_{n,k}^{(\alpha)}(x)=\frac{1}{n}\Bigg[V_n^{(\alpha)}(e_1;x)$$
$$+(n+1)\sum_{k=2}^{\infty}\binom{n+k-1}{n+1}\frac{x^{[2,-\alpha]}(x+2\alpha)^{[k-2,-\alpha]}1^{[2,-\alpha]}(1+2\alpha)^{[n-2,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}\Bigg]$$
$$=\frac{x}{n(1-\alpha)}+\left(\frac{n+1}{n}\right)\frac{x(x+\alpha)(1+\alpha)(1+2\alpha)^{[n-2,-\alpha]}}{(1-2\alpha)^{[n+2,-\alpha]}}$$
$$=\frac{1}{(1-\alpha)(1-2\alpha)}\left[x^2+\frac{x(x+1)}{n}+\alpha\left(1-\frac{1}{n}\right)x\right].$$

□

Lemma 2.2 [16]. *For $0\le\alpha<1$, we have*

$$\frac{\left\langle b_{n,k}^{(\alpha)}(t),t^r\right\rangle}{\left\langle b_{n,k}^{(\alpha)}(t),1\right\rangle}=P_r(k;\alpha),$$

where $P_r(k;\alpha)$ is a polynomial of order r in variable k and $\langle f,g\rangle=\int_0^{\infty}f(t)g(t)dt$.

In particular

$$P_0(k;\alpha)=1,$$
$$P_1(k;\alpha)=\frac{1}{n}\left[(1-\alpha)k+\frac{1}{1-\alpha}\right],$$
$$P_2(k;\alpha)=\frac{1}{n^2}\left[(1-\alpha)^2k^2+3k+\frac{2!}{1-\alpha}\right],$$
$$P_3(k;\alpha)=\frac{1}{n^3}\left[(1-\alpha)^3k^3+6(1-\alpha)k^2+\frac{(11-8\alpha)k}{1-\alpha}+\frac{3!}{1-\alpha}\right],$$
$$P_4(k;\alpha)=\frac{1}{n^4}\left[(1-\alpha)^4k^4+10(1-\alpha)^2k^3+5(7-4\alpha)k^2+\frac{10(5-3\alpha)k}{1-\alpha}+\frac{4!}{1-\alpha}\right],$$
$$P_5(k;\alpha)=\frac{1}{n^5}\Bigg[(1-\alpha)^5k^5+15(1-\alpha)^3k^4+5(1-\alpha)(17-8\alpha)k^3+\frac{15(15-20\alpha+6\alpha^2)k^2}{1-\alpha}$$
$$+\frac{(274-144\alpha)k}{1-\alpha}+\frac{5!}{1-\alpha}\Bigg].$$

Lemma 2.3. *For the operators $L_n^{(\alpha)}$ given by (1.7), the moments up to second order are given by*

$$L_n^{(\alpha)}(e_0;x)=1,\quad L_n^{(\alpha)}(e_1;x)=x+\frac{1}{n(1-\alpha)},$$

and

$$L_n^{(\alpha)}(e_2;x) = \frac{1-\alpha}{1-2\alpha}\left[x^2 + \frac{x(x+1)}{n} + \alpha\left(1-\frac{1}{n}\right)x\right] + \frac{3x}{n(1-\alpha)} + \frac{2}{n^2(1-\alpha)}.$$

Proof. Starting from $r-th$ order moment and making use of Lemma 2.2, we get

$$L_n^{(\alpha)}(e_r;x) = \sum_{k=0}^{\infty} \frac{\left\langle b_{n,k}^{(\alpha)}(t), t^r\right\rangle}{\left\langle b_{n,k}^{(\alpha)}(t), 1\right\rangle} v_{n,k}^{(\alpha)}(x) = \sum_{k=0}^{\infty} P_r(k,\alpha) v_{n,k}^{(\alpha)}(x). \tag{2.2}$$

