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Direct result on exponential-type operators ☆

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ABSTRACT

In this paper, we obtain a direct result and Voronovskaya type asymptotic formula for the exponential-type operators in simultaneous approximation with polynomial growth. In the end, we obtain the recurrence formulae for the central moments, direct results and Voronovskaya type asymptotic formulae for the various mixed summation-integral type operators.

1. Introduction

We denote $C^k[a,b], k = 0,1,2,\ldots$ the set of all real-valued functions, which are defined and k-times continuously differentiable on the interval (to be closed, half-open and open) $[a,b](b>0)$ of the real axis and $f^{(k)}(x) = O(x^\alpha)$, (α is a positive integer) as $x \to \infty$.

In [1], Baskakov introduced the sequence of positive linear operators $\{L_n\}, n \in N$, $L_n : C^0[a,\infty) \to C^0[0,R], R > 0$, which is defined as follows:

$$(L_n f)(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \phi_n^{(k)}(x) f\left(\frac{k}{n}\right) \tag{1.1}$$

and is generated by a sequence of functions $\{\phi_n\}$, $\phi_n : C \to C$, possess the following properties:

(i) ϕ_n is analytic on the interval $[0,R]$;
(ii) $\phi_n(0) = 1$;
(iii) $(-1)^k \phi_n^{(k)}(x) \geqslant 0$ if $k = 0,1,2,\ldots$ and $x \in [0,R]$, i.e., ϕ_n is completely monotone on $[0,R]$;
(iv) there exists a positive integer $m(n)$, not depending on k, such that

$$\phi_n^{(k)}(x) = -n\phi_{m(n)}^{(k-1)}(x)\{1 + \alpha_{n,k}(x)\}, x \in [0,R] \quad \text{and } k = 1,2,\ldots$$

where $\alpha_{n,k}(x)$ converges to zero uniformly in k and x on $[0,R]$ when $n \to \infty$;
(v) $\lim_{n\to\infty} \frac{n}{m(n)} = 1$.

It may be shown, by means of some suitable substitutions:

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$$\phi_n(x) = (1-x)^n, \quad x \in [0,1], \quad c = -1,$$
$$\phi_n(x) = e^{-nx}, \quad x \in [0,\infty), \quad c = 0,$$
$$\phi_n(x) = (1+x)^{-\frac{n}{c}}, \quad x \in [0,\infty), \quad c > 0,$$

that the operators (1.1) generalize some well-known operators.

From now on we always denote corresponding intervals by I, i.e., $I = [0,\infty)$ in the case when $c \geqslant 0$ and $I = [0,1]$ in the case when $c = -1$. With these notations, we can defined the nth general exponential-type operators M_n by:

$$M_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad \text{where } p_{n,k}(x) = \frac{(-x)^k}{k!}\phi_n^{(k)}(x), \tag{1.2}$$

for $x \in I$ and $n > max\{0,-c\}$. It is easy to see that $M_n(f,x)$ are Bernstein [2], Szász-Mirakian [12] and Baskakov [1] operators for $c = -1$, $c = 0$ and $c > 0$, respectively.

The discrete values $f(\frac{k}{n})$ in (1.2) are replaced by an integral over the weighted function. This concept has been given by Durrmeyer [5], where, he has introduced integral modification of Bernstein operators. In the similar manner, integral modification of other two classical operators (Szász-Mirakian and Baskakov) have been given by Mazhar & Totik [10], Kasana et al. [9], Sahai & Prasad [11], and Heilmann [6][1].

Definition 1.1. Let $c \in N \cup \{0\} \cup \{-1\}, n \in N, n > c$, and

$$I = \begin{cases} [0,1] & \text{in case } c = -1, \\ [0,\infty) & \text{in case } c \geqslant 0. \end{cases}$$

