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On approximation by a class of new Bernstein type operators ☆

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Abstract

This paper is concerned with a new type of the classical Bernstein operators where the function is evaluated at intervals $[0, 1 - \frac{1}{n+1}]$. We also make extensive study simultaneous approximation by the linear combination $L_n(f, k, x)$ of these new Bernstein type operators $L_n(f)$. At the end of this paper we have given an other modification of these operators.

Keywords: Bernstein operators; Linear combinations

1. Introduction

If $f(x)$ is a function defined on $[0, 1]$, the well known Bernstein operators $B_n(f)$ as

$$(B_nf)(x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, 2, \ldots, n.$$

Durrmeyer [7] studied the integral modification of Bernstein operator and Bernstein–Durrmeyer operators

$$(D_nf)(x) = (n+1) \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) \mathrm{d}t. \tag{1.2}$$

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In the last decade an interesting generalization viz. q-Bernstein polynomials were proposed by Phillips [14], which for each positive integer n and $f \in C[0,1]$ are defined as

$$B_{n,q}(f,x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) p_{nk}(q;x),$$

where

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k}.$$

Also for each non-negative integer k, by $[k]$ we mean the q-integer and $(1-x)_q^n = \prod_{j=0}^{n-1}(1-q^j x)$. Approximation properties of these operators were studied recently by Ostrovska [13] and Wang [17].

Very recently based on the q-analog of integration and for $f \in C[0,1]$, Gupta [8] proposed a simple q-analogue of the well known Durrmeyer operators as

$$D_{n,q}(f;x) = [n+1] \sum_{k=0}^{n} q^{-k} p_{nk}(q;x) \int_0^1 f(t) p_{nk}(q;qt) \mathrm{d}_q t.$$

It can be easily verified that in case $q = 1$, the operators $D_{n,q}(f;x)$ reduce to the well known Durrmeyer operators $D_n(f;x)$.

Cheney and Sharma [2] generalized the Bernstein polynomials by the relation

$$(G_n f)(x) = (1+nt_n)^{-n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x(x+kt_n)^{k-1}(1-x+(n-k)t_n)^{n-k},$$

where $\{t_n\}_{n\in N}$ is a sequence of positive real numbers.

Several modifications of Bernstein polynomials have been introduced and studied by many researchers. We cite the works on such operators (see [1,4,5,13,14,16]). Gupta and Maheshwari [11] estimated the rate of convergence for the Bezier variant of some other Bernstein–Durrmeyer operators.

Now we introduce a Bernstein type special operator V_n defined as

$$(V_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.3}$$

where

$$p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k}, \quad \frac{n}{n+1} \geqslant x \text{ and } x \in \left[0, 1-\frac{1}{n+1}\right].$$

It is a generalized form of Bernstein operators, i.e., if n is sufficient large then our operators convert in the original form of Bernstein operators (1.1).

By considering the integral modification of Bernstein operators, one can approximate Lebesgue integrable function on the interval $[0,1]$. Derriennic [6] was the first, who studied the operators (1.2) in detail. Motivated by the earlier works on Bernstein operators, we now propose a certain new integral modification of the operators (1.3) which are defined as

$$(L_n f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t) \mathrm{d}t. \tag{1.4}$$

It turns out that the order of approximation to $f(x)$ by $L_n(f,x)$ is at best $O(n^{-1})$. The existence of derivatives of higher order of the function does not improve this order of approximation. May [12] and Rathore [15] have described a method for forming linear combinations of positive linear operators so as to improve the order of approximation.

Wood [18] applied this technique of linear combinations to the operators L_n to improve the order of approximation. It is described as follows:

Let $d_0, d_1, \ldots, d_k$ be $k+1$ arbitrary but fixed distinct positive integers. Then, the linear combinations $L_n(f,k,x))$ of $L_{d_jn}(t,x),\ j=0,1,\ldots,k$ are defined as

$$L_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} L_{d_0n}(f,x) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ L_{d_1n}(f,x) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots & \ldots \\ L_{d_kn}(f,x) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix}, \quad \text{where } \Delta = \begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ . & \ldots & \ldots & \ldots & \ldots \\ . & \ldots & \ldots & \ldots & \ldots \\ 1 & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix}.$$

We may write these linear combinations in alternative form as follows:

$$L_n(f,k,x) = \sum_{j=0}^{k} C(j,k) L_{d_jn}(f,x), \tag{1.5}$$

where

$$C(j,x) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i} \quad \text{for } k \neq 0 \text{ and } C(0,0) = 1.$$

Now in this paper, we establish a Voronovskaya type asymptotic formula and obtain an estimate of error in terms of modulus of continuity in simultaneous approximation by the linear combinations of the operators (1.4). Very recently Deo [3] studied direct and inverse theorems in ordinary approximation for hybrid type operators. Gupta and Noor [10] also studied another type hybrid operators.

