COEFFICIENT ESTIMATES AND SUBORDINATION FOR UNIVALENT FUNCTIONS

THESIS SUBMITTED FOR THE AWARD
OF THE DEGREE OF

DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

BY

KANIKA KHATTER

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
DELHI 110 042

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DECLARATION

I, Kanika Khatter declate that the research work reported in this thesis entitled "*Coefficient Estimates and Subordination for Univalent Functions*" submitted by me to the Department of Applied Mathematics, Delhi Technological University, Delhi for the award of the degree of *Doctor of Philosophy in Mathematics* is a research work carried out by me under the joint supervision and guidance of *Dr. S. Sivaprasad Kumar* and *Prof. V. Ravichandran*.

The research work embodied in it is original and has not been submitted earlier in part or full or in any other form to any university or institute, here or elsewhere, for the award of any degree or diploma.

April, 2018

Kanika Khatter
(Research Scholar)

CERTIFICATE

This is to certify that this thesis entitled "*Coefficient Estimates and Subordination for Univalent Functions*" submitted by **Kanika Khatter** to the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of the degree of *Doctor of Philosophy in Mathematics* is a record of bonafide research work carried out by her under our joint supervision. The work embodied in it is original and has not been submitted earlier in part or full or in any other form to any university or institute, here or elsewhere, for the award of any degree or diploma.

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(Kanika Khatter)

Abstract

Univalent function theory is a branch of geometric function theory which comprises of the various geometric properties of analytic functions. The first milestone in the field of univalent functions theory was achieved by Bieberbach in the year 1916, wherein he proved the second coefficient bound for a function $f \in \mathcal{S}$ of normalised analytic univalent functions. He also proposed a conjecture for the n^{th} coefficient of the function in the class $\mathcal{S}$ in the same year. Bieberbach's Conjecture states that the coefficients of the the function $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, 4, \cdots$ with equality if and only if holds if f is some rotation of the famous Koebe function. Bieberbach's conjecture paved way for many mathematicians to work in the area of univalent functions and a vast literature is available now.

The present research work focusses on investigating the various types of coefficient estimate problems in geometric function theory such as computing the bounds on the second and the third Hankel determinants, the Fekete- Szegö coefficient functional. The thesis also aims at computing the sharp radius estimates and various inclusion relationships between certain classes of analytic functions. To begin with, Chapter 1 introduces some basic concepts and results in the theory of univalent functions which will be required later in our investigations.

Chapter 2, entitled "Initial Coefficients of Starlike Functions w.r.t. Symmetric Points" aims at studying the functions which are starlike with respect to symmetric points. It is well known that the class of analytic functions f defined on the unit disk satisfying

$\mathrm{Re}(zf'(z)/(f(z)-f(-z)))>0$ is a subclass of close-to-convex functions and the n^{th} Taylor coefficient of these functions are bounded by one. However, no bounds are known for the n^{th} coefficients of functions $f\in\mathcal{S}_s^*(\varphi)$ satisfying $2zf'(z)/(f(z)-f(-z))\prec\varphi(z)$, except for $n=2,3$. Thus, the sharp bound for the fourth coefficient of analytic univalent functions f satisfying the following subordination $2zf'(z)/(f(z)-f(-z))\prec\varphi(z)$ has been obtained. The bound for the fifth coefficient has also been obtained in certain special cases of φ including e^z and $\sqrt{1+z}$.

Chapter 3, entitled "Fekete-Szegö Coefficient Functional", deals with obtaining the bound for the Fekete-Szegö coefficient functional. Let φ be an analytic function with the positive real part satisfying $\varphi(0)=1$ and $\varphi'(0)>0$. Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be an analytic function satisfying the subordination $\alpha f'(z)+(1-\alpha)zf'(z)/f(z)\prec\varphi(z)$, $(f'(z))^\alpha(zf'(z)/f(z))^{(1-\alpha)}\prec\varphi(z)$, $(f'(z))^\alpha(1+zf''(z)/f'(z))^{(1-\alpha)}\prec\varphi(z)$, $(f(z)/z)^\alpha(zf'(z)/\ f(z))^{(1-\alpha)}\prec\varphi(z)$ or $(f(z)/z)^\alpha(1+zf''(z)/f'(z))^{(1-\alpha)}\prec\varphi(z)$. For functions satisfying the above subordination, the bounds of Fekete-Szegö coefficient functional have been obtained.

In Chapter 4 entitled "Hankel Determinant of Certain Analytic Functions", we have obtained the bounds for the second Hankel determinant $H_2(2)=a_2a_4-a_3^2$ for the function f satisfying $\alpha f'(z)+(1-\alpha)zf'(z)/f(z)\prec\varphi(z)$, $(f'(z))^\alpha(zf'(z)/f(z))^{(1-\alpha)}\prec\varphi(z)$, $(f'(z))^\alpha(1+zf''(z)/f'(z))^{(1-\alpha)}\prec\varphi(z)$, $(f(z)/z)^\alpha(zf'(z)/f(z))^{(1-\alpha)}\prec\varphi(z)$ or $(f(z)/z)^\alpha\ (1+zf''(z)/f'(z))^{(1-\alpha)}\prec\varphi(z)$. Here φ is an analytic function with the positive real part, $\varphi(0)=1$ and $\varphi'(0)>0$. We have also determined the third Hankel determinant $H_3(1)=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2)$ for an analytic function f of the form $f(z)=z+\sum a_nz^n$ satisfying either $\mathrm{Re}\left((f'(z))^\alpha(zf'(z)/f(z))^{(1-\alpha)}\right)>0$ or $\mathrm{Re}\left((f'(z))^\alpha(1+zf''(z)/f'(z))^{(1-\alpha)}\right)>0$. Our results include some previously known results.

In Chapter 5, entitled "Janowski Starlikeness and Convexity", certain necessary and sufficient conditions have been determined for the functions $f(z)=z-\sum_{n=2}^\infty a_nz^n\in\mathcal{T}$, $a_n\geq 0$, defined on $\mathbb{D}$, to belong to renowned subclasses of Janowski starlike and convex functions. In the same chapter, we have also discussed certain sufficient conditions for the normalised analytic functions f satisfying $(z/f(z))^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n$, $\mu \in \mathbb{C}$ to be in the class $\mathcal{S}^*[A,B]$ of Janowski starlike functions.

In Chapter 6, named "The classes $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$", we have attempted to study the function f defined on $\mathbb{D}$, with normalisations $f(0) = 0 = f'(0) - 1$, satisfying the subordinations $zf'(z)/f(z) \prec \alpha + (1-\alpha)e^z$ or $zf'(z)/f(z) \prec \alpha + (1-\alpha)\sqrt{1+z}$ respectively, where $0 \leq \alpha < 1$. The sharp radii has been determined for these functions to belong to several known subclasses of analytic functions. In addition, some inclusion relations and coefficient problems including the bounds for the first four coefficient estimates and the Fekete-Szegö functional have also been obtained.

Contents

Chapter 1

Introduction

Univalent function theory is an alluring branch of geometric function theory which comprises of the various geometric properties of analytic functions. The first paper on univalent functions dates back to 1907 and was written by Koebe [40]. Since then, there has been an extensive study in this field and a vast literature is available now. Among the other textbooks on univalent functions, there are excellent books by Pommerenke [68], Goodman [23], Duren [17], Goluzin [22] and Graham and Kohr [26] which provide an enormous amount of theory of univalent functions.

1.1. Univalent Functions

A function f is said to be univalent in some domain $D \subset \mathbb{C}$ if it doesn't take the same value twice. Mathematically, we can say that for two distinct $z_1, z_2 \in D$, $f(z_1) \neq f(z_2)$. In other words, a one to one function which is analytic except at the most one simple pole is said to be univalent. The function f is said to be locally univalent at some point $z_0 \in D$ if f is univalent in some neighbourhood of z_0. The necessary and sufficient condition for local univalence of a function f at the point z_0 is that $f'(z_0) \neq 0$.

Let $\mathcal{A}$ be the class of all normalised analytic functions defined on the open unit disk $\mathbb{D} := \{z : |z| < 1\}$ subject to normalisation $f(0) = 0$ and $f'(0) = 1$. Our primary concern here is to study a subclass of $\mathcal{A}$ which consists of normalised analytic univalent functions

f defined over a unit disc $\mathbb{D}$. Such a class is denoted by $\mathcal{S}$. We restrict the domain under consideration to the open unit disc because of the famous Riemann Mapping Theorem which was given as early as in 1851 and is one of the most remarkable results in Complex Analysis. It states that any simply connected domain in $\mathbb{C}$ can be mapped conformally onto a unit disc $\mathbb{D} = \{z : |z| < 1\}$. The following version of Riemann Mapping Theorem is from the textbook of Complex Analysis by Ahlfors.

THEOREM 1.1. (Riemann Mapping Theorem) [1] *Let $D \subsetneq \mathbb{C}$ be a simply connected domain, and $z_0 \in D$ be any given point. Then there must exist an analytic, one to one function $f : D \to \mathbb{D}$ which is unique, maps D onto $\mathbb{D}$ and has the properties that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Taylor series expansion for a normalised analytic and univalent function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad |z| < 1.$$

The best example of a function in the class $\mathcal{S}$ is the Koebe function

$$\mathcal{K}(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots.$$

Koebe function also acts as an extremal function for the class $\mathcal{S}$. It maps the unit disk $\mathbb{D}$ onto the entire complex plane except the slit from -1/4 to infinity on the negative real axis.