With the help of Lemma 2.1 and statement (2.2) we shall find the first three moments

$$L_n^{(\alpha)}(e_0;x) = \sum_{k=0}^{\infty} P_0(k,\alpha) v_{n,k}^{(\alpha)}(x) = 1,$$

$$\begin{aligned} L_n^{(\alpha)}(e_1;x) &= \sum_{k=0}^{\infty} P_1(k,\alpha) v_{n,k}^{(\alpha)}(x) = \sum_{k=0}^{\infty} \frac{1}{n}\left[(1-\alpha)k + \frac{1}{1-\alpha}\right] v_{n,k}^{(\alpha)}(x) \\ &= (1-\alpha)V_n^{(\alpha)}(e_1;x) + \frac{1}{n(1-\alpha)} V_n^{(\alpha)}(e_0;x) = x + \frac{1}{n(1-\alpha)}, \end{aligned}$$

and

$$\begin{aligned} L_n^{(\alpha)}(e_2;x) &= \sum_{k=0}^{\infty} P_2(k,\alpha) v_{n,k}^{(\alpha)}(x) \\ &= \sum_{k=0}^{\infty} \frac{1}{n^2}\left[(1-\alpha)^2 k^2 + 3k + \frac{2!}{1-\alpha}\right] v_{n,k}^{(\alpha)}(x) \\ &= (1-\alpha)^2 V_n^{(\alpha)}(e_2;x) + \frac{3}{n} V_n^{(\alpha)}(e_1;x) + \frac{2}{n^2(1-\alpha)} V_n^{(\alpha)}(e_0;x) \\ &= \frac{1-\alpha}{1-2\alpha}\left[x^2 + \frac{x(x+1)}{n} + \alpha\left(1-\frac{1}{n}\right)x\right] + \frac{3x}{n(1-\alpha)} + \frac{2}{n^2(1-\alpha)}. \end{aligned}$$

□

Remark 2.1. Taking Lemma 2.3 into the account, we get the following central moments:

$$L_n^{(\alpha)}(e_1 - x;x) = \frac{1}{n(1-\alpha)}, \tag{2.3}$$

and

$$\begin{aligned} L_n^{(\alpha)}\left((e_1-x)^2;x\right) &= \frac{1-\alpha}{1-2\alpha}\left[\frac{\alpha}{1-\alpha}x^2 + \frac{x(x+1)}{n} + \alpha\left(1-\frac{1}{n}\right)x\right] \\ &\quad + \frac{x}{n(1-\alpha)} + \frac{2}{n^2(1-\alpha)}. \end{aligned} \tag{2.4}$$

Lemma 2.4. *For the operators $L_n^{(\alpha)}$, we have*

$$\left|L_n^{(\alpha)}(f;x)\right| \le \|f\|,$$

where $f \in C[0,\infty)$ and $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$.

Proof. The definition of Stancu–Jain operators (1.7) and Lemma 2.3 yield

$$\begin{aligned} \left|L_n^{(\alpha)}(f;x)\right| &\le \left|\sum_{k=0}^{\infty}\left(\int_0^{\infty} b_{n,k}^{(\alpha)}(t)dt\right)^{-1} v_{n,k}^{(\alpha)}(x)\int_0^{\infty} b_{n,k}^{(\alpha)}(t)f(t)dt\right| \\ &\le \sum_{k=0}^{\infty}\left(\int_0^{\infty} b_{n,k}^{(\alpha)}(t)dt\right)^{-1} v_{n,k}^{(\alpha)}(x)\int_0^{\infty} b_{n,k}^{(\alpha)}(t)|f(t)|dt \le \|f\|. \end{aligned}$$

□

3. Direct results

Using the well-known Bohman–Korovkin–Popoviciu theorem (see [23]) we get the uniform convergence of the Stancu–Jain operators (1.7).