2. Basic results

In this section, we shall mention some definitions and certain lemmas to prove our main theorems.

Lemma 2.1. *Let the mth order moment for the operator (1.3) be defined by*

$$\mu_{n,m}(x) = \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m, \quad m = 0,1,2,\ldots,$$

then we have $\mu_{n,0}(x) = 1,\ \mu_{n,1}(x) = 0$ *and*

$$n\mu_{n,m+1}(x) = x\left(\frac{n}{n+1} - x\right)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)] \quad \textit{for } n \in N.$$

Consequently, for every $x \in [0, 1-\frac{1}{n+1}]$, *we have*

(i) $\mu_{n,m}(x)$ *is a polynomial in* x *of degree* $\leqslant m$;
(ii) $\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$, *where* $[\lambda]$ *denotes the integral part of* λ.

Proof. First we prove recurrence relation, by using the following relation:

$$x\left(\frac{n}{n+1} - x\right)p'_{n,k}(x) = n\left(\frac{k}{n+1} - x\right)p_{n,k}(x). \tag{2.1}$$

We have from the definition of $\mu_{n,m}(x)$

$$\mu'_{n,m}(x) = \sum_{k=0}^{n} p'_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m - m\mu_{n,m-1}(x),$$

then we obtain from (2.1)

$$x\left(\frac{n}{n+1}-x\right)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]=\sum_{k=0}^{n}x\left(\frac{n}{n+1}-x\right)p'_{n,k}(x)\left(\frac{k}{n+1}-x\right)^{m}$$
$$=n\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n+1}-x\right)^{m+1}=n\mu_{n,m+1}(x).\qquad\square$$

Lemma 2.2. *Let the mth order moments are defined by*

$$\mu_{n,m}(x)=(n+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}p_{n,k}(x)\int_0^{\frac{n}{n+1}}p_{n,k}(t)(t-x)^m\mathrm{d}t,$$

then, we have

$$\mu_{n,0}(x)=1,\quad \mu_{n,1}(x)=\frac{n-(n+1)2x}{(n+1)(n+2)},\tag{2.2}$$

and for $m\geqslant 1$

$$(m+n+2)\mu_{n,m+1}(x)=x\left(\frac{n}{n+1}-x\right)\left[2m\mu_{n,m-1}(x)+\mu'_{n,m}(x)\right]+\left(\frac{n}{n+1}-2x\right)(m+1)\mu_{n,m}(x).$$

Consequently,

(i) $\mu_{n,m}(x)$ *is a polynomial in x of degree* $\leqslant m$;
(ii) $\mu_{n,m}(x)=O(n^{-[\frac{m+1}{2}]})$, *where* $[\alpha]$ *denotes the integral part of* α.

Proof. The values of $\mu_{n,0}(x)$ and $\mu_{n,1}(x)$ can easily follows from the definition. We prove the recurrence relation as follows:

$$\mu'_{n,m}(x)=(n+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}p'_{n,k}(x)\int_0^{\frac{n}{n+1}}p_{n,k}(t)(t-x)^m\mathrm{d}t-m\mu_{n,m-1}(x).$$

From (2.1), we get

$$\begin{aligned}&x\left(\frac{n}{n+1}-x\right)\left\{\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right\}\\&=(n+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}p_{n,k}(x)\int_0^{\frac{n}{n+1}}n\left(\frac{k}{n+1}-x\right)p_{n,k}(t)(t-x)^m\mathrm{d}t\\&=(n+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}p_{n,k}(x)\int_0^{\frac{n}{n+1}}t\left(\frac{n}{n+1}-t\right)p'_{n,k}(t)(t-x)^m\mathrm{d}t+n\mu_{n,m+1}(x)\\&=(n+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}p_{n,k}(x)\int_0^{\frac{n}{n+1}}\left[x\left(\frac{n}{n+1}-x\right)+\left(\frac{n}{n+1}-2x\right)(t-x)-(t-x)^2\right]\cdot p'_{n,k}(t)(t-x)^m\mathrm{d}t\\&\quad+n\mu_{n,m+1}(x)=-mx\left(\frac{n}{n+1}-x\right)\mu_{n,m-1}(x)-(m+1)\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)\\&\quad+(m+2)\mu_{n,m+1}(x)+n\mu_{n,m+1}(x).\end{aligned}$$