The first milestone in the field of univalent function theory was the estimation of the second coefficient bound for a function in the class $\mathcal{S}$. It was given by Bieberbach in the year 1916 wherein he proved that

THEOREM 1.2. (Bieberbach's Theorem) [12]. *If $f \in \mathcal{S}$, $|a_2| \leq 2$ with equality if and only if the function f is a rotation of the Koebe function.*

Bieberbach's coefficient theorem suggested the general coefficient estimate problem which deals in finding $A_n = \sup_{f \in \mathcal{S}} |a_n|$, $n = 2, 3, 4, \cdots$. Bieberbach also proposed a conjecture for the n^{th} coefficient of the function in the class $\mathcal{S}$ in the same year which is as follows

Bieberbach Conjecture. [12] The coefficients of the the function $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, 4, \cdots$. Equality holds if f is a rotation of the Koebe function.

For many years this conjecture remained as a challenge for all mathematicians and has motivated a lot of studies in the theory of univalent functions. Among other results obtained in trying to prove the Bieberbach conjecture, the following are worthy of note. In the year 1923, Loewner proved that $|a_3| \leq 3$ using his parametric method. It was in the year 1955 that Garabedian and Schiffer proved the Bieberbach conjecture for $n = 4$ by the simultaneous use of variational and parametric methods. Later on, in 1960, with the aid of Grunsky's univalence condition, the estimate was obtained in a much more simple manner. Pederson and Ozava demonstrated the validity of Bieberbach's conjecture for $n = 6$ in the year 1968 with the help of the Grunsky inequalities; and in the year 1972, it was proved for $n = 5$ by Pederson and Schiffer with the aid of Garabedian-Schiffer inequalities. After 68 years of extensive research in the geometric theory of functions of a complex variable, the Bieberbach's conjecture was finally proved in the year 1985 by Louis de Branges [13] of Purdue University. He gave a proof for this conjecture using certain inequalities for special functions. Exact references to these papers and future discussion can be found in the book by Duren [17].

While going about proving the sharp coefficient bounds for functions in the class $\mathcal{S}$, Fekete and Szegö [20] in the year 1933, proved that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & (\mu \geq 1); \\ 1 + exp(\frac{-2\mu}{1-\mu}), & (0 \leq \mu \leq 1); \\ 3 - 4\mu, & (\mu \leq 0). \end{cases}$$

holds for the functions $f \in \mathcal{S}$ and the result is sharp.

Let us now examine certain applications of the Bieberbach's Theorem. Long back in 1907, Koebe [40] stated that the range of any function $f \in \mathcal{S}$ under the unit disk contains a disk centred at the origin of radius 1/4. That is, $\mathbb{D}_{1/4} = \{z : |z| \leq 1/4\} \subseteq f(\mathbb{D})$. This theorem came to be known as Koebe's One-Quarter Theorem and is given as

THEOREM 1.3. (Koebe One-Quarter Theorem) [40]. *The range of every function of the class $\mathcal{S}$ contains the disk $\{z : |z| \leq 1/4\}$. Equality holds if and only if $f(z)$ is a rotation of the function $K(z) = z/(1-z)^2$.*

Another interesting application of the Bieberbach's Theorem is the famous Koebe's Distortion Theorem. This theorem provides the sharp lower and upper bounds for $|f'(z)|$ where $f \in \mathcal{S}$.

THEOREM 1.4. (Distortion Theorem) [24, Theorem 3, p. 65] *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad |z| = r < 1.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is some rotation of the Koebe function.

Furthermore, there were some very interesting consequences of the distortion theorem, namely the growth theorem and the rotation theorem. Growth theorem gives the sharp bounds for $|f(z)|$ whereas the rotation theorem deals with the computation of the bounds for $|\arg f'(z)|$. Rotation theorem was given by Goluzin [21] in the year 1936 by skilfully using the Loewner differential equation. It is called so, since the quantity $\arg f'(z)$ geometrically represents the local rotation factor under the conformal mapping f; the two theorems are given as follows.

THEOREM 1.5. (Growth Theorem) [24, Theorem 8, p. 68] *For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad |z| = r < 1$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is some rotation of the Koebe function.

THEOREM 1.6. (Rotation Theorem) [21]. *If $f \in \mathcal{S}$, then*

$$|\arg f'(z)| \leq \begin{cases} 4arcsin^{-1}r, & \text{if } r \leq 1/\sqrt{2}; \\ \pi + \ln\frac{r^2}{1-r^2}, & \text{if } 1/\sqrt{2} < r < 1. \end{cases}$$

The bound is sharp for each $z \in \mathbb{D}$.

1.2. Fundamentals in Univalent Function Theory

A domain D is said to be starlike with respect to an interior point w_0 if every point in that domain is visible from w_0. A function is starlike with respect to some point z_0 if it maps $\mathbb{D}$ onto a domain which is starlike with respect to z_0. When $z_0 = 0$, we say that f is starlike. The class of all starlike functions is denoted by $\mathcal{S}^*$. The following theorem which gives the analytic characterisation of starlike functions was given by Nevanlinna in 1921.

THEOREM 1.7. [24, Theorem 1, p. 111] *Let f be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in \mathcal{S}^*$ if and only if* $\operatorname{Re}\left(zf'(z)/f(z)\right) > 0$

If line joining any two points in the domain lies entirely within the domain, the domain D is said to be convex. In other words, a domain is said to be convex, if it is starlike with respect to each of its interior points. A function f is said to be convex, if it maps the unit disk $\mathbb{D}$ onto a convex domain. The class of all convex functions is denoted by $\mathcal{K}$. The following analytic characterisation for the class of convex functions was given by Study in the year 1913.

THEOREM 1.8. [24, Theorem 1, p. 111] *Let f be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in \mathcal{K}$ if and only if* $\operatorname{Re}\left(1 + zf''(z)/f'(z)\right) > 0$

There is a close analytic relationship between starlike and convex functions. Alexander gave a two-way bridge relationship between the class of starlike and convex functions. He proved that

THEOREM 1.9. (Alexander's Theorem) [24, Theorem 5, p. 115] *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in \mathcal{K}$ if and only if $F \in \mathcal{S}^*$ where $F(z) = zf'(z)$.*

In other words, F is starlike $\Leftrightarrow$ the function f defined by $f(z) = \int_0^z F(\xi)/\xi d\xi$ is convex.

1.2.1. Caratheodory Class. Closely related to the class of starlike and convex functions is the Caratheodory class which consists of analytic functions with real part positive. We denote this class by $\mathcal{P}$ and $p \in \mathcal{P}$ if $\operatorname{Re}(p(z)) > 0$. Taylor series expansion for functions in the class $\mathcal{P}$ is given by $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$. We can reformulate the starlike and the convex functions in terms of the functions with the positive real part by saying that a function $f(z)$ is said to starlike or convex if it satisfies $zf'(z)/f(z) \in \mathcal{P}$ or $1 + zf''(z)/f'(z) \in \mathcal{P}$, respectively. The class $\mathcal{P}$ is also closely related to the class of univalent functions because of the beautiful result by Noshiro and Warschawski which states that

THEOREM 1.10. (The Noshiro- Warschawski Theorem.) [65, 93] *Suppose that for some real α we have $\operatorname{Re}(e^{\iota\alpha} f'(z)) > 0$ for all z in a convex domain D. Then $f(z)$ is univalent in D.*

Alexander also gave a relationship between the class $\mathcal{S}$ and $\mathcal{P}$ way back in 1915, which is merely a special case of the Noshiro- Warschawski Theorem. The result goes as follows.

THEOREM 1.11. [24, Theorem 12, p. 88] *If f' is in $\mathcal{P}$, then f is univalent in $\mathbb{D}$.*

1.2.2. Sharp bounds for coefficients. In the year 1921, Nevanlinna gave the following sharp bound for n^{th} coefficient of function $f \in \mathcal{S}^*$.

THEOREM 1.12. (R. Nevanlinna) [24, Theorem 6, p. 116] *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in $\mathcal{S}^*$, then for each positive integer n, $|a_n| \leq n$, $n = 1, 2, 3, \cdots$. Furthermore, equality holds if f is some rotation of the Koebe function.*

Loewner gave the sharp bound for n^{th} coefficient of functions $f \in \mathcal{K}$.

THEOREM 1.13. (C. Loewner) [24, Theorem 7, p. 117] *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in $\mathcal{K}$, then for each positive integer n, $|a_n| \leq 1$, $n = 1, 2, 3, \cdots$. Furthermore, equality holds if f is some rotation of the function $z/(1-z)$.*

Carathéodory gave the following lemma for the functions in the class $\mathcal{P}$. This lemma is quite useful for proving various coefficient problems in the theory of univalent functions.

LEMMA 1.14. (Carathéodory's Lemma.) [17, p. 41] *If $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \in \mathcal{P}$, then $|c_n| \leq 2$, $n = 1, 2, 3, \cdots$. Equality holds for the function $(1+z)/(1-z) \in \mathcal{P}$.*

1.2.3. Subordination. The concept of subordination was first studied long back in 1908 by Lindelöf [46] but the term subordination was only coined later in the year 1925 by Littlewood [47, 48] and studied in a more detailed fashion by Rogosinski [76, 77]. A substantial amount of theory has been developed over the years and subordination plays a very crucial role in the study of complex analysis.

Let the functions f and g be analytic on $\mathbb{D}$. Then, f is said to be subordinate to g if $f(z) = g(w(z))$, $|z| < 1$, where the function $w \in \Omega$ is analytic and satisfies $w(0) = 0$ and $|w(z)| \leq 1$. We denote is as $f(z) \prec g(z)$. We can also say that $g(z)$ is superordinate to $f(z)$ in $\mathbb{D}$. When the function g is subject to an additional condition of univalence along with analyticity, then the definition of subordination changes. In this case, $f(z)$ is subordinate to $g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. According to Lindelöf, the following condition holds for $f(z) \prec F(z)$.

