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A Note on Equivalent Theorem for Beta Operators

Naokant Deo

Abstract. In this paper, we give an equivalent theorem concerning on the whole interval $[0, +\infty)$. Both the direct and converse theorems are derived. These results bridge the gap between the point-wise conclusions and global conclusions.

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1. Introduction

Let $C[0, +\infty)$ be the set of continuous and bounded functions defined on $[0, +\infty)$. For $f \in C[0, +\infty)$ and $n \in \mathbb{N} = \{1, 2, ...\}$ the Beta operators are given by

$$(B_n f)(x) = \sum_{v=0}^{+\infty} p_{n,v}(x) f\left(\frac{v}{n+1}\right), \tag{1.1}$$

where

$$p_{n,v}(x) = \frac{1}{n} \cdot \frac{1}{\beta(v+1, n)} \cdot \frac{x^v}{(1+x)^{n+v+1}}, \qquad x \in [0, +\infty),$$

and $\beta(v+1, n)$ denotes the Beta function defined as: $\beta(v+1, n) = \dfrac{\Gamma(v+1).\Gamma n}{\Gamma(v+n+1)}$.

Very recently Deo [2] has studied simultaneous approximation for the modified Beta operators and Gupta and Deo [4] have studied the rate of convergence for bivariate Beta operators.

The purpose of this note is to prove an equivalence theorem concerning on the whole interval $[0, +\infty)$ for the Beta operators. Corresponding to the unbounded

Permanent address of author: Department of Applied Mathematics, Delhi College of Engineering, Bawana Road, Delhi-110042, India.

interval the functions are indeed allowed to be unbounded, with some restrictions however concerning the growth of f at infinity. Here we discuss functions of polynomial growth. To be precise, we consider spaces C_N defined via the weight ω_N as follows ($N \in N^0 := \mathbb{N} \cup \{0\}$):

$$\omega_0(x) = 1, \qquad \omega_N(x) = \frac{1}{1+x^N}, \quad (x \geq 0,\ N \in \mathbb{N}),$$
$$C_N = \{f \in C\,[0,+\infty)\,;\ \omega_N f \text{ uniformly continuous and bounded on } [0,+\infty)\},$$
$$\left\|f\right\|_N = \sup_{x\geq 0}\ \omega_N(x)\left|f(x)\right|.$$

The corresponding Lipschitz classes are given for $0 < \alpha \leq 2$ by ($h > 0$)

$$\triangle_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x),$$
$$\omega_N^2\,(f,\delta) = \sup_{0<h\leq\delta} \left\|\triangle_h^2 f\right\|_N,$$
$$Lip_N^2\alpha = \left\{f \in C_N;\ \omega_N^2(f,\delta) = O(\delta^\alpha),\ \delta \to 0+\right\}.$$

Actually the direct theorem provides the rate of convergence for functions of specified smoothness; on the other hand, a converse theorem infers the nature of smoothness of function. In the present paper we study an equivalence result, which includes the direct as well as the converse part for the Beta operators. Our main result is the following:

Theorem 1.1. *Let $N \in N^0$, $f \in C_N$, $0 < \alpha \leq 2$, then the following statements are equivalent:*

$$f \in Lip_N^2\alpha, \tag{1.2}$$

$$\omega_N(x)\left|B_n f(x) - f(x)\right| \leq M_N \left[\frac{\phi(x)}{n+1}\right]^{\alpha/2}, \qquad (x \geq 0,\ n \in \mathbb{N}), \tag{1.3}$$

the constant M_N being independent of n and x, where $\phi(x) = x(1+x)$.

Throughout this note, M_N denotes a constant independent of n and x, but it is not necessarily the same in different cases.

2. Basic Results

Let us introduce some basic properties of these operators in the first three Lemmas, which we shall apply to the proofs of the main theorems.

Lemma 2.1 ([2]). *Let the function $\mu_{n,m}$ be defined on $[0,+\infty)$ for positive integers m and n and the mth moments*

$$\mu_{n,m}(x) = B_n\big(\,(t-x)^m\,; x\big) = \sum_{v=0}^{+\infty} \left(\frac{v}{n+1} - x\right)^m p_{n,v}(x),$$

then

$$(n+1)\mu_{n,m+1}(x) = x\,(1+x)\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right], \quad \left(x \geq 0,\ m \in N^0\right). \tag{2.1}$$