This complete the proof of the recurrence relation. $\square$

Lemma 2.3 [6, Lemma II.7]. *There exists the polynomials* $q_{i,j,r}(x)$ *independent of n and k such that*

$$x^r\left(\frac{n}{n+1}-x\right)^r\frac{\mathrm{d}^r}{\mathrm{d}x^r}p_{n,k}(x)=\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^{i+j}\left(\frac{k}{n+1}-x\right)^j q_{i,j,r}(x)p_{n,k}(x).$$

3. Main results

In this section, we shall prove the following main result.

Theorem 3.1. *Let $f \in C[0, 1 - \frac{1}{n+1}]$, if $f^{(2k+r+2)}$ exists at a point $x \in [0, 1 - \frac{1}{n+1}]$ then*

$$\lim_{n\to\infty} n^{k+1}[L_n^{(r)}(f,k,x) - f^{(r)}(x)] = \sum_{i=r}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x), \tag{3.1}$$

$$\lim_{n\to\infty} n^{k+1}[L_n^{(r)}(f,k+1,x) - f^{(r)}(x)] = 0, \tag{3.2}$$

where $Q(i,k,r,x)$ are certain polynomials in x. Furthermore (3.1) and (3.2) hold uniformly on $[0, 1 - \frac{1}{n+1}]$ if $f^{(2k+r+2)} \in C[0, 1 - \frac{1}{n+1}]$.

Proof. By Taylor's expansion of $f(t)$, we have

$$f(t) = \sum_{i=1}^{2k+r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + g(t-x)(t-x)^{2k+r+2}, \tag{3.3}$$

where $g(t-x) \to 0$ as $t \to x$ and is bounded and integrable function on $[-x, 1 - \left(\frac{1}{n+1} + x\right)]$.

Using (3.3), we get

$$\begin{aligned}
L_n^{(r)}(f,k,x) &= \sum_{j=0}^{k} C(j,k)(d_jn+1)\left(1+\frac{1}{d_jn}\right)\sum_{k=0}^{d_jn} p_{d_jn,k}^{(r)}(x)\int_0^{\frac{d_jn}{d_jn+1}} p_{n,k}(t)f(t)\mathrm{d}t \\
&= \sum_{j=0}^{k} C(j,k)(d_jn+1)\left(1+\frac{1}{d_jn}\right)\sum_{k=0}^{d_jn} p_{d_jn,k}^{(r)}(x)\int_0^{\frac{d_jn}{d_jn+1}} p_{d_jn,k}(t)\cdot\left\{\sum_{i=1}^{2k+r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i\right. \\
&\quad \left. + g(t-x)(t-x)^{2k+r+2}\right\}\mathrm{d}t \\
&= \sum_{i=1}^{2k+r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{k} C(j,k)L_{d_jn}^{(r)}(x)((t-x)^i,x) + \sum_{j=0}^{k} C(j,k)L_{d_jn}^{(r)}(x)(g(t-x)(t-x)^{2k+r+2},x) \\
&= I_1 + I_2.
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
I_1 &= \sum_{i=r}^{2k+r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{k} C(j,k)L_{d_jn}^{(r)}(x)((t-x)^i,x) \\
&= \sum_{i=r}^{2k+r+2}\frac{f^{(i)}(x)}{i!}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}x^{i-l}L_n^{(r)}(x)(t^l,k,x) \\
&= \sum_{i=r}^{2k+r+2}\frac{f^{(i)}(x)}{i!}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}x^{i-l}\cdot\left[\frac{\mathrm{d}^r}{\mathrm{d}x^r}x^l + n^{-(k+1)}\left\{\sum_{j=1}^{2k+2}\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\frac{Q(j,k,x)}{j!}\frac{\mathrm{d}^j}{\mathrm{d}x^j}x^l\right) + \mathrm{O}(1)\right\}\right] \\
&= \sum_{i=r}^{2k+r+2}\frac{f^{(i)}(x)}{i!}r!\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}\binom{l}{r}x^{i-r} + n^{-(k+1)}\sum_{i=r}^{2k+r+2}Q(i,k,r,x)f^{(i)}(x) + \mathrm{O}(n^{-(k+1)}) \\
&= f^{(r)}(x) + n^{-(k+1)}\sum_{i=r}^{2k+r+2}Q(i,k,r,x)f^{(i)}(x) + \mathrm{O}(n^{-(k+1)}).
\end{aligned}$$