THEOREM 1.15. (The Lindelöf Principle. 1908.) [24, Theorem 10, p. 86] *Suppose that $f(z) \prec F(z)$ in $\mathbb{D}$. Then for each r in [0,1], $f(\mathbb{D}_r) \subset F(\mathbb{D}_r)$. Further, if $f(re^{\iota\theta})$ is on the boundary of $F(\mathbb{D}_r)$ for any one point $z_0 = re^{\iota\theta_0}$, with $0 < r < 1$, then there is a real α such that $f(z) = F(e^{\iota\alpha}z)$ and $f(re^{\iota\theta})$ is on the boundary of $F(\mathbb{D}_r)$ for every point $z = re^{\iota\theta}$ in $\mathbb{D}$.*

1.2.4. Ma- Minda Subclasses of Analytic Functions. Let φ be a univalent function with positive real part which maps $\mathbb{D}$ onto a domain which is starlike with respect to $\varphi(0) = 1$, symmetric with respect to the real line and $\varphi'(0) > 0$. For such a function φ, in the year 1992, Ma and Minda defined and studied unified classes of starlike and convex functions denoted by $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ respectively [51] which are defined as

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

Ma-Minda classes generalise various subclasses by taking suitable choice of φ. For instance, when $\varphi = (1 + Az)/(1 + Bz)$, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduces to the class $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$, respectively, which are the classes of the Janowski starlike and convex functions respectively. Similarly, on replacing $\varphi = (1+(1-2\alpha)z)/(1-z)$, where $0 \leq \alpha < 1$ in $\mathcal{S}^*(\varphi)$ we get the classes $S^*(\alpha)$ which is the familiar class of s-tarlike functions of order α and $\mathcal{K}(\varphi)$ reduces to the class $\mathcal{K}(\alpha)$ which is the class of convex functions of order α. For $0 < \alpha \leq 1$, on substituting $\varphi = \big((1+z)/(1-z)\big)^{\alpha}$ in $\mathcal{S}^*(\varphi)$, we get the subclass $\mathcal{SS}^*(\alpha)$ which is the renowned class of strongly starlike functions of order α. The class $\mathcal{SS}^*(\alpha)$ consists of the functions f in $\mathcal{S}$ with $\big|\arg\big((zf'(z))/f(z)\big)\big| < \alpha\pi/2$ for $z \in \mathbb{D}$. Finally, replacing φ by $(1+z)/(1-z)$, the class $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduces to the classes $\mathcal{S}^*$ and $\mathcal{K}$ respectively. Note that, on replacing φ with $\alpha+(1-\alpha)e^z$ and $\alpha+(1-\alpha)\sqrt{1+z}$ in $\mathcal{S}^*(\varphi)$, we have defined two extremely important subclasses of analytic functions, which have been studied extensively in the sixth chapter of the thesis.

$$\mathcal{S}^*_{\alpha,e} = \mathcal{S}^*(\alpha+(1-\alpha)e^z) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \alpha+(1-\alpha)e^z \right\},$$

and

$$\mathcal{SL}^*(\alpha) = \mathcal{S}^*(\alpha+(1-\alpha)\sqrt{1+z}) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \alpha+(1-\alpha)\sqrt{1+z} \right\}.$$

1.3. Starlike Functions w.r.t. Symmetric Points

A function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\mathbb{D}$ if for every r less than and sufficiently close to one and every ζ on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z = \zeta$ as z traverses the circle $|z| = r$ in the positive direction. Analytically, a function $f \in \mathcal{A}$ is starlike with respect to symmetric points if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0, \qquad z \in \mathbb{D}.$$

The class of all starlike functions with respect to symmetric points is denoted by $\mathcal{S}^*_s$. The class $\mathcal{S}^*_s$ was introduced by Sakaguchi [79]. The functions belonging to this class are close-to-convex and therefore univalent. It is a well known fact that this class $\mathcal{S}^*_s$ includes the class of convex functions $\mathcal{K}$ and the class of odd starlike functions [79]; the functions in $\mathcal{S}^*_s$ also satisfy the sharp coefficient inequality $|a_n| \leq 1$, see [16] and [74] for other related classes. Let φ be a univalent function with positive real part which maps $\mathbb{D}$ onto a domain which is starlike with respect to $\varphi(0)=1$, symmetric with respect to the real line and $\varphi'(0)>0$. For such φ, Ravichandran [74] introduced the following generalised class of Ma-Minda starlike functions with respect to symmetric points:

$$S_s^*(\varphi) := \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi(z) \right\}$$

and later in [80], the sharp bound for the Fekete-Szegö coefficient functional $|a_3-\mu a_2^2|$ were obtained. This immediately gives the bound for the first two coefficients of functions in the above classes.

1.4. Fekete-Szegö Problem

We have already stated that in the year 1933, Fekete and Szegö proved that

$$|a_3-\mu a_2^2| \leq \begin{cases} 4\mu-3, & (\mu \geq 1); \\ 1+exp(\frac{-2\mu}{1-\mu}), & (0 \leq \mu \leq 1); \\ 3-4\mu, & (\mu \leq 0). \end{cases}$$

holds for functions $f \in \mathcal{S}$ and the result is sharp. Thereafter, computing the bound for the quantity $|a_3-\mu a_2^2|$ came to be known as the Fekete-Szegö problem or the Fekete-Szegö coefficient problem. Keogh and Merkes [58], in 1969, obtained the sharp upper bound of the Fekete-Szegö functional $|a_3-\mu a_2^2|$ for functions in some subclasses of $\mathcal{S}$. The Fekete-Szegö functional problem for close to convex functions was investigated among others by Koepf [41], Kim, Choi and Sugawa [15, 39] and Cho *et al.* [14]. The problem for starlike and convex functions were investigated in a more general settings by Ma and Minda [51]. For other general classes of p-valent functions, the Fekete-Szegö functional problem was investigated by Ali *et al.* [5, 7]. For classes defined by quasi-subordination, see Mohd and Darus [73]. Jakubowski and Zyskowska [33] obtained the estimate for $|a_2-ca_2^2|+c|a_2|^n$ for $c \in \mathbb{R}$, $f \in \mathcal{S}$. Kiepiela, Pietrzyk and Szynal [38] obtained bounds for certain combination of initial coefficients of bounded functions; these results were used later for estimating fourth coefficients of many classes [7]. The results related to this functional can be seen in [5, 7].

1.5. HANKEL DETERMINANT.

The qth Hankel determinant, denoted by $H_q(n)$, for $q = 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$ of the function f is given by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Therefore, the second Hankel determinant is given by $H_2(2) := a_2a_4 - a_3^2$. Among the first few papers on determination of second Hankel determinant, theres one by Pommerenke [69] in the year 1966. He investigated the second Hankel determinant $H_2(2) := a_2a_4 - a_3^2$ of univalent functions, starlike functions and class of areally p-valent functions. In addition to that, Pommerenke [70] also established that Hankel determinant of univalent functions satisfies the following relation: $|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$ $(n = 1, 2, \ldots; q = 1, 2, \ldots)$, where $\beta > 1/40000$ and K depends on q. The second Hankel determinants of areally mean p-valent functions were investigated by Noonan in [59–61]. Noor studied the bounds for Hankel determinant for the class of close-to-convex functions in [62–64]. Later, Hayman [30] proved that $|H_2(n)| < An^{1/2}$ for areally mean univalent functions($n \in \mathbb{N}$; A is an absolute constant). In 1986, Elhosh [18, 19] computed the bound for the second Hankel determinant for univalent functions having positive Hayman index α, functions which are k-fold symmetric and the class of close-to-convex functions. The bound for the second Hankel determinant for the class of starlike and convex functions, the class of close-to-starlike and close-to-convex functions with respect to symmetric points and the class of functions whose derivative has a positive real part has been studied in [29, 34]. Lee *et al.* [45] obtained bounds for the second Hankel determinant for the unified classes of Ma-Minda starlike and convex functions with respect to φ and two other similar subclasses. One may refer to the survey given by Liu *et al.* [50] for the other work done in the research of second Hankel determinant for univalent functions. Hankel determinants have been studied by several other authors for various other classes of analytic functions and can be referred to in [2, 11, 89].

Similarly, the third Hankel Determinant is defined as

$$H_3(1) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$
$$:= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

The third Hankel determinant $H_3(1)$ for the class of starlike and convex functions was studied by Babalola [10]. Shanmugam *et al.* [82] obtained the third Hankel determinant $H_3(1)$ for the class of α- starlike functions. The third Hankel determinant for the class of close to convex functions can be referred to in [71], for a subclass of p- valent functions has been studied in [90], for a class of analytic functions associated with the lemniscate of Bernoulli in [73] and for starlike and convex functions with respect to symmetric points in [57]. One can refer to [92] for the third Hankel determinant $H_3(1)$ for the inverse of a function whose derivative has real part positive and [82] for α- starlike functions.

1.6. Radius Problem.

In this problem, we are interested in finding the maximal radius $r \leq 1$ such that if a function f is in some class imply that f is in some other subclass whenever $|z| < r \leq 1$. For instance, if f is a starlike function, then it need not be convex. The problem of finding value for r $(0 < r \leq 1)$ so that $f(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r \leq 1\}$reduces to a convex domain from a starlike domain is known as radius of convexity for the class $\mathcal{S}^*$. The radius of convexity and starlikeness for the class of all normalised analytic univalent functions $\mathcal{S}$ was studied by Nevanlinna and Grunsky respectively and are given as

THEOREM 1.16. (Nevanlinna 1920.) [24, Theorem 10, p. 119] *Let $f(z) \in \mathcal{S}$. Then for each $r \leq 2 - \sqrt{3}$, the image of $|z| = r$ is a simple closed convex curve. The number $R_{\mathcal{K}} = 2 - \sqrt{3}$ is sharp. There is a function in $\mathcal{S}$ such that for each $r > R_{\mathcal{K}}$ the image of $|z| = r$ is not a convex curve. The number $R_{\mathcal{K}} = 2 - \sqrt{3}$ is called the radius of convexity for the set $\mathcal{S}$.*

THEOREM 1.17. (Grunsky 1933.) [24, Theorem 11, p. 121] *The the radius of starlikeness of the class $\mathcal{S}$ is the root of the equation*

$$\ln \frac{1+r}{1-r} = \frac{\pi}{2},$$

namely

$$R_{ST} = \frac{e^{\pi/2}-1}{e^{\pi/2}+1} = \tanh \frac{\pi}{4} = 0.65579.$$

1.7. SUMMARY OF THE THESIS

This thesis comprises of six chapters. The first chapter being the introductory chapter contains the basic definitions and fundamentals of the univalent function theory which will be needed in the subsequent chapters. The second chapter deals with the estimation of the fourth and the fifth coefficient bound for the class of functions starlike with respect to symmetric points. The third chapter focuses on the Fekete-Szegö inequality whereas in the fourth chapter we have computed the second and the third Hankel determinants for various interesting subclasses. In the fifth chapter, we have studied the Janowski starlike and Janowski convex classes in detail. And lastly, the final chapter deals with the radius problems and coefficient estimates for two very interesting subclasses. Given below is the chapter - wise brief summary of the work we executed.