As a first consequence we have

$$B_n(t^i;x) = x^i, \qquad (i \in \{0,1\}), \tag{2.2}$$

$$\mu_{n,2}(x) = B_n\big((t-x)^2;x\big) = \frac{x(1+x)}{n+1}. \tag{2.3}$$

Lemma 2.2. **(**[5, p. 475]**).** *For $f \in C[0,+\infty)$ and $n \in \mathbb{N}$, by* [3, (9.4.3)], *we know that*

$$(B_n f)^r(x) = \frac{(n+r)!}{n!}\sum_{v=0}^{+\infty} \triangle^r_{(n+1)^{-1}} f\left(\frac{v}{n+1}\right) p_{n+r,v}(x).$$

Lemma 2.3. **(**[6, p. 335]**).** *Let $m \geq 2$, $\delta_m := 0$ if m is even, $\delta_m := 1$ if m is odd. Then one has for the mth moment of the Beta operators that*

$$\mu_{n,m}(x) = \sum_{j=1}^{[m/2]} p_{n,m,j}\left\{\frac{x(1+x)}{n+1}\right\}^j \left\{\frac{1+2x}{n+1}\right\}^{\delta_m}, \tag{2.4}$$

with positive coefficients $p_{n,m,j}$, bounded with respect to n. In particular, $\mu_{n,m}(x)$ is a polynomial of degree m without a constant term.

Lemma 2.4. **(**[3, (10.5.3)]**).** *For each $N \in N^0$ there is a constant M_N such that uniformly for $n \in \mathbb{N}$, $x \geq 0$*

$$\omega_N(x) B_n\Big(\frac{1}{\omega_N(t)};x\Big) \leq M_N. \tag{2.5}$$

In particular, for any $f \in C_N$,

$$\|B_n f\|_N \leq M_N \|f\|_N. \tag{2.6}$$

Lemma 2.5. **(**[3, (10.5.3)]**).** *For each $N \in N^0$ there is a constant M_N such that for all $x \geq 0$, $n \in \mathbb{N}$*

$$\omega_N(x) B_N\left(\frac{(t-x)^2}{\omega_N(t)};x\right) \leq M_N \frac{x(1+x)}{n+1}. \tag{2.7}$$

Furthermore, one has

$$(1+2x)\omega_N(x) B_N\left(\frac{(t-x)}{\omega_N(t)};x\right) \leq M_N \frac{x(1+x)}{n+1}. \tag{2.8}$$

Lemma 2.6 **(**[1]**).** *Let $N \in N^0$, $g \in C_N^2 := \{f \in C_N; f'' \in C_N\}$. Then there exists a constant M_N such that for all $x \geq 0$, $n \in \mathbb{N}$*

$$\omega_N(x)\big|B_n g(x) - g(x)\big| \leq M_N \|g''\|_N \frac{x(1+x)}{n+1}. \tag{2.9}$$

Lemma 2.7. *For $x, \delta > 0$, $f \in C_N$ there holds*

$$\omega_N(x)\big|(B_n f)''(x)\big| \leq M_N \omega_N^2(f,\delta)\left\{\frac{n+1}{x(1+x)} + \delta^{-2}\right\}. \tag{2.10}$$

Lemma 2.8. *Let $\phi(x) = x(1+x)$, then one has for $x \geq 0,\ 0 < h \leq 1$*

$$\int\int_0^h \frac{\mathrm{d}s\,\mathrm{d}t}{\phi(x+s+t)} \leq \frac{Mh^2}{\phi(x+2h)}. \tag{2.11}$$

Since the proofs of Lemma 2.7 and Lemma 2.8 are easy, we leave them to the readers.

3. Direct Theorem

The proof of the direct theorem follows along standard lines using the Steklov means, the Jackson-type inequality and approximate estimates of the moments of the Beta operators.

Theorem 3.1. *For any $N \in N^0,\ f \in C_N$ there holds for all $x > 0, n \in \mathbb{N}$*

$$\omega_N(x)\big|B_n f(x) - f(x)\big| \leq M_N \omega_N^2\left(f, \sqrt{\frac{x(1+x)}{n+1}}\right). \tag{3.1}$$

In particular, if $f \in Lip_N^2\alpha$ for some $\alpha \in (0,2]$, then

$$\omega_N(x)\big|B_n f(x) - f(x)\big| \leq M_N \left[\frac{x(1+x)}{n+1}\right]^{\alpha/2}.$$