Using the identities

$$\sum_{l=0}^{i}(-1)^l\binom{i}{l}\binom{l}{r} = \begin{cases} 0, & i > r, \\ (-1)^r, & i = r. \end{cases}$$

To prove the the first assertion (3.1) it is sufficient to show that

$$n^{k+1}L_n^{(r)}(g(t-x)(t-x)^{2k+r+2},x)\to 0 \quad \text{as } n\to\infty.$$

Now

$$L_n^{(r)}(g(t-x)(t-x)^{2k+r+2},x)=(n+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}p_{n,k}^{(r)}(x)\int_0^{\frac{n}{n+1}}p_{n,k}(t)g(t-x)(t-x)^{2k+r+2}\mathrm{d}t$$

and from Lemma 2.3, we obtain

$$I_2=(n+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n}x^r\left(\frac{n}{n+1}-x\right)^r p_{n,k}^{(r)}(x)\int_0^{\frac{n}{n+1}}p_{n,k}(t)g(t-x)(t-x)^{2k+r+2}\mathrm{d}t$$
$$=(n+1)\left(1+\frac{1}{n}\right)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^{i+j}q_{i,j,r}(x)\sum_{k=0}^{n}p_{n,k}(x)\left|\frac{k}{n+1}-x\right|^j\int_0^{\frac{n}{n+1}}p_{n,k}(t)g(t-x)(t-x)^{2k+r+2}\mathrm{d}t.$$

Put $M=\sup_{x\in[0,1-\frac{1}{n+1}]}\sup_{2i+j\leqslant r}|q_{i,j,r}(x)|$ then apply Schwartz inequality summation to have

$$I_2=M(n+1)\left(1+\frac{1}{n}\right)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^{i+j}\left\{\sum_{k=0}^{n}\left(\frac{k}{n+1}-x\right)^{2j}p_{n,k}(x)\right\}^{1/2}$$
$$\cdot\left\{\sum_{k=0}^{n}p_{n,k}(x)\left(\int_0^{\frac{n}{n+1}}p_{n,k}(t)g(t-x)(t-x)^{2k+r+2}\mathrm{d}t\right)^2\right\}^{1/2}. \tag{3.4}$$

From Lemma 2.1, we get

$$\frac{1}{n^{i+j}}\sup_{x\in\left[0,1-\frac{1}{n+1}\right]}\left\{\sum_{k=0}^{n}\left(\frac{k}{n+1}-x\right)^{2j}p_{n,k}(x)\right\},$$

bounded for all values of j.

Let $K=\sup_{u\in[-x,1-(\frac{1}{n+1}+x)]}|g(u)|$ and let $\varepsilon>0$ be arbitrary. Choose $\delta>0$ such that $|g(u)|<\varepsilon$ when $|u|\leqslant\delta$. So for all $t\in[0,1-\frac{1}{n+1}]$, we have $(g(t-x))^2<\varepsilon^2+\frac{K^2(t-x)^2}{\delta^2}$. By Schwartz inequality, we get

$$\left(\int_0^{\frac{n}{n+1}}p_{n,k}(t)g(t-x)(t-x)^{2k+r+2}\mathrm{d}t\right)^2\leqslant\frac{n}{(n+1)^2}\int_0^{\frac{n}{n+1}}p_{n,k}(t)\left\{\varepsilon^2+\frac{K^2(t-x)^2}{\delta^2}\right\}(t-x)^{2(2k+r+2)}\mathrm{d}t.$$

Applying Lemma 2.2, we get

$$\sum_{k=0}^{n}p_{n,k}(x)\left(\int_0^{\frac{n}{n+1}}p_{n,k}(t)g(t-x)(t-x)^{2k+r+2}\mathrm{d}t\right)^2$$
$$\leqslant\sum_{k=0}^{n}p_{n,k}(x)\frac{n}{(n+1)^2}\int_0^{\frac{n}{n+1}}p_{n,k}(t)\left\{\varepsilon^2+\frac{K^2(t-x)^2}{\delta^2}\right\}(t-x)^{2(2k+r+2)}\mathrm{d}t$$
$$=\frac{n}{(n+1)^2}\left\{\sum_{k=0}^{n}p_{n,k}(x)\int_0^{\frac{n}{n+1}}p_{n,k}(t)\varepsilon^2(t-x)^{2(2k+r+2)}\mathrm{d}t+\frac{K^2}{\delta^2}\sum_{k=0}^{n}p_{n,k}(x)\int_0^{\frac{n}{n+1}}p_{n,k}(t)(t-x)^{2(2k+r+3)}\mathrm{d}t\right\}$$
$$=\frac{n}{(n+1)^2}\left\{\varepsilon^2\mathrm{O}(n^{-(2k+r+3)})+\frac{K^2}{\delta^2}\mathrm{O}(n^{-(2k+r+4)})\right\}=\varepsilon^2\mathrm{O}(n^{-(2k+r+4)}),$$