For $f : I \to R$ the nth Durrmeyer operators for above mentioned three operators is defined as:

$$(V_n f)(x) = (n-c)\sum^{\otimes}(-x)^k \frac{\phi_n^{(k)}(x)}{k!}\int_I (-t)^k \frac{\phi_n^{(k)}(t)}{k!} f(t)\,dt = (n-c)\sum^{\otimes} p_{n,k}(x)\int_I p_{n,k}(t) f(t)\,dt, \quad x \in I, \tag{1.3}$$

whenever the right-hand side makes sense. The weight functions $p_{n,k}(x)$ are given as:

$$\text{if } c = -1, \quad p_{n,k}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k}, & k \leqslant n, \\ 0, & k > n; \end{cases}$$
$$\text{if } c = 0, \quad p_{n,k}(x) = \tfrac{(nx)^k}{k!} e^{-nx};$$
$$\text{if } c > 0, \quad p_{n,k}(x) = \binom{n+k-1}{k} \tfrac{x^k}{(1+x)^{n+k}}.$$

Very recently similar type of mixed Szász-Mirakian-Beta operators were studied by Gupta and Noor [8] and Deo et al. [3] have given an other modification of Bernstein operators.

The main motivation of this paper is to obtain a direct result as well as Voronovskaya type asymptotic formula in a unified form, via Derriennic [4] type modified Baskakov-type operators, which generalize some well-known exponential operators (Bernstein, Szász-Mirakian and Baskakov) in simultaneous approximation. In the last section of this paper we have given central moments in the recurrence relations form for the various mixed summation-integral type operators.

2. Auxiliary results

In this section, we give some preliminary results which will be used in the sequel:

Lemma 2.1. [4,7]. *Let $r, m \in N \cup \{0\} = \{0,1,2,\ldots\}$, then for $n > cr$, we get*

$$\mu_{r,n,m}(x) = \{n - c(r+1)\}\sum^{\otimes} p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)(t-x)^m\,dt, \quad x \in I,$$

then the recurrence relation holds:

$$\{n - c(r+m+2)\}\mu_{r,n,m+1}(x) = \phi^2(x)\{\mu'_{r,n,m}(x) + 2m\mu_{r,n,m-1}(x)\} + (r+m+1)(1+2cx)\mu_{r,n,m}(x), \tag{2.1}$$

where $\phi(x) = \sqrt{x(1+cx)}$ and $n > c(r+m+2)$. Consequently,

$$\mu_{r,n,0}(x) = 1, \mu_{r,n,1}(x) = \tfrac{(r+1)(1+2cx)}{\{n-c(r+2)\}} \text{ and}$$
$$\mu_{r,n,2}(x) = \tfrac{2(n-c)\phi^2(x) + (r+1)(r+2)(1+2cx)^2}{\{n-c(r+2)\}\{n-c(r+3)\}}.$$

For all $x \in I$, $\mu_{r,n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integral part of α.

[1] When $c = -1$; $\sum^{\otimes}$ stand for $\sum_{k=0}^{n-r}$ and when $c \geqslant 0$; $\sum^{\otimes}$ stand for $\sum_{k=0}^{\infty}$.

Proof. We first prove (2.1) by using $\phi^2(x)p'_{n,k}(x)=(k-nx)p_{n,k}(x)$. Now by definition of $\mu_{r,n,m}(x)$, we get

$$\begin{aligned}
&\phi^2(x)[\mu'_{r,n,m}(x)+m\mu_{r,n,m-1}(x)]\\
&=\{n-c(r+1)\}\sum^{\otimes}\phi^2(x)p'_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)(t-x)^m\mathrm{d}t\\
&=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I\{k-(n+cr)x\}p_{n-cr,k+r}(t)(t-x)^m\mathrm{d}t\\
&=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I[\{(k+r)-(n-cr)t\}-r(1+2cx)+(n-cr)(t-x)]p_{n-cr,k+r}(t)(t-x)^m\mathrm{d}t\\
&=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I\phi^2(t)p'_{n-cr,k+r}(t)(t-x)^m\mathrm{d}t-r(1+2cx)\mu_{r,n,m}(x)+(n-cr)\mu_{r,n,m+1}(x)\\
&=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I\{(1+2cx)(t-x)+c(t-x)^2+x(1+cx)\}p'_{n-cr,k+r}(t)(t-x)^m\mathrm{d}t\\
&\quad-r(1+2cx)\mu_{r,n,m}(x)+(n-cr)\mu_{r,n,m+1}(x).
\end{aligned}$$

Integrating by part the first term on the right-hand side leads to the required result (2.1).