Proof. For $x = 0$ the assertion is trivial. For $f \in C_N, h > 0$ by Lemmas 2.4 and 2.6, we get

$$\begin{aligned}
&\omega_N(x)\big|B_n f(x) - f(x)\big| \\
&\leq \omega_N(x)\big|B_n(f-f_h)(x)\big| + \omega_N(x)\big|B_n f_h(x) - f_h(x)\big| + \omega_N(x)\big|f_h(x) - f(x)\big| \\
&\leq \big\|f - f_h\big\|_N \left[\omega_N(x) B_n\left(\frac{1}{\omega_N(t)}; x\right) + 1\right] + M_N\big\|f_h''\big\|_N \left\{\frac{x(1+x)}{n+1}\right\} \\
&\leq M_N \omega_N^2(f,h)\left[1 + \left\{\frac{x(1+x)}{h^2(n+1)}\right\}\right],
\end{aligned}$$

so that (3.1) follows upon setting $h = \sqrt{\frac{x(1+x)}{(n+1)}}$. In particular, for each $f \in C_N$, $x \geq 0$, we get $\lim\limits_{n\to\infty} \omega_N(x)\big|B_n f(x) - f(x)\big| = 0$. □

4. Inverse Theorem

The main tool for the proof of the inverse theorem in the nonoptimal case $0 < \delta < 2$ is an appropriate Bernstein-type inequality.

Theorem 4.1. *Let $N \in N^0$. If $f \in C_N$ satisfies for some $\alpha \in (0,2)$ and all $n \in \mathbb{N}$, $x \geq 0$*

$$\omega_N(x)\big|B_n f(x) - f(x)\big| \leq M_N \left[\frac{x(1+x)}{n+1}\right]^{\alpha/2}, \tag{4.1}$$

then $f \in Lip_N^2\alpha$.

Proof. It is sufficient to show that for $0 < h,\ \delta \leq 1,\ \delta < \sqrt{h}$ (see [7])

$$\omega_N^2(f,h) \leq M_N\left[\delta^\alpha + \left(\frac{h}{\delta}\right)^2 \omega_N^2(f,\delta)\right], \tag{4.2}$$

where $0 < h,\ \delta \leq 1,\ \delta < \sqrt{h},\ x \geq 0$. By Lemmas 2.7 and 2.8 and observing that

$$\frac{\omega_N(x)}{\omega_N(x+2h)} \leq 3^N \quad \text{as } h \leq 1,$$

then from (4.1) for all $n \in \mathbb{N}$

$$\begin{aligned}
&\omega_N(x)\left|\triangle_h^2 f(x)\right| \\
&\quad\leq \omega_N(x)\left|f(x+2h) - B_n f(x+2h)\right| + 2\omega_N(x)\left|B_n f(x+h) - f(x+h)\right| \\
&\quad\quad +\omega_N(x)\left|f(x) - B_n f(x)\right| + \omega_N(x)\left|\triangle_h^2(B_n f)(x)\right| \\
&\quad\leq M_N\left\{\frac{\phi(x+2h)}{n+1}\right\}^{\alpha/2}\left\{\frac{\omega_N(x)}{\omega_N(x+2h)} + \frac{2\omega_N(x)}{\omega_N(x+h)} + 1\right\} \\
&\quad\quad +\omega_N(x)\int\int_0^h \left|(B_n f)''(x+s+t)\right| \mathrm{d}s\,\mathrm{d}t \\
&\quad\leq M_N\left\{\frac{\phi(x+2h)}{n+1}\right\}^{\alpha/2} + M_N\omega_N^2(f,\delta)\left(\frac{\omega_N(x)}{\omega_N(x+2h)}\right) \\
&\quad\quad \cdot\left\{(n+1)\int\int_0^h \frac{\mathrm{d}s\,\mathrm{d}t}{\phi(x+s+t)} + \left(\frac{h}{\delta}\right)^2\right\} \\
&\quad\leq M_N\left[\left\{\frac{\phi(x+2h)}{n+1}\right\}^{\alpha/2} + \left\{\frac{(n+1)Mh^2}{\phi(x+2h)} + \left(\frac{h}{\delta}\right)^2\right\}\omega_N^2(f,\delta)\right].
\end{aligned}$$

For $x = 0$ let us only note that the estimate holds true in view of the existence of the integrals for $x = 0$ and the continuity of the expressions involved. Now choose n such that

$$\sqrt{\frac{\phi(x+2h)}{n+1}} \leq \delta < \sqrt{\frac{\phi(x+2h)}{n}} \leq \sqrt{2\frac{\phi(x+2h)}{n+1}},$$

the last expression being $\geq 2\sqrt{\frac{h}{n+1}}$. Then

$$\omega_N(x)\left|\triangle_h^2 f(x)\right| \leq M_N\left[\delta^\alpha + \left(\frac{h}{\delta}\right)^2 \omega_N^2(f,\delta)\right],$$

proving (4.2). This completes the proof of the theorem. □

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Naokant Deo
School of Information Science and Engineering
Graduate School of the Chinese Academy of Sciences
No. 3 Nanyitiao, Zhongguancun Haidian District
Beijing-100080
P. R. China.
e-mail: dr_naokant_deo@yahoo.com

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