Theorem 3.1. *Let $f \in C[0,\infty) \cap E$ and $\alpha(n)$ be such that $\alpha \to 0$ as $n \to \infty$, then we have*

$$\lim_{n\to\infty} L_n^{(\alpha)}(f;x) = f(x)$$

uniformly on each compact subset of $[0,\infty)$, where $C[0,\infty)$ is the space of all real-valued continuous functions on $[0,\infty)$ and

$$E := \left\{ f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{is convergent as } x \to \infty \right\}$$

Proof. Taking Lemma 2.3 into the account and the fact that $\alpha \to 0$ as $n \to \infty$, it is clear that

$$\lim_{n\to\infty} L_n^{(\alpha)}(e_i;x) = x^i, \tilde{}\tilde{}i = 0,1,2$$

uniformly on each compact subset of $[0,\infty)$. Hence, applying the well-known Korovkin-type theorem [5] regarding the convergence of a sequence of positive linear operators, we get the desired result.□

Modulus of continuity is the main tool to measure the degree of approximation of linear positive operators towards the identity operators.

Definition 3.1. *Let $f \in C_B[0,\infty)$ be given and $\delta > 0$. The modulus of continuity of the function f is defined by*

$$\omega(f,\delta) := \sup\{|f(x)-f(y)| \,:\, x,y \in [0,\infty),\; |x-y| \le \delta\}, \tag{3.1}$$

where $C_B[0,\infty)$ is the space of all real-valued continuous and bounded functions on $[0,\infty)$.

Definition 3.2. *For $f \in C[0,\infty)$ and $\delta > 0$*

$$\omega_2(f,\delta) := \sup\{|f(x+h)-2f(x)+f(x-h)| \,:\, x, x\pm h \in [0,\infty),\; 0 \le h \le \delta\} \tag{3.2}$$

is the modulus of smoothness of second order.

Definition 3.3. *Let be f from the space $C_B[0,\infty)$ endowed with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$ and let us consider Peetre's K-functional*

$$K_2(f,\delta) = \inf_{g\in W_\infty^2} \left\{ \|f-g\| + \delta \left\|g''\right\| \right\}, \tag{3.3}$$

where $\delta > 0$ and $W_\infty^2 = \left\{ g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty) \right\}$. Also, from ([10], p. 177, Theorem 2.4), there exists an absolute constant $A > 0$ such that

$$K_2(f,\delta) \le A\omega_2\left(f,\sqrt{\delta}\right), \tag{3.4}$$

Theorem 3.2. *If $f \in C_B[0,\infty)$, then for $x \in [0,\infty)$, it follows:*

$$\left|L_n^{(\alpha)}(f;x) - f(x)\right| \le 2\cdot\omega(f,\delta), \quad \text{with} \quad \delta = \left(L_n^{(\alpha)}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}.$$

Proof. From (1.7), Lemma 2.3 and property of modulus of continuity, we have

$$\begin{aligned} &\left|L_n^{(\alpha)}(f;x) - f(x)\right| \\ &\le \sum_{k=0}^{\infty} \left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{-1} v_{n,k}^{(\alpha)}(x) \int_0^\infty b_{n,k}^{(\alpha)}(t)|f(t)-f(x)|dt \\ &\le \left[1 + \tfrac{1}{\delta}\sum_{k=0}^{\infty} \left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{-1} v_{n,k}^{(\alpha)}(x) \int_0^\infty b_{n,k}^{(\alpha)}(t)|t-x|dt\right]\omega(f,\delta). \end{aligned}$$

Applying Cauchy–Schwarz inequality for integration, we get

$$\begin{aligned} \left|L_n^{(\alpha)}(f;x) - f(x)\right| \le &\left[1 + \tfrac{1}{\delta}\sum_{k=0}^{\infty} \left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{-1} v_{n,k}^{(\alpha)}(x) \left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{1/2}\right. \\ &\left.\times \left(\int_0^\infty b_{n,k}^{(\alpha)}(t)(t-x)^2dt\right)^{1/2}\right]\omega(f,\delta). \end{aligned}$$