The proofs of the other consequences easily follow by the definition of $\mu_{r,n,m}(x)$ and (2.1). This completes the proof of the lemma. □

Lemma 2.2. *For $r\in N\cup\{0\}=\{0,1,2,\ldots\}$, we have*

$$(V_n^{(r)}f)(x)=(n-c)\beta(n,r)\sum^{\otimes}p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)\frac{\mathrm{d}^rf(t)}{\mathrm{d}t^r}\mathrm{d}t,\quad x\in I,\tag{1.2}$$

where $I=[0,1]$ and $\beta(n,r)=\frac{(n!)^2}{(n-r)!(n+r)!}<1$, for $r\leqslant n$ (see [4,7]), *in the case when $c=-1$ and $I=[0,\infty)$ and $\beta(n,r)=\prod_{j=0}^{r-1}\frac{n+cj}{n-c(j+1)}$, in the case when $c\geqslant 0$.*

Proof. On differentiating (1.3) r-times with respect to x and applying Leibnitz's theorem, we get

$$\begin{aligned}
(V_n^{(r)}f)(x)&=(n-c)\sum^{\otimes}\frac{(-1)^r(-x)^k\phi_n^{(k+r)}(x)}{k!}\int_I\left\{\sum_{i=0}^r\binom{r}{i}\frac{(-1)^{r-i}(-t)^{k+i}\phi_n^{(k+i)}(t)}{(k+i)!}\right\}f(t)\mathrm{d}t\\
&=(n-c)\sum^{\otimes}\frac{(-1)^r(-x)^k\phi_n^{(k+r)}(x)}{k!}\int_I\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left\{\frac{(-t)^{k+r}\phi_n^{(k+r)}(t)}{(k+i)!}\right\}f(t)\mathrm{d}t.
\end{aligned}$$

On integrating r-times by parts, the right hand side of the above expression, we obtain the required result by analogous manner of Lemma II.6 of [4] □.

3. Main results

Theorem 3.1. *If $f^{(r)}(t), r\geqslant 0$ is bounded and integrable in I and admits $(r+2)$th derivative at a point $x\in I$, and $f^{(r)}(t)=O(t^\alpha)$ as $t\to\infty$ for some $\alpha>0$, then we get*

$$\lim_{x\to\infty}n\left\{\frac{n-c(r+1)}{(n-c)\beta(n,r)}(V_n^{(r)}f)(x)-f^{(r)}(x)\right\}=(r+1)(1+2cx)f^{(r+1)}(x)+\phi^2(x)f^{(r+2)}(x).$$

Proof. Using Taylor's formula, we have

$$f^{(r)}(t)-f^{(r)}(x)=(t-x)f^{(r+1)}(x)+\frac{(t-x)^2}{2}f^{(r+2)}(x)+\frac{(t-x)^2}{2}\zeta(u),\tag{3.1}$$

where

$$\zeta(u)\equiv\zeta(t,x)=\frac{f^{(r)}(t)-f^{(r)}(x)-(t-x)f^{(r+1)}(x)-\frac{(t-x)^2}{2}f^{(r+2)}(x)}{\frac{(t-x)^2}{2}}\quad\text{if } x\neq t$$

$$=0\quad\text{if } x=t.$$

i.e., $\zeta(u)\to 0$ as $u\to 0$ and ζ is bounded as well as integrable in I. Now applying (3.1) to this and using Lemma (2.1), we obtain

$$\begin{aligned}
&\tfrac{n-c(r+1)}{(n-c)\beta(n,r)}(V_n^{(r)}f)(x)-f^{(r)}(x)=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)f^{(r)}(t)dt-f^{(r)}(x)\\
&\quad=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)\{f^{(r)}(t)-f^{(r)}(x)\}\mathrm{d}t=\mu_{r,n,1}f^{(r+1)}(x)+\tfrac{\mu_{r,n,2}}{2}f^{(r+2)}(x)+R_{n,r}(x),
\end{aligned}$$

where

$$R_{n,r}(x)=\frac{1}{2}\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)(t-x)^2\zeta(t,x)\mathrm{d}t.$$

We shall show that $nR_{n,r}(x)\to 0$ as $n\to\infty$. Let $\varepsilon>0$ and $A>0$ be arbitrary there exists a $\delta>0$, such that

$$|\zeta(u)|\leqslant\varepsilon,\quad \text{when } |u|\leqslant\delta,\ x\leqslant A. \tag{3.2}$$

