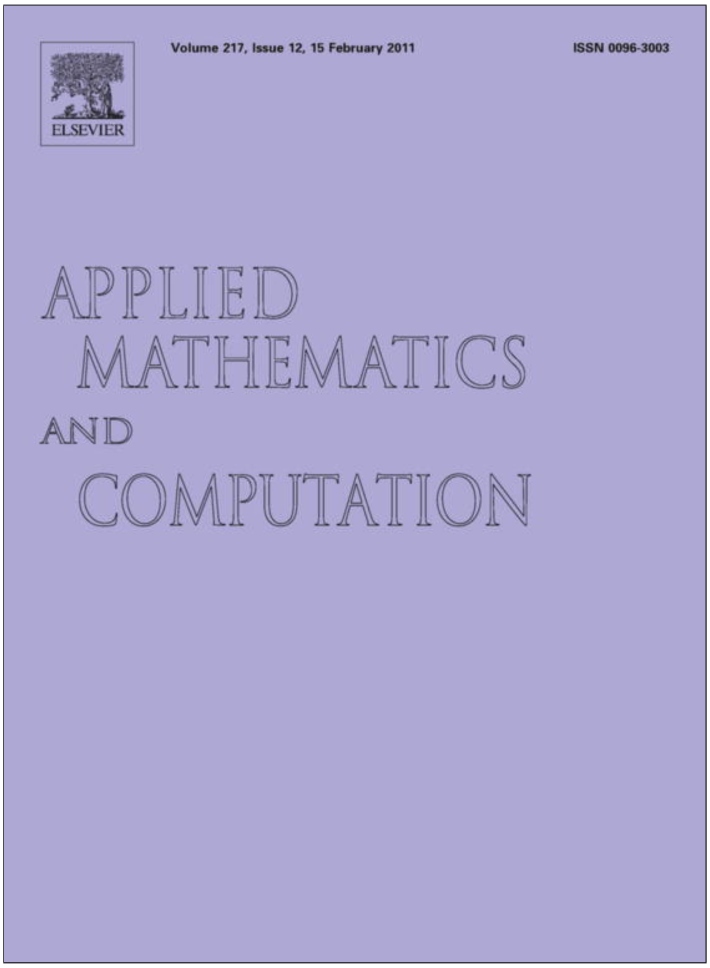
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Some approximation results for Durrmeyer operators

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ABSTRACT

In the present paper, we obtain a sequence of positive linear operators which has a better rate of convergence than the Szász–Mirakian Durrmeyer and Baskakov Durrmeyer operators and their Voronovskaya type results.

1. Introduction

In 2003, King [12] introduced an exotic sequence of positive linear operators $V_n : C([0,1]) \to C([0,1])$, which modifies the Bernstein operators:

$$(V_nf)(x) = \sum_{k=0}^{\infty} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right), \quad f \in C([0,1]),\ x \in [0,1],$$

where $r_n(x) : [0,1] \to [0,1]$ are continuous function,

$$r_n(x) = \begin{cases} x^2, & n = 1, \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & n = 2,3,\ldots. \end{cases} \tag{1.1}$$

This sequence preserves two test functions e_0, e_2 and $(V_ne_1)(x) = r_n(x)$. He also proved that the operators V_n have a better rate of convergence than the classical Bernstein polynomials whenever $0 \leqslant x \leqslant 1/3$. After this several researchers have studied that many approximating operators, L, possess these properties, i.e., $L(e_i, x) = e_i(x)$ where $e_i(x) = x^i (i = 0, 1)$ or $(i = 0, 2)$ for example Bernstein, Szász–Mirakian, Baskakov, Meyer–König and Zeller, Post-Widder and Stancu operators (see [6–9,12,14,17]).

Very recently Deo and Singh [4] have given another modification of Baskakov operators and studied Voronovskaya type results. The first author [1,3] and Pop [15,16] have studied Voronovskaya results for other positive linear operators.