Again using Cauchy–Schwarz inequality for sum, we have

$$\left|L_n^{(\alpha)}(f;x)-f(x)\right|$$
$$\leq \left[1+\frac{1}{\delta}\left\{\sum_{k=0}^{\infty}\left(\int_0^{\infty} b_{n,k}^{(\alpha)}(t)dt\right)^{-1} v_{n,k}^{(\alpha)}(x)\int_0^{\infty} b_{n,k}^{(\alpha)}(t)dt\right\}^{1/2}\right.$$
$$\left.\times\left\{\sum_{k=0}^{\infty}\left(\int_0^{\infty} b_{n,k}^{(\alpha)}(t)dt\right)^{-1} v_{n,k}^{(\alpha)}(x)\int_0^{\infty} b_{n,k}^{(\alpha)}(t)(t-x)^2dt\right\}^{1/2}\right]\omega(f,\delta)$$
$$=\left[1+\frac{1}{\delta}\left\{L_n^{(\alpha)}(e_0;x)\right\}^{1/2}\left\{L_n^{(\alpha)}\left((e_1-x)^2;x\right)\right\}^{1/2}\right]\omega(f,\delta).$$

If we choose $\delta=\left(L_n^{(\alpha)}\left((e_1-x)^2;x\right)\right)^{1/2}$, then we obtain

$$\left|L_n^{(\alpha)}(f;x)-f(x)\right|\leq 2\omega(f,\delta).$$

□

Theorem 3.3. *Let be $f\in C[0,\infty)$, then for any $x\in[0,\infty)$ gives*

$$\left|L_n^{(\alpha)}(f;x)-f(x)\right|\leq A\omega_2\left(f,\tfrac{1}{2}\delta_n(x,\alpha)\right)+\omega(f,\delta_\omega),$$

where A is an absolute constant, $\delta_\omega=L_n^{(\alpha)}(e_1-x;x)$ and

$$\delta_n(x,\alpha)=\left(L_n^{(\alpha)}\left((e_1-x)^2;x\right)+\left(L_n^{(\alpha)}(e_1-x;x)\right)^2\right)^{\frac{1}{2}}.$$

Proof. For $x\in[0,\infty)$ consider the following operators:

$$\tilde{L}_n^{(\alpha)}(f;x)=L_n^{(\alpha)}(f;x)-f\left(x+\frac{1}{n(1-\alpha)}\right)+f(x). \quad (3.5)$$

From Lemma 2.3, we get that

$$\tilde{L}_n^{(\alpha)}(e_0;x)=1$$

and

$$\tilde{L}_n^{(\alpha)}(e_1;x)=x$$

i.e. the operators $\tilde{L}_n^{(\alpha)}$ preserve constants and linear functions. Therefore

$$\tilde{L}_n^{(\alpha)}(e_1-x;x)=0. \quad (3.6)$$

Let $g\in W_\infty^2$ and $t\in[0,\infty)$. By Taylor's expansion, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Applying $\tilde{L}_n^{(\alpha)}$ to above expression and using (3.6), we get

$$\tilde{L}_n^{(\alpha)}(g;x)-g(x)=g'(x)\cdot\tilde{L}_n^{(\alpha)}(e_1-x;x)+\tilde{L}_n^{(\alpha)}\left(\int_x^t(t-u)g''(u)du;x\right)$$
$$=L_n^{(\alpha)}\left(\int_x^t(t-u)g''(u)du;x\right)-\int_x^{x+\frac{1}{n(1-\alpha)}}\left(x+\frac{1}{n(1-\alpha)}-u\right)g''(u)du.$$

Also, we consider following:

$$\left|\int_x^t(t-u)g''(u)du\right|\leq(t-x)^2\|g''\|,$$

then

$$\left|\tilde{L}_n^{(\alpha)}(g;x)-g(x)\right|\leq\left(L_n^{(\alpha)}((e_1-x)^2;x)+\left(L_n^{(\alpha)}(e_1-x;x)\right)^2\right)\|g''\|.$$