Now

$$nR_{n,r}(x)=Q_{n,r,1}(x)+Q_{n,r,2}(x)$$

where

$$Q_{n,r,1}(x)=\frac{n\{n-c(r+1)\}}{2}\sum^{\otimes}p_{n+cr,k}(x)\int_{|t-x|\leqslant\delta}p_{n-cr,k+r}(t-x)^2\zeta(t,x)\,\mathrm{d}t$$

and

$$Q_{n,r,2}(x)=\frac{n\{n-c(r+1)\}}{2}\sum^{\otimes}p_{n+cr,k}(x)\int_{|t-x|>\delta}p_{n-cr,k+r}(t-x)^2\zeta(t,x)\,\mathrm{d}t.$$

From Lemma (2.1) and (3.2), we obtain

$$\begin{aligned}|Q_{n,r,1}(x)| &< \tfrac{\varepsilon n\{n-c(r+1)\}}{2}\sum^{\otimes}p_{n+cr,k}(x)\int_{|t-x|\leqslant\delta}p_{n-cr,k+r}(t)(t-x)^2\,\mathrm{d}t\\ &\leqslant \varepsilon\phi^2(x)\quad \text{as}\quad n\to\infty.\end{aligned} \tag{3.3}$$

Finally, we estimate $Q_{n,r,2}(x)$, using the assumption of theorem,

$$\begin{aligned}Q_{n,r,2}(x)&=O\left(\frac{n\{n-c(r+1)\}}{2}\sum^{\otimes}p_{n+cr,k}(x)\int_{|t-x|>\delta}p_{n-cr,k+r}(t)t^{\alpha}\mathrm{d}t\right)\\
&=O\left(\frac{n\{n-c(r+1)\}}{2}\sum^{\otimes}p_{n+cr,k}(x)\int_{|t-x|>\delta}p_{n-cr,k+r}(t)\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t-x)^i x^{\alpha-i}\right)\mathrm{d}t\right)\\
&=O\left(\frac{n\{n-c(r+1)\}}{2}\sum^{\otimes}p_{n+cr,k}(x)\int_{|t-x|>\delta}p_{n-cr,k+r}(t)\frac{(t-x)^3}{\delta^3}\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t-x)^i x^{\alpha-i}\right)\mathrm{d}t\right)\\
&=O\left(\frac{n\{n-c(r+1)\}}{2\delta^3}\sum^{\otimes}p_{n+cr,k}(x)\int_0^{\infty}p_{n-cr,k+r}(t)\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t-x)^{i+3}x^{\alpha-i}\right)\mathrm{d}t\right)\\
&=O\left(\frac{1}{n}\right),\quad \text{in view of Lemma (2.1).}\end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4), we have

$$\lim_{n\to\infty}|nR_{n,r}(x)|\leqslant\varepsilon\phi^2(x).$$

Since ε is arbitrary, therefore

$$\lim_{n\to\infty}nR_{n,r}(x)=0.$$

This completes the proof. □

Theorem 3.2. *Let $f^{(r+1)}\in C[0,\infty)$ and $[0,\lambda]\subseteq[0,\infty)$ and let $\omega(f^{(r+1)};.)$ be the modulus of continuity of $f^{(r+1)}$ then for $r=0,1,2,\ldots$*

$$\left\|\frac{n-c(r+1)}{(n-c)\beta(n,r)}(V_n^{(r)}f)-f^{(r)}\right\|_{C[0,\lambda]}\leqslant\frac{(r+1)(1+2c\lambda)}{[n-c(r+2)]}\|f^{(r+1)}\|+C(n,r)\left(\sqrt{\eta}+\frac{\eta}{2}\right)\omega(f^{(r+1)};C(n,r)),$$

where the norm is sup-norm over $[0,\lambda]$,

$$\eta=2\lambda^2\{c^2(2r^2+6r+3)+cn\}+2\lambda\{2c(r^2+3r+1)+n\}+(r^2+3r+2)$$

and

$$C(n,r)=\frac{1}{\{n-c(r+2)\}\{n-c(r+3)\}}.$$

Proof. Applying the Taylor formula:

$$f^{(r)}(t)-f^{(r)}(x)=(t-x)f^{(r+1)}(x)+\int_x^t\{(f^{(r+1)}(y)-f^{(r+1)}(x)\}\mathrm{d}y.$$