Now we consider Heilmann's operator which is defined as:

Definition 1.1 (*[2,10]*). The nth operator D_n of Baskakov–Durrmeyer operator, $n \in N, c \in N_0, n > c$, is defined by

$$(D_nf)(x) = (n - c) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) dt, \tag{1.2}$$

with $x \in [0,\infty), p_{n,k}(x) = (-1)^k \frac{x^k}{k!} \phi_n^{(k)}(x)$, where

$$\phi_n(x) = \begin{cases} e^{-nx}, & \text{for the interval } [0,\infty) \text{ with } c = 0, \\ (1 + cx)^{-n/c}, & \text{for the interval } [0,\infty) \text{ with } c > 0, \end{cases}$$

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and f is a function for which the right side of (1.2) makes sense. It is easy to see that D_n are Szász–Durrmeyer operators [11,13], Lupaş–Durrmeyer [18] and Baskakov–Durrmeyer operators [10] for $c = 0, c = 1$ and $c > 0$, respectively.

Lemma 1.2 ([2,10]). *Let $e_i(x) = x^i,\ i = 0, 1, 2$, then for $x \in [0, \infty), n \in N$ and $n > 3c$, we have*

(i) $(D_n e_0)(x) = 1$,
(ii) $(D_n e_1)(x) = \frac{nx+1}{n-2c}$,
(iii) $(D_n e_2)(x) = \frac{n(n+c)x^2+4nx+2}{(n-2c)(n-3c)}$.

Lemma 1.3. *For $x \in [0, \infty), n \in N, n > 3c$ and $\varphi_x(t) = t - x$, we have*

(i) $(D_n \varphi_x)(x) = -\frac{1+2cx}{n-2c}$,
(ii) $(D_n \varphi_x^2)(x) = \frac{2\{(n+3c)x(1+cx)+1\}}{(n-2c)(n-3c)}$.

The linear functions, i.e., for $h(t) = ct + d$, where c, d any real constants, we get $\left(\widehat{D}_n h\right)(x) = h(x)$.

2. Construction of the operators and basic results

Let $\{r_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0, \infty)$ with $0 \leqslant r_n(x) \leqslant \infty$, for $x \in [0, \infty), n \in N$ then we have

$$\left(\widehat{D}_n f\right)(x) = (n - c)\sum_{k=0}^{\infty} p_{n,k}(r_n(x)) \int_0^{\infty} p_{n,k}(t) f(t) dt \tag{2.1}$$

with $x \in [0, \infty), p_{n,k}(r_n(x)) = (-1)^k \frac{(r_n(x))^k}{k!} \phi_n^{(k)}(r_n(x))$, where

$$\phi_n(r_n(x)) = \begin{cases} e^{-nr_n(x)}, & \text{for the interval } [0,\infty) \text{ with } c = 0, \\ (1 + cr_n(x))^{-n/c}, & \text{for the interval } [0,\infty) \text{ with } c > 0, \end{cases}$$

and

$$r_n(x) = \frac{(n-2c)x - 1}{n}.$$

We obtain the following results at once.

Lemma 2.1. *Let $e_i(x) = x^i,\ i = 0, 1, 2$ then for each $x \geqslant 0$ and $n > 3c$, we have*

(i) $\left(\widehat{D}_n e_0\right)(x) = 1$,
(ii) $\left(\widehat{D}_n e_1\right)(x) = x$,
(iii) $\left(\widehat{D}_n e_2\right)(x) = \frac{(n+c)(n-2c)^2x^2+2(n-c)(n-2c)x-(n-c)}{n(n-2c)(n-3c)}$.

Lemma 2.2. *For $x \in [0, \infty),\ n \in N,\ n > 3c$ and $\varphi_x(t) = t - x$, we have*

(i) $\left(\widehat{D}_n \varphi_x\right)(x) = 0$,
(ii) $\left(\widehat{D}_n \varphi_x^2\right)(x) = \frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}$,
(iii) $\left(\widehat{D}_n \varphi_x^m\right)(x) = O\left(n^{-[\frac{m+1}{2}]}\right)$.