In Chapter 2, we have studied the class of functions starlike with respect to symmetric points defined by

$$\mathcal{S}_s^*(\varphi) := \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi(z) \right\} \tag{1.1}$$

Taylor series expansion for the function f in this subclass is $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots$. This class was introduced by Ravichandran [74] in the year 2004. We have also examined certain special subclasses of $\mathcal{S}_s^*(\varphi)$. On replacing φ by e^z, $\sqrt{1+z}$ and $\sqrt{2} - (\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}$, we get the following subclasses.

(1) $\mathcal{S}_{s,e}^* := S_s^*(e^z)$,
(2) $\mathcal{S}_{s,L}^* := S_s^*(\sqrt{1+z})$,

(3) $\mathcal{S}^*_{s,RL} := S^*_s\left(\sqrt{2}-(\sqrt{2}-1)\sqrt{\dfrac{1-z}{1+2(\sqrt{2}-1)z}}\right).$

Sharp bounds for the fourth coefficient a_4 of the functions belonging to $\mathcal{S}^*_s(\varphi)$ have been obtained. Also, since the bounds for the fifth coefficient a_5 for the generalised class $\mathcal{S}^*_s(\varphi)$ was coming out to be highly non linear. Therefore, the bounds for the fifth coefficient a_5 for the functions belonging to certain special subclasses of $\mathcal{S}^*_s(\varphi)$ have been determined namely $\mathcal{S}^*_{s,e}$, $\mathcal{S}^*_{s,L}$ and $\mathcal{S}^*_{s,RL}$.

In Chapter 3, motivated by [5,7,58], we have defined five extremely interesting subclasses of $\mathcal{S}$ and estimated the bound for the Fekete-Szegö coefficient functional $|a_3-\mu a_2^2|$ for them. The classes are $\mathcal{V}_\alpha(\varphi)$, $\mathcal{M}_\alpha(\varphi)$, $\mathcal{L}_\alpha(\varphi)$, $\mathcal{K}_\alpha(\varphi)$ and $\mathcal{T}_\alpha(\varphi)$ respectively, where $0\leq\alpha\leq 1$, and are defined as below:

$$\mathcal{V}_\alpha(\varphi) := \left\{f\in\mathcal{S} : \alpha f'(z)+(1-\alpha)\frac{zf'(z)}{f(z)} \prec \varphi(z)\right\},$$

$$\mathcal{M}_\alpha(\varphi) := \left\{f\in\mathcal{S} : (f'(z))^\alpha\left(\frac{zf'(z)}{f(z)}\right)^{(1-\alpha)} \prec \varphi(z)\right\}.$$

Clearly, when $\alpha=0$, both the classes $\mathcal{V}_\alpha(\varphi)$ and $\mathcal{M}_\alpha(\varphi)$ reduce to the Ma - Minda unified class of starlike functions $\mathcal{S}^*(\varphi)$, whereas when $\alpha=1$, both the classes reduce to the class $\mathcal{R}(\varphi)$ which is a subclass of close-to-convex functions.

$$\mathcal{L}_\alpha(\varphi) := \left\{f\in\mathcal{S} : (f'(z))^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)} \prec \varphi(z)\right\}.$$

Here, when $\alpha=0$, the class $\mathcal{L}_\alpha(\varphi)$ reduces to the Ma - Minda class of convex functions $\mathcal{K}(\varphi)$, whereas when $\alpha=1$, $\mathcal{L}_\alpha(\varphi)$ reduces to the subclass of close-to-convex functions $\mathcal{R}(\varphi)$.

$$\mathcal{K}_\alpha(\varphi) := \left\{f\in\mathcal{S} : \left(\frac{f(z)}{z}\right)^\alpha\left(\frac{zf'(z)}{f(z)}\right)^{(1-\alpha)} \prec \varphi(z)\right\},$$

$$\mathcal{T}_\alpha(\varphi) := \left\{f\in\mathcal{S} : \left(\frac{f(z)}{z}\right)^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)} \prec \varphi(z)\right\}.$$

Again, on substituting $\alpha=0$ in the classes $\mathcal{K}_\alpha(\varphi)$ and $\mathcal{T}_\alpha(\varphi)$, we get the Ma - Minda generalised subclasses of starlike and convex functions, respectively.

Note that the Fekete-Szegö coefficient functional directly yields the bounds for the first two coefficients. Therefore, the sharp bounds for the second and the third coefficients a_2 and a_3 for the functions belonging to the same classes have also been estimated.

Chapter 4 focuses on the Hankel determinants. The qth Hankel determinant (denoted by $H_q(n)$) for $q = 1, 2, \dots$ and $n = 1, 2, 3, \dots$ of the function $f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$ is the determinant of the $q \times q$ matrix given by $H_q(n) := \det(a_{n+i+j-2})$. Here $a_{n+i+j-2}$ denotes the entry for the i^{th} row and j^{th} column of the matrix. Motivated by the paper by Lee *et al.* [45], who obtained bounds for the second Hankel determinant for the classes of Ma-Minda starlike and convex functions with respect to φ, we defined five subclasses of analytic functions and obtained the bound for the second Hankel determinant $H_2(2) = |a_2 a_4 - a_3^2|$ for them. The classes respectively are consisting of the function $f \in \mathcal{S}$ satisfying $\alpha f'(z) + (1-\alpha) z f'(z)/f(z) \prec \varphi$, $(f'(z))^\alpha (zf'(z)/f(z))^{(1-\alpha)} \prec \varphi$, $(f'(z))^\alpha (1 + zf''(z)/f'(z))^{(1-\alpha)} \prec \varphi$, $(f(z)/z)^\alpha (zf'(z)/f(z))^{(1-\alpha)} \prec \varphi$ or $(f(z)/z)^\alpha (1 + zf''(z)/f'(z))^{(1-\alpha)} \prec \varphi$, where φ is a univalent function with the positive real part and satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$.

Apart from this, motivated by the paper by Babalola [10], who estimated the third Hankel determinant $H_3(1)$ for the class of starlike and convex functions, we have also obtained the third Hankel determinant for two very interesting subclasses of starlike and convex functions defined as follows:

$$\mathcal{M}_\alpha := \left\{ f \in \mathcal{S} : \operatorname{Re}\left((f'(z))^\alpha \left(\frac{zf'(z)}{f(z)} \right)^{1-\alpha} \right) > 0 \right\},$$
$$\mathcal{L}_\alpha := \left\{ f \in \mathcal{S} : \operatorname{Re}\left((f'(z))^\alpha \left(1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} \right) > 0 \right\}.$$

Note that for $\alpha = 0$, the classes $\mathcal{M}_\alpha$ and $\mathcal{L}_\alpha$ reduce to the classes of the starlike and the convex functions respectively, whereas substituting $\alpha = 1$ in the above subclasses again yield the subclass of close to convex functions whose derivative has real part positive, satisfying $\{f \in \mathcal{S} : \operatorname{Re}(f'(z)) > 0\}$.

Chapter 5 deals mainly with the functions with the negative coefficients which is a subclass of $\mathcal{S}$ which consists of functions whose coefficients from second onwards are negative. We denote this class by $\mathcal{T}$. A function $f \in \mathcal{T}$ if it is of the form $f(z) = z - a_2z^2 - a_3z^3 - a_4z^4 - \cdots, a_n \geq 0$. Certain necessary and sufficient conditions have been investigated for the functions in the class $\mathcal{T}$ to be in classes $\mathcal{TS}^*[A,B]$, $\mathcal{TK}[A,B]$ and $TR(A,B,\alpha)$ defined as follows:

$$\mathcal{TS}^*[A,B] :=\left\{f \in \mathcal{T} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}\right\},$$
$$\mathcal{TK}[A,B] :=\left\{f \in \mathcal{T} : 1+\frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}\right\},$$

and

$$\mathcal{TR}(A,B,\alpha) := \left\{f \in \mathcal{T} : \frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) \prec \frac{1+Az}{1+Bz}\right\}.$$

Also discussed are some sufficient conditions for the functions belonging to the intersection of the above defined classes. In Section 5.4, motivated by [44], we defined another class of functions f satisfying $(z/f(z))^{\mu} = 1+\sum_{n=1}^{\infty} b_nz^n, \quad \mu \in \mathbb{C}$ and obtained the necessary and sufficient conditions for such functions to be in the class $\mathcal{S}^*[A,B]$.

In Chapter 6, two very fascinating subclasses $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$, $0 \leq \alpha < 1$ of $\mathcal{S}^*$ have been defined using the concept of subordination. They are given by

$$\mathcal{S}^*_{\alpha,e} = \mathcal{S}^*(\alpha+(1-\alpha)e^z) := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \alpha+(1-\alpha)e^z\right\},$$

and

$$\mathcal{SL}^*(\alpha) = \mathcal{S}^*(\alpha+(1-\alpha)\sqrt{1+z}) := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \alpha+(1-\alpha)\sqrt{1+z}\right\}.$$

and have been thoroughly examined and studied in this chapter. An observation leads to the fact that when $\alpha = 0$, $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$ reduce to the classes $\mathcal{S}^*_e$ and $\mathcal{SL}$ respectively.