Thus

$$\begin{aligned}\frac{n-c(r+1)}{(n-c)\beta(n,r)}\left(V_n^{(r)}f\right)(x)-f^{(r)}(x)&=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)\{f^{(r)}(t)-f^{(r)}(x)\}\mathrm{d}t\\&=\{n-c(r+1)\}\sum^{\otimes}p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)[(t-x)f^{(r+1)}(x)+\int_x^t\{(f^{(r+1)}(y)-f^{(r+1)}(x)\}\mathrm{d}y]\mathrm{d}t.\end{aligned}$$

Since,

$$|f^{(r+1)}(y)-f^{(r+1)}(x)|<\left(1+\frac{|y-x|}{\delta}\right)\omega(f^{(r+1)};\delta).$$

Hence, by Schwartz's inequality

$$\left|\frac{n-c(r+1)}{(n-c)\beta(n,r)}(V_n^{(r)}f)(x)-f^{(r)}(x)\right|\leqslant|\mu_{r,n,1}||f^{(r+1)}(x)|+\left(|\sqrt{\mu_{r,n,2}}|+\frac{|\mu_{r,n,2}|}{2\delta}\right)\omega(f^{(r+1)};\delta).$$

Further, choosing $\delta=C(n,r)$ and using Lemma 2.1, we get the required result. □

4. Examples

Suppose $s_{n,k}(x), v_{n,k}(x)$ and $b_{n,k}(x)$ are Szász-Mirakian, Baskakov and Beta basis functions, respectively, which are defined as:

$$s_{n,k}(x)=\frac{(nx)^k}{k!}e^{-nx},\quad v_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\quad\text{and}\quad b_{n,v}(x)=\frac{1}{B(v+1,n)}\frac{x^v}{(1+x)^{n+v+1}}.$$

To approximate integrable functions on the intervals $[0,\infty)$, we are giving here some recurrence relations of some mixed summation-integrable type operators.

Szász-Mirakian-Baskakov type operators:

$$(G_nf)(x)=(n-1)\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}v_{n,k}(t)f(t)\,\mathrm{d}t.$$

If $G_{n,m}(x)\equiv G_n((t-x)^m,x)$, then we have the recurrence relation:

$$(n-m-r-2)\mu_{r,n,m+1}(x)=x\mu'_{r,n,m}(x)+[(m+1)(1+2x)+r(1+x)]\mu_{r,n,m}(x)+mx(2+x)\mu_{n,m-1}(x),$$

and

$$(G_n^{(r)}f)(x)=\gamma(n,r)\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}v_{n-r,k+r}(t)f^{(r)}(t)\,\mathrm{d}t,$$

where

$$\gamma(n,r)=\frac{n^r(n-r-1)!}{(n-2)!}.$$

Baskakov-Szász-Mirakian type operators:

$$(H_nf)(x)=n\sum_{k=0}^{\infty}v_{n,k}(x)\int_0^{\infty}s_{n,k}(t)f(t)\,\mathrm{d}t.$$

If $H_{n,m}(x)\equiv H_n((t-x)^m,x)$, then we have the recurrence relation:

$$n\mu_{r,n,m+1}(x)=x(1+x)\mu'_{r,n,m}(x)+[(m+1)+r(1+x)]\mu_{r,n,m}(x)+mx(2+x)\mu_{r,n,m-1}(x),$$

and

$$(H_n^{(r)}f)(x)=\gamma(n,r)\sum_{k=0}^{\infty}v_{n+r,k}(x)\int_0^{\infty}s_{n,k+r}(t)f^{(r)}(t)\,\mathrm{d}t,$$

where

$$\gamma(n,r)=\frac{(n+r-1)!}{n^{r-1}(n-1)!}.$$

Baskakov-Beta type operators:

$$(K_nf)(x) = \sum_{k=0}^{\infty} v_{n,k}(x) \int_0^{\infty} b_{n,k}(t)f(t)\,\mathrm{d}t.$$