Again using the expression (3.5) for the operators $\tilde{L}_n^{(\alpha)}$ and Lemma 2.4, it follows:

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\le \left|\tilde{L}_n^{(\alpha)}(f-g;x)\right|+\left|\tilde{L}_n^{(\alpha)}(g;x)-g(x)\right| \\ &\quad+|g(x)-f(x)|+\left|f\left(x+\frac{1}{n(1-\alpha)}\right)-f(x)\right| \\ &\le 4\|f-g\|+\delta_n^2(x,\alpha)\left\|g''\right\|+\omega(f,\delta_\omega),\end{aligned}$$

with $\delta_n^2(x,\alpha)=L_n^{(\alpha)}((e_1-x)^2;x)+(L_n^{(\alpha)}(e_1-x;x))^2$ and $\delta_\omega=L_n^{(\alpha)}(e_1-x;x)$.

Now, taking infimum on the right-hand side over all $g\in W_\infty^2$ and using the relation (3.4), we get

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\le 4K_2\left(f,\frac{{\delta_n}^2(x,\alpha)}{4}\right)+\omega(f,\delta_\omega) \\ &\le A\omega_2\left(f,\tfrac{1}{2}\delta_n(x,\alpha)\right)+\omega(f,\delta_\omega).\end{aligned}$$

□

Now we present ordinary approximation in terms of Lipschitz constant defined by

$$Lip_M(\beta)=\left\{f\in C_B(0,\infty]:|f(t)-f(x)|\le M\frac{|t-x|^\beta}{(t+x)^{\beta/2}}\right\}, \tag{3.7}$$

where M is a positive constant and $0<\beta\le 1$.

Theorem 3.4. *Let be $f\in C_B[0,\infty)$, then for any $x\in[0,\infty)$, the following inequality holds:*

$$\left|L_n^{(\alpha)}(f;x)-f(x)\right|\le M\left(\frac{\lambda_n(x,\alpha)}{x}\right)^{\beta/2},$$

where

$$\lambda_n(x,\alpha)=L_n^{(\alpha)}\left((t-x)^2;x\right).$$

Proof. By using the fact that $f\in Lip_M(\beta)$, $0<\beta\le 1$ it follows:

$$\begin{aligned}&\left|L_n^{(\alpha)}(f;x)-f(x)\right| \\ &\le \sum_{k=0}^{\infty}\left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{-1}v_{n,k}^{(\alpha)}(x)\int_0^\infty b_{n,k}^{(\alpha)}(t)|f(t)-f(x)|dt \\ &\le M\sum_{k=0}^{\infty}\left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{-1}v_{n,k}^{(\alpha)}(x)\int_0^\infty b_{n,k}^{(\alpha)}(t)\frac{|t-x|^\beta}{(t+x)^{\beta/2}}dt.\end{aligned}$$

Applying Hölder inequality for integration and sum respectively by taking $p=\frac{2}{\beta}$ and $q=\frac{2}{2-\beta}$,

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\le M\left\{\sum_{k=0}^{\infty}\left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{-1}v_{n,k}^{(\alpha)}(x)\left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)\right\}^{(2-\beta)/2} \\ &\quad\times\left\{\sum_{k=0}^{\infty}\left(\int_0^\infty b_{n,k}^{(\alpha)}(t)dt\right)^{-1}v_{n,k}^{(\alpha)}(x)\int_0^\infty b_{n,k}^{(\alpha)}(t)\frac{(t-x)^2}{t+x}dt\right\}^{\beta/2}.\end{aligned}$$

Using the fact that $\frac{1}{t+x}\le\frac{1}{x}$, we finally get

$$\left|L_n^{(\alpha)}(f;x)-f(x)\right|\le M\left(\frac{\lambda_n(x,\alpha)}{x}\right)^{\beta/2}.$$

where

$$\lambda_n(x,\alpha)=L_n^{(\alpha)}\left((t-x)^2;x\right).$$

□

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