Apart from obtaining the bound for the Fekete-Szegö inequality, we have also studied the relationship between $\mathcal{S}^*_{\alpha,e}$ and other subclasses of analytic functions such as the class $\mathcal{M}(\beta)$, the class of k- starlike functions $k-\mathcal{S}^*$ and the class $\mathcal{S}^*(\beta)$ of functions starlike of order β defined as follows:

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} < \beta \right\}, \qquad z \in \mathbb{D}$$
$$k - \mathcal{S}^* := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}, \qquad z \in \mathbb{D}$$

and

$$\mathcal{S}^*(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta \right\}, \qquad z \in \mathbb{D}.$$

We have also determined the $\mathcal{S}^*(\beta)$, $\mathcal{M}(\beta)$, $k - \mathcal{S}^*$ and $\mathcal{K}$- radii for functions in these two subclasses. In addition, the $\mathcal{S}^*_{\alpha,e}$ - radius for various subclasses such as $\mathcal{S}^*[A, B]$, $\mathcal{W}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ has also been determined. The classes $\mathcal{W}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined as under:

$$\mathcal{W} := \left\{ f \in \mathcal{A} : \frac{f(z)}{z} \in \mathcal{P} \right\}, \qquad z \in \mathbb{D}$$
$$\mathcal{F}_1 := \left\{ f \in \mathcal{A} : \frac{f(z)}{g(z)} \in \mathcal{P}, g \in \mathcal{W} \right\}, \qquad z \in \mathbb{D}$$

and

$$\mathcal{F}_2 := \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1, g \in \mathcal{W} \right\}, \qquad z \in \mathbb{D}.$$

Finally, many coefficient inequalities have been determined and the bounds for the first four coefficient estimates have been obtained for both the classes $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$.

Chapter 2

Initial Coefficients of Starlike Functions w.r.t. Symmetric Points

Recall that an analytic function f is subordinate to F, written $f \prec F$ or $f(z) \prec F(z) \quad (z \in \mathbb{D})$ if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ satisfying $w(0) = 0$ and $f(z) = F(w(z))$ for $z \in \mathbb{D}$. Let φ be a univalent function with positive real part which maps $\mathbb{D}$ onto a domain which is symmetric with respect to the real line and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Taylor series expansion of the function φ is given as $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. Let $\mathcal{S}^*(\varphi)$ be the class of functions $f \in \mathcal{S}$ for which $zf'(z)/f(z) \prec \varphi(z)$ and $\mathcal{K}(\varphi)$ be the class of functions $f \in \mathcal{S}$ for which $1 + zf''(z)/f'(z) \prec \varphi(z)$. The above classes were introduced and studied by Ma and Minda [51]. Similar to Ma-Minda classes, for φ as defined above, Ravichandran [74] introduced the following subclass:

$$\mathcal{S}_s^*(\varphi) = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z) \right\}$$

The contents of this chapter appeared in K. Khatter, V. Ravichandran and S. Sivaprasad Kumar, Estimates for initial coefficients of certain starlike functions with respect to symmetric points, in *Applied analysis in biological and physical sciences*, 385–395, Springer Proc. Math. Stat., 186, Cushing J., Saleem M., Srivastava H., Khan M., Merajuddin M. (eds) Springer, New Delhi.

and later Shanmugam [80] obtained the sharp bound for the Fekete-Szegö coefficient functional $|a_3 - \mu a_2^2|$ for the class $\mathcal{S}_s^*(\varphi)$ were obtained. This immediately gives the bound for the first two coefficients of functions in the above classes.

Let $\mathcal{P}$ be the class of all analytic functions $p(z) = 1 + \sum_{n=0}^{\infty} c_n z^n$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$ and Ω be the class of all analytic functions $w : \mathbb{D} \to \mathbb{D}$ of the form $w(z) = w_1 z + w_2 z^2 + \cdots$. In this chapter, our aim is to determine the bound for the fourth coefficient of functions belonging to the class $S_s^*(\varphi)$. This is done by first expressing the coefficients of f in terms of the coefficients B_n of φ and the coefficient c_n of a function with a positive real part. The coefficient estimate for a_4 also follows from a result of Prokhorov and Szynal [72]. The bound for the fifth coefficient of functions in $S_s^*(\varphi)$ is highly non-linear. We are able to estimate a_5 in certain important special cases of φ:

$$\mathcal{S}_{s,e}^* := S_s^*(e^z), \quad \mathcal{S}_{s,L}^* := S_s^*(\sqrt{1+z}),$$

and

$$\mathcal{S}_{s,RL}^* := S_s^* \left(\sqrt{2} - (\sqrt{2}-1) \sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}} \right).$$

These classes are analogues of the corresponding classes of starlike functions introduced and studied respectively in [53, 55, 86].

2.1. Fourth Coefficient

To prove our results, we need the following results; the results in (a)–(c) of Lemma 2.1 are respectively in [3, 51, 75].

LEMMA 2.1. *Let* $p(z) = 1 + \sum_{n=0}^{\infty} c_n z^n \in \mathcal{P}$. *Then,*

(a) $|c_2 - v c_1^2| \leq 2 \max\{1, |2v - 1|\}$,
(b) $|c_3 - 2\beta c_1 c_2 + \delta c_1^3| \leq 2$ *if* $0 \leq \beta \leq 1$ *and* $\beta(2\beta - 1) \leq \delta \leq \beta$.
(c) $|\gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - (3/2)\beta c_1^2 c_2 - c_4| \leq 2$, *when* $0 < \alpha < 1, 0 < a < 1$ *and* $8a(1-a)((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4\alpha^2(1-\alpha)^2 a(1-a)$.

LEMMA 2.2. [72] *If $w \in \Omega$, then for any real numbers q_1 and q_2, the following sharp estimate $|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \leq H(q_1, q_2)$ holds, where*

$$H(q_1,q_2) = \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2 \\ |q_2| & \text{for } (q_1,q_2) \in \cup_{k=3}^{7} D_k \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9 \\ \frac{1}{3}q_2\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} - \{\pm 2, 1\} \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12} \end{cases} \tag{2.1}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{z([(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \quad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z}$$

$$|\varepsilon_1| = |\varepsilon_2| = 1, \quad \varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}(a \mp b), \quad \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2}, \quad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \quad \lambda = \frac{b \pm a}{2b}$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}}, \quad \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right].$$

The sets D_k, $k = 1, 2, \cdots, 12$, are defined as follows

$$\begin{aligned} D_1 &= \{(q_1,q_2) : |q_1| \leq \frac{1}{2}, |q_2| \leq 1\}, \\ D_2 &= \{(q_1,q_2) : \frac{1}{2} \leq |q_1| \leq 2, \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \leq q_2 \leq 1\}, \end{aligned}$$

$$\begin{aligned}
D_3 &= \{(q_1,q_2) : |q_1| \le \frac{1}{2}, q_2 \le -1\},\\
D_4 &= \{(q_1,q_2) : |q_1| \ge \frac{1}{2}, q_2 \le -\frac{2}{3}(|q_1|+1)\},\\
D_5 &= \{(q_1,q_2) : |q_1| \le 2, q_2 \ge 1\},\\
D_6 &= \{(q_1,q_2) : 2 \le |q_1| \le 4, q_2 \ge \frac{1}{12}(q_1^2+8)\},\\
D_7 &= \{(q_1,q_2) : |q_1| \ge 4, q_2 \ge \frac{2}{3}(|q_1|-1)\},\\
D_8 &= \{(q_1,q_2) : \frac{1}{2} \le |q_1| \le 2, -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\},\\
D_9 &= \{(q_1,q_2) : |q_1| \ge 2, -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\},\\
D_{10} &= \{(q_1,q_2) : 2 \le |q_1| \le 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{1}{12}(q_1^2+8)\},\\
D_{11} &= \{(q_1,q_2) : |q_1| \ge 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\},\\
D_{12} &= \{(q_1,q_2) : |q_1| \ge 4, \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \le q_2 \le \frac{2}{3}(|q_1|-1)\}.
\end{aligned}$$

By using Lemma 2.1 and 2.2, we have proved the following bound for the fourth coefficient of functions in $S_s^*(\varphi)$.

THEOREM 2.3. *Let the function* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_s^*(\varphi)$ *where* $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$. *Then*

$$|a_4| \le \frac{B_1}{4}H(q_1,q_2),$$

where $H(q_1,q_2)$ *is as defined in* (2.1),

$$q_1 := \frac{4B_2 + B_1^2}{2B_1}; \qquad \text{and} \qquad q_2 := \frac{2B_3 + B_1B_2}{2B_1}. \tag{2.2}$$

PROOF. If $f \in S_s^*(\varphi)$, then there is an analytic function $w(z) = w_1z + w_2z^2 + \cdots \in \Omega$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \varphi(w(z)). \tag{2.3}$$

Since

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1+2a_2z+2a_3z^2+(-2a_2a_3+4a_4)z^3+(-2a_3^2+4a_5)z^4+\cdots$$

and

$$\varphi(w(z)) = 1+B_1w_1z+(B_2w_1^2+B_1w_2)z^2+(B_3w_1^3+2B_2w_1w_2+B_1w_3)z^3+\cdots$$

we get, from (2.3),

$$\begin{aligned} a_2 &= \frac{1}{2}B_1w_1, \\ a_3 &= \frac{1}{2}(B_2w_1^2+B_1w_2), \\ a_4 &= \frac{1}{4}\left((B_3+\frac{1}{2}B_1B_2)w_1^3+(2B_2+\frac{1}{2}B_1^2)w_1w_2+B_1w_3\right). \end{aligned}$$

The coefficient a_4 can be rewritten as

$$a_4 = \frac{B_1}{4}(w_3+q_1w_1w_2+q_2w_1^3)$$

where q_1 and q_2 are as given in the equation 2.2. Lemma 2.2 immediately yields the desired estimate $|a_4| \leq B_1H(q_1,q_2)/4$. ∎

2.2. FIFTH COEFFICIENT

Our next theorems provide sharp bound on $|a_5|$ for three different choices of φ. The bounds for a_2, a_3, a_4 have also been included here for completeness.