3. Voronovskaya type results & better error estimation

In this section we compute the rates of convergence and Voronovskaya type results of these operators $\widehat{D}_n$ given by (2.1).

Let $f \in C_B[0, \infty)$ be the space of all real valued continuous bounded functions on $[0, \infty)$, equipped with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(t)|$. The Peetre's K–functional is defined by

$$K_2 = \inf\left\{\|f - g\| + \delta\|g''\| : g \in W_\infty^2\right\}, \quad \delta > 0,$$

where $W_\infty^2 = \{g \in C_B[0, \infty) : g',\ g'' \in C_B[0, \infty)\}$. From [5], there exists a positive constant C such that

$$K_2(f,\delta) \leqslant C\omega_2\left(f,\sqrt{\delta}\right) \tag{3.1}$$

and

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0<h\leqslant\delta}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Theorem 3.1. *Let $f \in C_B[0,\infty)$, then for every $x \in [0,\infty)$ and for $C > 0$, we have*

$$\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| \leqslant C\omega_2\left(f,\sqrt{\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}}\right), \quad n > 3c. \tag{3.2}$$

Proof. Let $g \in W^2_\infty$. Using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du$$

from Lemma 2.2, we have

$$\left(\widehat{D}_n g\right)(x) - g(x) = \left(\widehat{D}_n \int_x^t (t-u)g''(u)du\right)(x).$$

We know that

$$\left|\int_x^t (t-u)g''(u)du\right| \leqslant (t-u)^2\|g''\|.$$

Therefore

$$\left|\left(\widehat{D}_n g\right)(x) - g(x)\right| \leqslant \left(\widehat{D}_n (t-u)^2\right)(x)\|g''\| = \frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}\|g''\|.$$

By Lemma 2.1, we have

$$\left|\left(\widehat{D}_n f\right)(x)\right| \leqslant (n-c)\sum_{k=0}^{\infty} p_{n,k}(r_n(x)) \int_0^\infty p_{n,k}(t)|f(t)|dt \leqslant \|f\|.$$

Hence

$$\begin{aligned}\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| &\leqslant \left|\left(\widehat{D}_n (f-g)\right)(x) - (f-g)(x)\right| + \left|\left(\widehat{D}_n g\right)(x) - g(x)\right| \\ &\leqslant 2\|f-g\| + \frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}\|g''\|\end{aligned}$$

taking the infimum on the right side over all $g \in W^2_\infty$ and using (3.1), we get the required result. □

Theorem 3.2. *If a function f is such that its first and second derivative are bounded in $[0,\infty)$, then we get*

$$\lim_{n\to\infty} n\left\{\left(\widehat{D}_n f\right)(x) - f(x)\right\} = x(1+cx)f''(x). \tag{3.3}$$

Proof. Using Taylor's theorem we write that

$$f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \frac{(t-x)^2}{2!}\xi(t,x), \tag{3.4}$$

where $\xi(t,x)$ is a bounded function $\forall t,x$ and $\lim_{t\to x}\xi(t,x) = 0$

Now applying (2.1) and (3.4), we get

$$\left(\widehat{D}_n f\right)(x) - f(x) = f'(x)\widehat{D}_n(\varphi_x, x) + \frac{f''(x)}{2}\widehat{D}_n\left(\varphi_x^2, x\right) + I_1,$$

where

$$I_1 = \frac{1}{2}\widehat{D}_n\left(\varphi_x^2, x\right)\xi(t,x).$$

Using Lemma 2.2, we get

$$n\left\{\left(\widehat{D}_n f\right)(x)-f(x)\right\}=\frac{f''(x)}{2}\left\{\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}\right\}+nI_1.$$