Thus finally in (3.4), we get

$$|I_2|\leqslant= M(n+1)\left(1+\frac{1}{n}\right)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}C_3n^{\frac{i+j}{2}}\varepsilon C_4n^{-\frac{1}{2(2k+r+4)}}.$$

Since $\varepsilon>0$ is arbitrary, therefore $n^{k+1}I_2\to 0$ as $n\to\infty$.

The last assentation (3.2) can be proved along similar lines using $L_n((t-x)^i, k+1, x) = O(n^{-(k+2)})$, $i = 1, 2, \ldots$.

The last assertion follows due to the uniform continuity of $f^{(2k+r+2)}$ on $[0, 1-\frac{1}{n+1}]$ (enabling δ to become independent of $x \in [0, 1-\frac{1}{n+1}]$) and the uniformity of $O(n^{-(k+1)})$ term in the estimate of I_1 (because in fact, it is a polynomial in x).

This completes the proof. □

Theorem 3.2. *Let* $1 \leqslant p \leqslant 2k+2$ *and* $f \in C[0, 1-\frac{1}{n+1}]$. *If* $f^{(p+r)}$ *exists and is continuous on* $\langle a, b\rangle \subset \left(0, 1-\frac{1}{n+1}\right)$ *having the modulus of continuity* $\omega_{f^{(p+r)}}(\delta)$ *on* $\langle a, b\rangle$ *then, for n sufficiently large*

$$\|L_n^{(r)}(f, k, x) - f^{(r)}(x)\| \leqslant \mathrm{Max}\{C_1 n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2}), C_2 n^{-(k+1)}\},$$

where $C_1 = C_1(k, p, r)$ *and* $C_2 = C_2(k, p, r, f)$.

Proof. For every $x \in [a, b]$, by the hypothesis, we have

$$f(t) = \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{(p+r)!}(t-x)^{p+r} + h(t,x)\chi(t), \quad t < \xi < x, \tag{3.5}$$

where $\chi(t)$ is the characteristic function of the set $[0, 1-\frac{1}{n+1}]/\langle a, b\rangle$ and $t \in [0, 1-\frac{1}{n+1}]$.

The function $h(t,x)$ for $x \in [a,b]$ is bounded by $M|t-x|^{p+r}$ for some constant M. From (3.5), we get

$$\begin{aligned} L_n^{(r)}(f,k,x) &= \sum_{j=0}^{k} C(j,k)(d_jn+1)\left(1+\frac{1}{d_jn}\right)\sum_{k=0}^{d_jn} p_{d_jn,k}^{(r)}(x)\int_0^{\frac{d_jn}{d_jn+1}} p_{d_jn,k}(t) \\ &\quad \cdot \left\{\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(p+r)}(\xi)-f^{(p+r)}(x)}{(p+r)!}(t-x)^{p+r} + h(t,x)\chi(t)\right\}\mathrm{d}t \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Now $I_1 = f^{(r)}(x) + O(n^{-(k+1)})$, uniformly in $x \in [a,b]$.

To estimate I_2, we have for every $\delta > 0$

$$|f^{(p+r)}(\xi) - f^{(p+r)}(x)| \leqslant \omega_{f^{(p+r)}}(|\xi - x|) \leqslant \omega_{f^{(p+r)}}(|t-x|) \leqslant \left(1 + \frac{|t-x|}{\delta}\right)\omega_{f^{(p+r)}}(\delta). \tag{3.6}$$