THEOREM 2.4. *Let $f(z) = z+a_2z^2+a_3z^3+\cdots$ and $f \in \mathcal{S}^*_{s,L}$. Then*

$$|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{8} \quad \textit{and} \quad |a_5| \leq \frac{1}{8}.$$

All the bounds here are sharp.

PROOF. For the function $f(z) = z+a_2z^2+a_3z^3+\cdots \in \mathcal{S}^*_s(\varphi)$, we first express a_n $n=1,2,3,\cdots$ in terms of the coefficients of the functions $\varphi(z) = 1+B_1z+B_2z^2+\cdots$ and $p_1(z) = 1+c_1z+c_2z^2+\cdots \in \mathcal{P}$. In terms of the coefficients b_n of the function p

defined by

$$p(z) := \frac{2zf'(z)}{f(z)-f(-z)} = 1 + b_1 z + b_2 z^2 + \cdots,$$

the coefficients a_n are expressed by

$$na_n = \sum_{k=1}^{\lceil n/2 \rceil} b_{n+1-2k} a_{2k-1}. \tag{2.4}$$

From equation (2.4), we have

$$2a_2 = b_1, \quad 2a_3 = b_2, \quad -2a_2a_3 + 4a_4 = b_3, \quad -2a_3^2 + 4a_5 = b_4. \tag{2.5}$$

Since φ is univalent and $p \prec \varphi$, the function

$$p_1(z) = \frac{1+\varphi^{-1}(p(z))}{1-\varphi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

belongs to $\mathcal{P}$. Equivalently,

$$p(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

Using the previous equation, we obtain each b_i's in terms of c_i's and B_i's as follows:

$$\begin{aligned}
b_1 &= \frac{1}{2}B_1c_1,\\
b_2 &= \frac{1}{4}\Big((B_2-B_1)c_1^2 + 2B_1c_2\Big),\\
b_3 &= \frac{1}{8}\Big((B_1-2B_2+B_3)c_1^3 + 4(B_2-B_1)c_1c_2 + 4B_1c_3\Big),\\
b_4 &= \frac{1}{16}\Big((-B_1+3B_2-3B_3+B_4)c_1^4 + 6(B_3-2B_2+B_1)c_1^2c_2\\
&\qquad + 4(B_2-B_1)c_2^2 + 8(B_2-B_1)c_1c_3 + 8B_1c_4\Big).
\end{aligned} \tag{2.6}$$

Thus, from equations (2.5) and (2.6), we get

$$a_2 = \frac{1}{4}B_1c_1, \tag{2.7}$$

$$a_3 = \frac{1}{8}\Big((B_2-B_1)c_1^2 + 2B_1c_2\Big), \tag{2.8}$$

$$a_4 = \frac{1}{64}\Big((2B_1 - B_1^2 - 4B_2 + B_1B_2 + 2B_3)c_1^3 + (2B_1^2 + 8B_2 - 8B_1)c_1c_2 + 8B_1c_3\Big) \tag{2.9}$$

and

$$a_5 = \frac{1}{128}\Big((-2B_1 + B_1^2 + 6B_2 - 2B_1B_2 + B_2^2 - 6B_3 + 2B_4)c_1^4 \tag{2.10}$$
$$+ (12B_1 - 4B_1^2 - 24B_2 + 4B_1B_2)c_1^2c_2 + (4B_1^2 + 8B_2 - 8B_1)c_2^2$$
$$+ (16B_2 - 16B_1)c_1c_3 + 16B_1c_4\Big).$$

Since $f \in \mathcal{S}^*_{s,L}$, therefore

$$\varphi(z) = \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{128}z^4 + \cdots .$$

Thus $B_1 = 1/2$, $B_2 = -1/8$, $B_3 = 1/16$ and $B_4 = -5/128$. On substituting these values in (2.7), (2.8), (2.9) and (2.10) as in the previous theorem, we get

$$a_2 = \frac{1}{8}c_1,$$
$$a_3 = \frac{1}{64}(-5c_1^2 + 8c_2),$$
$$a_4 = \frac{1}{1024}(21c_1^3 - 72c_1c_2 + 64c_3),$$
$$a_5 = \frac{1}{8192}(-116c_1^4 + 544c_1^2c_2 - 256c_2^2 - 640c_1c_3 + 512c_4).$$

Since $|c_n| \leq 2$ for $n \geq 1$, we have $|a_2| \leq 1/4$. By using Lemma 2.1(a) we obtain $|a_3| \leq 1/4$. Since

$$a_4 = \frac{1}{16}(c_3 - 2\beta c_1c_2 + \delta c_1^3)$$

where $\beta = 9/16$ and $\delta = 21/64$, Lemma 2.1(b) shows that $|a_4| \leq 1/8$. Similarly,

$$a_5 = \frac{1}{16}(\gamma c_1^4 + ac_2^2 + 2\alpha c_1c_3 - (3/2)\beta c_1^2c_2 - c_4),$$

where $\gamma = 29/128$, $a = 1/2$, $\alpha = 5/8$, $\beta = 17/24$. Lemma 2.1(c) shows that $|a_5| \leq 1/8$. Define the functions f_k $(k = 1, 2, \cdots)$ by

$$\frac{2zf_k'(z)}{f_k(z) - f_k(-z)} = \sqrt{1+z^k} = 1 + \frac{z^k}{2} - \frac{z^{2k}}{8} + \frac{z^{3k}}{16} + \cdots , \quad (f_k(0) = 0, f_k'(0) = 1).$$

Then

$$f_1(z) = z + \frac{1}{4}z^2 + \cdots , \qquad f_2(z) = z + \frac{1}{4}z^3 + \cdots ,$$

$$f_3(z) = z + \frac{1}{8}z^4 + \cdots, \text{ and } \quad f_4(z) = z + \frac{1}{8}z^5 + \cdots.$$

Clearly the functions $f_k \in \mathcal{S}^*_{s,L}$. Moreover the kth coefficient is sharp for f_{k-1} where $k = 2, 3, 4, 5$. ∎

THEOREM 2.5. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and $f \in \mathcal{S}^*_{s,RL}$. Then*

$$|a_2| \leq \frac{5}{4} - \frac{3}{2\sqrt{2}}, \quad |a_3| \leq \frac{5}{4} - \frac{3}{2\sqrt{2}}, \quad |a_4| \leq \frac{5}{8} - \frac{3}{4\sqrt{2}} \quad \text{and} \quad |a_5| \leq \frac{5}{8} - \frac{3}{4\sqrt{2}}.$$

All the bounds here are sharp.

PROOF. Proceeding as in the proof of the previous theorem, the expressions (2.7), (2.8), (2.9) and (2.10) for the coefficients a_2-a_5 can be obtained. Now let $f \in \mathcal{S}^*_{s,RL}$. Then,

$$\begin{aligned}\varphi(z) &= \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}} \\ &= 1 + \frac{5-3\sqrt{2}}{2}z + \frac{71-51\sqrt{2}}{8}z^2 + \frac{589-415\sqrt{2}}{16}z^3 + \frac{20043-14179\sqrt{2}}{128}z^4 + \cdots.\end{aligned}$$

Thus $B_1 = (5-3\sqrt{2})/2$, $B_2 = (71-51\sqrt{2})/8$, $B_3 = (589-415\sqrt{2})/16$ and $B_4 = (20043-14179\sqrt{2})/128$. Using these values in (2.7), (2.8), (2.9) and (2.10), we get

$$\begin{aligned}a_2 &= \frac{1}{8}(-1+\sqrt{2})(-c_1+2\sqrt{2}c_1), \\ a_3 &= \frac{1}{64}(-1+\sqrt{2})\left((-27+12\sqrt{2})c_1^2 + (-8+16\sqrt{2})c_2\right), \\ a_4 &= \frac{1}{1024}\left((1179-818\sqrt{2})c_1^3 + 8(145-108\sqrt{2})c_1c_2 + 64(5-3\sqrt{2})c_3\right)\end{aligned}$$

and

$$\begin{aligned}a_5 = \frac{1}{8192}\Big(&(14638-10453\sqrt{2})c_1^4 - 48(-508+351\sqrt{2})c_1^2c_2 \\ &- 64(-94+69\sqrt{2})c_2^2 - 384(-17+13\sqrt{2})c_1c_3 - 512(-5+3\sqrt{2})c_4\Big).\end{aligned}$$

Since $|c_n| \leq 2$ for $n \geq 1$, we have

$$|a_2| \leq \frac{5}{4} - \frac{3}{2\sqrt{2}}.$$

Use of Lemma 2.1(a) shows that

$$|a_3| \leq \frac{5}{4} - \frac{3}{2\sqrt{2}}.$$

Since

$$a_4 = \frac{5-3\sqrt{2}}{16}\left(c_3 - 2\beta c_1 c_2 + \delta c_1^3\right)$$

where

$$\beta = (108\sqrt{2} - 145)/(16(5-3\sqrt{2})), \quad \delta = (1179 - 818\sqrt{2})/(64(5-3\sqrt{2})).$$