Now, we have to show that as $n\to\infty$, the value of $nI_1\to 0$. Let $\varepsilon>0$ be given since $\xi(t,x)\to 0$ as $t\to 0$, then there exists $\delta>0$ such that when $|t-x|<\delta$ we have $|\xi(t,x)|<\varepsilon$ and when $|t-x|\geqslant\delta$, we write

$$|\xi(t,x)|\leqslant C<C\frac{(t-x)^2}{\delta^2}.$$

Thus, for all $t,x\in[0,\infty)$

$$|\xi(t,x)|\leqslant\varepsilon+C\frac{(t-x)^2}{\delta^2}$$

and

$$nI_1\leqslant n\left(\widehat{D}_n\varphi_x^2\left(\varepsilon+\frac{C\varphi_x^2}{\delta^2}\right)\right)(x)\leqslant\varepsilon n\left(\widehat{D}_n\varphi_x^2\right)(x)+\frac{C}{\delta^2}n\left(\widehat{D}_n\varphi_x^4\right)(x).$$

Using Lemma 2.2, we get that,

$$nI_1\to 0 \text{ as } n\to\infty.$$

This leads to (3.5). □

Remark 3.3. We may note here that under the conditions of Theorem 3.2, we have

$$\lim_{n\to\infty} n\{(D_n f)(x)-f(x)\}=-(1+2cx)f'(x)+x(1+cx)f''(x). \tag{3.5}$$

Theorem 3.4. *If $g\in C_B^2[0,\infty)$ then we have*

$$\left|\left(\widehat{D}_n g\right)(x)-g(x)\right|\leqslant\alpha_n(x)\|g\|_{C_B^2}, \tag{3.6}$$

where

$$\alpha_n(x)=\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{2n(n-2c)(n-3c)},\quad n>3c.$$

Proof. We write that

$$g(t)g(x)=(t-x)g'(x)+\frac{1}{2}(t-x)^2g''(\zeta), \tag{3.7}$$

where $t\leqslant\zeta\leqslant x$. From Lemma 2.2 and (3.7), we have

$$\left|\left(\widehat{D}_n g\right)(x)-g(x)\right|\leqslant\|g'\|\left|\left(\widehat{D}_n\varphi_x\right)(x)\right|+\frac{1}{2}\|g''\|\left|\left(\widehat{D}_n\varphi_x^2\right)(x)\right|\leqslant\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{2n(n-2c)(n-3c)}\|g''\|=\alpha_n(x)\|g\|_{C_B^2}.\quad\square$$

Remark 3.5. Under the same conditions of Theorem 3.4, we obtain

$$|(D_n g)(x)-g(x)|\leqslant\alpha_n^*(x)\|g\|_{C_B^2}, \tag{3.8}$$

where

$$\alpha_n^*(x)=\frac{(n+3c)x(1+cx)+1}{(n-2c)(n-3c)},\quad n>3c.$$

Theorem 3.6. *For $f\in C_B[0,\infty)$, we obtain*

$$\left|\left(\widehat{D}_n f\right)(x)-f(x)\right|\leqslant A\left\{\omega_2\left(f,\frac{\sqrt{\alpha_n(x)}}{2}\right)+\min\left(1,\frac{\alpha_n(x)}{2}\right)\|f\|_{C_B}\right\}, \tag{3.9}$$

where $n>3c$ and constant A depends on f & $\alpha_n(x)$.

Proof. For $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$ we write

$$\left(\widehat{D}_n f\right)(x) - f(x) = \left(\widehat{D}_n f\right)(x) - \left(\widehat{D}_n g\right)(x) + \left(\widehat{D}_n g\right)(x) - g(x) + g(x) - f(x)$$

by using (3.6) and Peetre K-functions, we get

$$\begin{aligned}\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| &= \left|\left(\widehat{D}_n f\right)(x) - \left(\widehat{D}_n g\right)(x)\right| + \left|\left(\widehat{D}_n g\right)(x) - g(x)\right| + |g(x) - f(x)| \leqslant \left\|\widehat{D}_n f\right\| \|f-g\| + \alpha_n(x)\|g\|_{C_B^2} + \|f-g\| \\ &\leqslant 2\|f-g\| + \alpha_n(x)\|g\|_{C_B^2} \leqslant 2\left\{\|f-g\| + \frac{1}{2}\alpha_n(x)\|g\|_{C_B^2}\right\} \leqslant 2K_2\left\{f, \frac{1}{2}\alpha_n(x)\right\} \\ &\leqslant 2A\left\{\omega_2\left(f, \frac{1}{2}\sqrt{\alpha_n(x)}\right) + \min\left(1, \frac{1}{2}\alpha_n(x)\right)\|f\|_{C_B}\right\}.\end{aligned}$$