Therefore from (3.6) and Lemma 2.3, we have

$$\begin{aligned} I_2 &\leqslant \frac{1}{(p+r)!}\sum_{j=0}^{k}|C(j,k)|(d_jn+1)\left(1+\frac{1}{d_jn}\right)\sum_{k=0}^{d_jn}\left|p_{d_jn,k}^{(r)}(x)\right|\int_0^{\frac{d_jn}{d_jn+1}} p_{d_jn,k}(t)\cdot\left(1+\frac{|t-x|}{\delta}\right)(\delta)|t \\ &\quad - x|^{p+r}\omega_{f^{(p+r)}}(\delta)\mathrm{d}t \\ &\leqslant \frac{\omega_{f^{(p+r)}}(\delta)}{(p+r)!}\sum_{j=0}^{k}|C(j,k)|(d_jn+1)\left(1+\frac{1}{d_jn}\right)\sum_{\substack{2i+s\leqslant r\\ i,s\geqslant 0}}(d_jn)^i\frac{|q_{i,s,r}(x)|}{x^r\left(\frac{n}{n+1}-x\right)^r}\cdot\sum_{k=0}^{d_jn}p_{d_jn,k}(x)\left|d_jn\left(\frac{k}{d_jn+1}-x\right)\right|^s \\ &\quad \times\int_0^{\frac{d_jn}{d_jn+1}} p_{d_jn,k}(t)\left\{|t-x|^{p+r} + \frac{1}{\delta}|t-x|^{p+r+1}\right\}\mathrm{d}t. \end{aligned}$$

Putting $K = \sup_{x\in[a,b]}\sup_{\substack{2i+s\leqslant r\\ i,s\geqslant 0}}\frac{|q_{i,s,r}(x)|}{x^r\left(\frac{n}{n+1}-x\right)^r}$, then applying Schwartz inequalities for summation and for integral and Lemma 2.2, as in the proof of Theorem 3.1, we have

$$|I_2| \leqslant K\left[C_5 n^{-p/2} + \frac{1}{\delta}C_6 n^{-(p+1)/2}\right]\omega_{f^{(p+r)}}(\delta).$$

Now choosing $\delta = n^{-1/2}$, it follows that

$$I_2 \leqslant \omega_{f^{(p+r)}}(n^{-1/2})\mathrm{O}(n^{-p/2}),$$

where the O-term holds uniformly in $x \in [a,b]$. Finally choosing a positive number δ such that $|t-x| \geqslant \delta$, we get

$$|I_3| \leqslant \sum_{j=0}^{k} |C(j,k)|(d_jn+1)\left(1+\frac{1}{d_jn}\right) \sum_{\substack{2i+s\leqslant r\\ i,s\geqslant 0}} (d_jn)^i \frac{|q_{i,s,r}(x)|}{x^r\left(\frac{n}{n+1}-x\right)^r} \cdot \sum_{k=0}^{d_jn} p_{d_jn,k}(x)\left|d_jn\left(\frac{k}{d_jn+1}-x\right)\right|^s$$

$$\times \int_{\substack{[0,1-\frac{1}{n+1}]\\ \langle a,b\rangle}} p_{d_jn,k}(t)M|t-x|^{p+r}\mathrm{d}t$$

$$\leqslant M\sum_{j=0}^{k} |C(j,k)|(d_jn+1)\left(1+\frac{1}{d_jn}\right) \sum_{\substack{2i+s\leqslant r\\ i,s\geqslant 0}} (d_jn)^i \frac{|q_{i,s,r}(x)|}{x^r\left(\frac{n}{n+1}-x\right)^r} \cdot \sum_{k=0}^{d_jn} p_{d_jn,k}(x)\left|d_jn\left(\frac{k}{d_jn+1}-x\right)\right|^s$$

$$\times \int_0^{\frac{n}{n+1}} p_{d_jn,k}(t)\frac{(t-x)^{2m}}{\delta^{2m-p-r}}\mathrm{d}t.$$

Thus

$$I_3 = \mathrm{O}\left(n^{\left(\frac{r-2m}{2}\right)}\right), \quad m > \frac{2k+r+2}{2}.$$

Therefore

$$I_3 = \mathrm{O}\left(n^{-(k+1)}\right) \quad \text{uniformly in } x \in [a,b].$$

The theorem follows from the estimates of I_1, I_2 and I_3. □

Remark 3.3. Very recently Gupta and Ispir [9] introduced a new type of Bernstein–Durrmeyer operators. We can modify that operators as follows:

$$(M_nf)(x) = (n+1)\sum_{k=1}^{n} p_{n,k}(x)\int_0^{\frac{n}{n+1}} p_{n-1,k-1}(t)f(t)\mathrm{d}t + \left(1+\frac{1}{n}\right)^n\left(\frac{n}{n+1}-x\right)^n f(0), \tag{3.7}$$

where $x \in [0, 1-\frac{1}{n+1}]$. These operators (3.7) have different approx. properties, analysis is different we will discuss them elsewhere.

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