Lemma 2.1(b) shows that

$$|a_4| \leq \frac{5}{8} - \frac{3}{4\sqrt{2}}.$$

Similarly, a_5 can be rewritten as

$$a_5 = \frac{-5+3\sqrt{2}}{8}\left(\gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - (3/2)\beta c_1^2 c_2 - c_4\right)$$

where

$$\gamma = (14638 - 10453\sqrt{2})/(512(-5+3\sqrt{2})), \quad a = (94 - 69\sqrt{2})/(8(-5+3\sqrt{2})),$$
$$\alpha = (3(17 - 3\sqrt{2}))/(8(-5+3\sqrt{2})) \quad \text{and} \quad \beta = (-508 + 351\sqrt{2})/(16(-5+3\sqrt{2})).$$

Lemma 2.1(c) shows that

$$|a_5| \leq \frac{5}{8} - \frac{3}{4\sqrt{2}}.$$

Define the functions f_k $(k = 1, 2, \cdots)$ by

$$\frac{2zf_k'(z)}{f_k(z) - f_k(-z)} = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z^k}{1+2(\sqrt{2}-1)z^k}} \quad (f_k(0) = 0, f_k'(0) = 1).$$

Then

$$f_1(z) = z + \frac{5-3\sqrt{2}}{4}z^2 + \cdots, \qquad f_2(z) = z + \frac{5-3\sqrt{2}}{4}z^3 + \cdots,$$

$$f_3(z) = z + \frac{5-3\sqrt{2}}{8}z^4 + \cdots, \text{ and } \quad f_4(z) = z + \frac{5-3\sqrt{2}}{8}z^5 + \cdots.$$

Clearly, the functions $f_k \in \mathcal{S}^*_{s,RL}$ and the kth coefficient is sharp for f_{k-1} ($k = 2, 3, 4, 5$) ▮

THEOREM 2.6. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and $f \in \mathcal{S}^*_{s,e}$. Then*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{2}, \quad |a_4| \leq \frac{1}{4} \quad and \quad |a_5| \leq \frac{1}{4}.$$

All the bounds here are sharp.

PROOF. Proceeding as in the proof of the Theorem 2.4, the expressions for the coefficients a_2-a_5 can be easily obtained. Next let $f \in \mathcal{S}^*_{s,e}$. Then,

$$q(z) = e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots.$$

Thus, $B_1 = 1$, $B_2 = 1/2$, $B_3 = 1/6$ and $B_4 = 1/24$. Using these values in (2.7), (2.8), (2.9) and (2.10), we get

$$\begin{aligned} a_2 &= \frac{1}{4}c_1, \\ a_3 &= \frac{1}{16}(-c_1^2 + 4c_2), \\ a_4 &= \frac{1}{384}(-c_1^3 - 12c_1c_2 + 48c_3) \end{aligned}$$

and

$$a_5 = \frac{1}{384}(c_1^4 - 24c_1^2c_3 + 48c_4).$$

Since $|c_n| \leq 2$ for $n \geq 1$, therefore $|a_2| \leq 1/2$. Use of Lemma 2.1(a) shows that $|a_3| \leq 1/2$. Since

$$a_4 = \frac{1}{8}(c_3 - 2\beta c_1 c_2 + \delta c_1^3)$$

where $\beta = 1/8$ and $\delta = -1/48$, Lemma 2.1(b) shows that $|a_4| \leq 1/4$. Similarly,

$$a_5 = \frac{1}{8}(\gamma c_1^4 + ac_2^2 + 2\alpha c_1 c_3 - (3/2)\beta c_1^2 c_2 - c_4)$$

where $\gamma = -1/48$, $a = 0$, $\alpha = 1/4$, $\beta = 0$. Note that Lemma 2.2 holds for $a = 0$. Applying Lemma 2.1(c) with $a = 0$, we have $|a_5| \leq 1/4$. Define the functions f_k $(k = 1, 2, \cdots)$ by

$$\frac{2zf_k'(z)}{f_k(z) - f_k(-z)} = e^{kz} = 1 + z^k + \frac{z^{2k}}{2!} + \frac{z^{3k}}{3!} + \cdots \quad (f_k(0) = 0, f_k'(0) = 1).$$

Then

$$f_1(z) = z + \frac{1}{2}z^2 + \cdots \qquad f_2(z) = z + \frac{1}{2}z^3 + \cdots,$$

$$f_3(z) = z + \frac{1}{4}z^4 + \cdots, \text{ and } \quad f_4(z) = z + \frac{1}{4}z^5 + \cdots.$$

Clearly the functions $f_k \in \mathcal{S}^*_{s,e}$. Clearly the kth coefficient is sharp for f_{k-1} for $k = 2, 3, 4, 5$. ∎

Chapter 3

Fekete-Szegö Coefficient Functional

3.1. Fekete- Szegö Problem

Let φ be a univalent function having real part positive satisfying $\varphi(0)=1$ and $\varphi'(0)>0$. In this chapter, we determine the bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$ for the following subclasses

$$\begin{aligned}
\mathcal{V}_\alpha(\varphi) &:= \Big\{f : \alpha f'(z)+(1-\alpha)\frac{zf'(z)}{f(z)} \prec \varphi(z)\Big\},\\
\mathcal{M}_\alpha(\varphi) &:= \Big\{f : (f'(z))^{\alpha}\Big(\frac{zf'(z)}{f(z)}\Big)^{(1-\alpha)} \prec \varphi(z)\Big\},\\
\mathcal{L}_\alpha(\varphi) &:= \Big\{f : (f'(z))^{\alpha}\Big(1+\frac{zf''(z)}{f'(z)}\Big)^{(1-\alpha)} \prec \varphi(z)\Big\},\\
\mathcal{K}_\alpha(\varphi) &:= \Big\{f : \Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(\frac{zf'(z)}{f(z)}\Big)^{(1-\alpha)} \prec \varphi(z)\Big\}
\end{aligned}$$

and

$$\mathcal{T}_\alpha(\varphi) := \Big\{f : \Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(1+\frac{zf''(z)}{f'(z)}\Big)^{(1-\alpha)} \prec \varphi(z)\Big\}$$

Our results include some previously known results. In order to prove our results, we need the following lemma:

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LEMMA 3.1. [51] *Let* $p \in \mathcal{P}$ *be given by* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. *Then,*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if} \quad \nu \leq 0; \\ 2, & \text{if} \quad 0 \leq \nu \leq 1; \\ 4\nu - 2, & \text{if} \quad \nu \geq 1. \end{cases}$$

Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a function having real part positive with

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \qquad B_1 > 0; B_1, B_2, B_3 \in \mathbb{R}. \tag{3.1}$$

3.2. THE CLASS $\mathcal{V}_\alpha(\varphi)$

Our first theorem gives the Fekete-Szegö inequality for functions in the class $\mathcal{V}_\alpha(\varphi)$.

THEOREM 3.2. *Let* φ *be a function defined as in* (3.1) *and let the function* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{V}_\alpha(\varphi)$. *Then we have the following:*

(1) *If* B_1, B_2 *and* μ *satisfy the condition*

$$(2+\alpha)B_1^2\mu \leq (1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(2+\alpha)}\Big[B_2 + \frac{(1-\alpha)}{(1+\alpha)^2}B_1^2 - \frac{(2+\alpha)\mu}{(1+\alpha)^2}B_1^2\Big].$$

(2) *If* B_1, B_2 *and* μ *satisfy the condition*

$$(1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2 \leq (2+\alpha)B_1^2\mu \leq (1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2+\alpha}.$$

(3) *If* B_1, B_2 *and* μ *satisfy the condition*

$$(1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2 \leq (2+\alpha)B_1^2\mu,$$

then

$$|a_3-\mu a_2^2| \le \frac{1}{2+\alpha}\Big[-B_2-\frac{1-\alpha}{(1+\alpha)^2}B_1^2+\frac{(2+\alpha)\mu}{(1+\alpha)^2}B_1^2\Big].$$

PROOF. Since $f\in\mathcal{V}_\alpha(\varphi)$, there is an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$, such that

$$\alpha f'(z)+(1-\alpha)\frac{zf'(z)}{f(z)}=\varphi(w(z)). \tag{3.2}$$

Define $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots.$$

Then

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0)=1$ and $p_1\in\mathcal{P}$. Since $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots. \tag{3.3}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}\alpha f'(z)+(1-\alpha)z\frac{f'(z)}{f(z)}&=1+a_2(1+\alpha)z+\Big((2+\alpha)a_3-(1-\alpha)a_2^2\Big)z^2\\&\quad+\Big((3+\alpha)a_4-(1-\alpha)(3a_2a_3-a_2^3)\Big)z^3+\cdots.\end{aligned} \tag{3.4}$$

Then from (3.2), (3.3) and (3.4), we get

$$a_2=\frac{B_1c_1}{2(1+\alpha)}. \tag{3.5}$$

$$a_3=\frac{1}{4(2+\alpha)}\left[2B_1c_2+\Big(\frac{(1-\alpha)}{(1+\alpha)^2}B_1^2+(B_2-B_1)\Big)c_1^2\right]. \tag{3.6}$$

Equations (3.5) and (3.6) yield

$$a_3-\mu a_2^2=\frac{1}{(2+\alpha)}\left[\frac{c_2B_1}{2}-\frac{c_1^2B_1}{4}+\frac{c_1^2B_2}{4}+\frac{(1-\alpha)c_1^2B_1^2}{4(1+\alpha)^2}-\frac{\mu B_1^2c_1^2}{4(1+\alpha)^2}\right]$$

$$= \frac{B_1}{2(2+\alpha)}(c_2 - \nu c_1^2),$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{(1-\alpha)B_1}{(1+\alpha)^2} + \frac{(2+\alpha)B_1}{(1+\alpha)^2}\mu\right).$$

Finally, by using Lemma 3.1, we get the desired result. ∎

REMARK 3.3. Bounds for the second and the third coefficient for f in $\mathcal{V}_\alpha(\varphi)$ can be directly obtained from Theorem 3.2 and is given as follows:

$$|a_2| \leq \frac{B_1}{1+\alpha},$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_2 + \frac{1-\alpha}{(1+\alpha)^2}B_1^2}{2+\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2 \geq 0; \\ \dfrac{B_1}{2+\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2 \leq 0 \quad \text{or} \\ & (1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2 \geq 0; \\ \dfrac{-B_2 - \frac{1-\alpha}{(1+\alpha)^2}B_1^2}{2+\alpha}, & (1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2 \leq 0. \end{cases}$$

3.3. THE CLASS $\mathcal{M}_\alpha(\varphi)$

Our next theorem gives the Fekete-Szegö inequality for functions in the class $\mathcal{M}_\alpha(\varphi)$.