This completes the proof. □

Remark 3.7. Under the same conditions of Theorem 3.6, we get

$$|(D_n f)(x) - f(x)| \leqslant 2A\left\{\omega_2\left(f, \sqrt{\alpha_n^*(x)}\right) + \min(1, \alpha_n^*(x))\|f\|_{C_B}\right\}. \tag{3.10}$$

Theorem 3.8. *For every $f \in C[0,\infty),\ x \in [0,\infty)$, we obtain*

$$\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| \leqslant 2\omega(f, \delta_x), \tag{3.11}$$

where

$$\delta_x = \sqrt{\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}}, \quad n > 3c$$

and $\omega(f,\delta_x)$ is the modulus of continuity of f.

Proof. Let $f \in C[0,\infty)$ and $x \in [0,\infty)$. Using linearity and monotonicity of $\widehat{D}_n$, we easily obtain, for every $\delta > 0$ and $n \in N$, that

$$\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| \leqslant \omega(f,\delta)\left\{1 + \frac{1}{\delta}\sqrt{\widehat{D}_n(\varphi_x^2, x)}\right\}.$$

By using Lemma 2.2 and choosing $\delta = \delta_x$ the proof is completed. □

Remark 3.9. For the original operator D_n defined in, we may write that, for every $f \in C[0,\infty)$

$$|(D_n f)(x) - f(x)| \leqslant 2\omega(f, v_x), \tag{3.12}$$

where

$$v_x = \sqrt{\frac{2\{(n+3c)x(1+cx)+1\}}{(n-2c)(n-3c)}}, \quad n > 3c$$

and $\omega(f, v_x)$ is the modulus of continuity of f. The error estimate in Theorem 3.8 is better than that of (3.12) for $f \in C[0,\infty)$ and $x \in [0,\infty)$, we get $\delta_x \leqslant v_x$.

Now we compute rate of convergence of the operators of D_n by means of the Lipschitz class $Lip_M(\gamma)$, $(0 < \gamma \leqslant 1)$. As usual, we say that $f \in C_B[0,\infty)$ belongs to $Lip_M(\gamma)$ if the inequality

$$|f(t) - f(x)| \leqslant M|t-x|^{\gamma} \tag{3.13}$$

holds.

Theorem 3.10. *If $f \in Lip_M(\gamma),\ x \in [0,\infty)$ and $n > 3c$, we have*

$$\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| \leqslant M\left[\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}\right]^{\gamma/2}.$$

Proof. Since $f \in Lip_M(\gamma)$ and $x \geqslant 0$, from inequality (3.13) and applying the Hölder inequality with $p = \frac{2}{\gamma},\ q = \frac{2}{2-\gamma}$, we have

$$\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| \leqslant \left(\widehat{D}_n|f(t) - f(x)|\right)(x) \leqslant M\left(\widehat{D}_n|t - x|^{\gamma}\right)(x) \leqslant M\left\{\left(\widehat{D}_n\varphi_x^2\right)(x)\right\}^{\gamma/2} \leqslant M\left[\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}\right]^{\gamma/2}$$

proof is completed. □

Remark 3.11. If using Lemma 1.3, for the original operator D_n, then we get the following result

$$\left|\left(\widehat{D}_n f\right)(x) - f(x)\right| \leqslant M\left\{\frac{2\{(n+3c)x(1+cx)+1\}}{(n-2c)(n-3c)}\right\}^{\gamma/2}$$

for every $f \in Lip_M(\gamma),\ x \geqslant 0$ and $n > 3c$.

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