If $K_{n,m}(x) \equiv K_n((t-x)^m, x)$, then we have the recurrence relation:

$$(n-m-r-1)\mu_{r,n,m+1}(x) = x(1+x)[\mu'_{r,n,m}(x) + 2m\mu_{r,n,m-1}(x)] + [(m+r+1)(1+2x)-x]\mu_{r,n,m}(x)$$

and

$$(K_n^{(r)}f)(x) = \gamma(n,r)\sum_{k=0}^{\infty} v_{n+r,k}(x) \int_0^{\infty} b_{n-r,k+r}(t)f(t)\,\mathrm{d}t,$$

where

$$\gamma(n,r) = \frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}.$$

Theorem 4.1. *If f is integrable in $[0,\infty)$ and bounded at a point $x \in [0,\infty)$. If $f^{(r+2)}$ exists at a fixed point $x \in [0,\infty)$, and $f^{(r)}(x) = O(x^{\alpha})$ as $x \to \infty$ for some $\alpha > 0$, then we have*

(i) $\lim_{n\to\infty} n\left[\frac{1}{\gamma(n,r)}(G_n^{(r)}f)(x) - f^{(r)}(x)\right] = \{(1-r) + x(2+r)\}f^{(r+1)}(x) + x(2+x)f^{(r+2)}(x).$

(ii) $\lim_{n\to\infty} n\left[\frac{1}{\gamma(n,r)}(H_n^{(r)}f)(x) - f^{(r)}(x)\right] = \{1 + r(1+x)\}f^{(r+1)}(x) + x(2+x)f^{(r+2)}(x).$

(iii) $\lim_{n\to\infty} n\left[\frac{1}{\gamma(n,r)}(K_n^{(r)}f)(x) - f^{(r)}(x)\right] = \{(1+r) + x(1+2r)\}f^{(r+1)}(x) + 2x(1+x)f^{(r+2)}(x).$

Theorem 4.2. *Let $f^{(r+1)} \in C[0,\infty)$ and $[0,\lambda] \subseteq [0,\infty)$ and let $\omega(f^{(r+1)};.)$ be the modulus of continuity of $f^{(r+1)}$ then for $r = 0,1,2,\ldots$*

(i) $\left\|\frac{1}{\gamma(n,r)}(G_n^{(r)}f) - f^{(r)}\right\|_{C[0,\lambda]} \leqslant \frac{(1+r)+\lambda(2+r)}{(n-r-2)}\|f^{(r+1)}\| + C(n,r)\left(\sqrt{\eta} + \frac{\eta}{2}\right)\omega(f^{(r+1)}; C(n,r)),$

where $\eta = \lambda^2(n + r^2 + 5r + 6) + \lambda(2n + 2r^2 + 8r + 6) + (r^2 + 3r + 2)$ *and*

$$C(n,r) = \frac{1}{(n-r-2)(n-r-3)}.$$

(ii) $\left\|\frac{1}{\gamma(n,r)}(H_n^{(r)}f) - f^{(r)}\right\|_{C[0,\lambda]} \leqslant \frac{\{1+r(1+\lambda)\}}{n}\|f^{(r+1)}\| + C(n,r)\left(\sqrt{\eta} + \frac{\eta}{2}\right)\omega(f^{(r+1)}; C(n,r)),$

where $\eta = r\lambda(1+\lambda) + 1 + \{1 + r(1+\lambda)\}^2 + n\lambda(2+\lambda)$ *and* $C(n,r) = 1/n^2$.

(iii) $\left\|\frac{1}{\gamma(n,r)}(K_n^{(r)}f) - f^{(r)}\right\|_{C[0,\lambda]} \leqslant \frac{\{(1+r)+\lambda(1+2r)\}}{(n-r-1)}\|f^{(r+1)}\| + C(n,r)\left(\sqrt{\eta} + \frac{\eta}{2}\right)\omega(f^{(r+1)}; C(n,r)),$

where $\eta = 2(2r^2 + 4r + n + 1)\lambda^2 + 2(2r^2 + 5r + 2 + n)\lambda + (r^2 + 3r + 2)$ *and* $C(n,r) = \frac{1}{(n-r-1)(n-r-2)}$, *where the norm is sup-norm over* $[0,\lambda]$.

Remark 4.3. The similar manner we may obtain direct results and Voronovskaya type asymptotic formulae for other mixed summation-integrable type operators like Beta-Szász-Mirakian, Szász-Mirakian-Beta, Beta-Baskakov.

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