THEOREM 3.4. *Let φ be defined as in* (3.1) *and let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{M}_\alpha(\varphi)$. Then we have the following:*

(1) If B_1, B_2 and μ satisfy the condition

$$2(2+\alpha)B_1^2\mu \leq (2+\alpha)(1-\alpha)B_1^2 + 2(B_2 - B_1)(1+\alpha)^2,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(2+\alpha)}\left(B_2 + \frac{(2+\alpha)(1-\alpha)}{2(1+\alpha)^2}B_1^2 - \frac{(2+\alpha)\mu}{(1+\alpha)^2}B_1^2\right).$$

(2) If B_1, B_2 and μ satisfy the condition

$$(1-\alpha)(2+\alpha)B_1^2 + 2(B_2 - B_1)(1+\alpha)^2 \leq 2(2+\alpha)B_1^2\mu$$
$$\leq (1-\alpha)(2+\alpha)B_1^2 + 2(B_2 + B_1)(1+\alpha)^2,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2+\alpha}.$$

(3) If B_1, B_2 and μ satisfy the condition

$$(1-\alpha)(2+\alpha)B_1^2 + 2(B_2 + B_1)(1+\alpha)^2 \leq 2(2+\alpha)B_1^2\mu,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{2+\alpha}\Big(- B_2 - \frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1^2 + \frac{(2+\alpha)\mu}{(1+\alpha)^2}B_1^2\Big).$$

PROOF. Since $f \in \mathcal{M}_\alpha(\varphi)$, there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$(f'(z))^\alpha \Big(\frac{zf'(z)}{f(z)}\Big)^{1-\alpha} = \varphi(w(z)). \tag{3.7}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots. \tag{3.8}$$

Also, the Taylor series expansion of f gives

$$(f'(z))^\alpha \Big(\frac{zf'(z)}{f(z)}\Big)^{1-\alpha} = 1 + a_2(1+\alpha)z + \frac{1}{2}((2+\alpha)(2a_3 - (1-\alpha)a_2^2)z^2 \tag{3.9}$$

$$+\frac{1}{6}(3+\alpha)(6a_4-6(1-\alpha)a_2a_3+(1-\alpha)(2-\alpha)a_2^3))z^3+\cdots.$$

Then from (3.7), (3.8) and (3.9), we get

$$a_2=\frac{B_1c_1}{2(1+\alpha)}. \tag{3.10}$$

$$a_3=\frac{1}{2(2+\alpha)}\left[B_1c_2-\Big(\frac{2(1+\alpha)^2(B_1-B_2)-(1-\alpha)(2+\alpha)B_1^2}{4(1+\alpha)^2}\Big)c_1^2\right]. \tag{3.11}$$

Using (3.10) and (3.11) we get,

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{1}{(2+\alpha)}\Big(\frac{c_2B_1}{2}-\frac{c_1^2B_1}{4}+\frac{c_1^2B_2}{4}+\frac{(1-\alpha)(2+\alpha)}{4(1+\alpha)^2}c_1^2B_1^2-\frac{\mu(2+\alpha)B_1^2c_1^2}{4(1+\alpha)^2}\Big)\\ &= \frac{B_1}{2(2+\alpha)}\left(c_2-c_1^2\Big(\frac{1}{2}\Big(1-\frac{B_2}{B_1}\Big)-\frac{(1-\alpha)(2+\alpha)}{4(1+\alpha)^2}B_1+\frac{(2+\alpha)}{2(1+\alpha)^2}\mu B_1\Big)\right)\\ &= \frac{B_1}{2(2+\alpha)}(c_2-\nu c_1^2), \end{aligned}$$

where $\nu=\frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1+\frac{\mu(2+\alpha)}{(1+\alpha)^2}B_1\right)$.

Using Lemma 3.1, we get the desired result. ∎

REMARK 3.5. Bounds for the second and the third coefficients for f can be directly obtained from Theorem 3.4 and are given below:

$$|a_2|\le\frac{B_1}{(1+\alpha)},\quad \text{and}$$

$$|a_3|\le\begin{cases}\dfrac{B_2+\frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1^2}{2+\alpha}, & (1-\alpha)(2+\alpha)B_1^2+2(B_2-B_1)(1+\alpha)^2\ge 0;\\ \dfrac{B_1}{2+\alpha}, & (1-\alpha)(2+\alpha)B_1^2+2(B_2-B_1)(1+\alpha)^2\le 0 \quad \text{or}\\ & (1-\alpha)(2+\alpha)B_1^2+2(B_2+B_1)(1+\alpha)^2\ge 0;\\ \dfrac{-B_2-\frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1^2}{2+\alpha}, & (1-\alpha)(2+\alpha)B_1^2+2(B_2+B_1)(1+\alpha)^2\le 0.\end{cases}$$

3.4. THE CLASS $\mathcal{L}_\alpha(\varphi)$

Our next theorem gives the Fekete-Szegö inequality for functions in the class $\mathcal{L}_\alpha(\varphi)$.

THEOREM 3.6. *Let φ be defined as in equation* (3.1) *and let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{L}_\alpha(\varphi)$. Then we have the following:*

(1) *If B_1, B_2 and μ satisfy the condition*

$$3(2-\alpha)B_1^2\mu \leq 4(1-\alpha)B_1^2 + 4(B_2 - B_1),$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{3(2-\alpha)}\Big(B_2 + (1-\alpha)B_1^2 - \frac{3(2-\alpha)\mu}{4}B_1^2\Big).$$

(2) *If B_1, B_2 and μ satisfy the condition*

$$4(1-\alpha)B_1^2 + 4(B_2 - B_1) \leq 3(2-\alpha)B_1^2\mu \leq 4(1-\alpha)B_1^2 + 4(B_2 + B_1),$$

then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{3(2+\alpha)}.$$

(3) *If B_1, B_2 and μ satisfy the condition*

$$4(1-\alpha)B_1^2 + 4(B_2 + B_1) \leq 3(2-\alpha)B_1^2\mu,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{3(2+\alpha)}\Big(-B_2 - (1-\alpha)B_1^2 + \frac{3(2-\alpha)\mu}{4}B_1^2\Big).$$

PROOF. Since $f \in \mathcal{L}_\alpha(\varphi)$, there is an analytic function $w(z) = w_1z + w_2z^2 + \cdots \in \Omega$, such that

$$(f'(z))^\alpha\Big(1 + \frac{zf''(z)}{f'(z)}\Big)^{1-\alpha} = \varphi(w(z)). \tag{3.12}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots,$$

which then implies

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0)=1$ and $p_1\in\mathcal{P}$. Then, since $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots. \tag{3.13}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}(f'(z))^{\alpha}\Big(1+\frac{zf''(z)}{f'(z)}\Big)^{1-\alpha}&=1+2a_2z+(3(2-\alpha)a_3-4(1-\alpha)a_2^2)z^2 \qquad (3.14)\\ &\quad+(4(3-2\alpha)a_4-18(1-\alpha)a_2a_3+8(1-\alpha)a_2^3)z^3+\cdots.\end{aligned}$$

Then from (3.12), (3.13) and (3.14), we get

$$a_2=\frac{B_1c_1}{4}. \tag{3.15}$$

$$a_3=\frac{1}{6(2+\alpha)}\left[B_1c_2-\Big(\frac{(B_1-B_2)-(1-\alpha)B_1^2}{2}\Big)c_1^2\right]. \tag{3.16}$$

Using (3.15) and (3.16) we get,

$$\begin{aligned}a_3-\mu a_2^2&=\frac{1}{(2-\alpha)}\left(\frac{c_2B_1}{6}-\frac{c_1^2B_1}{12}+\frac{c_1^2B_2}{12}+\frac{(1-\alpha)}{12}c_1^2B_1^2-\frac{\mu(2-\alpha)B_1^2c_1^2}{16}\right)\\ &=\frac{B_1}{6(2-\alpha)}\left(c_2-\frac{c_1^2}{2}\Big(\Big(1-\frac{B_2}{B_1}\Big)-(1-\alpha)B_1+\frac{3(2-\alpha)}{4}\mu B_1\Big)\right)\\ &=\frac{B_1}{6(2+\alpha)}(c_2-\nu c_1^2),\end{aligned}$$

where $\nu=\dfrac{1}{2}\left(1-\dfrac{B_2}{B_1}-(1-\alpha)B_1+\dfrac{3\mu(2-\alpha)}{4}B_1\right)$. Using Lemma 3.1 we get the desired result. ∎

REMARK 3.7. Bounds for the second and the third coefficients for f can be directly obtained from Theorem 3.6 and are given as follows:

$$|a_2|\leq\frac{B_1}{2},$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_2 + (1-\alpha)B_1^2}{3(2-\alpha)}, & (1-\alpha)B_1^2 + (B_2 - B_1) \geq 0; \\ \dfrac{B_1}{3(2-\alpha)}, & (1-\alpha)B_1^2 + (B_2 - B_1) \leq 0 \quad \text{or} \\ & (1-\alpha)B_1^2 + (B_2 + B_1) \geq 0; \\ \dfrac{-B_2 - (1-\alpha)B_1^2}{3(2-\alpha)}, & (1-\alpha)B_1^2 + (B_2 + B_1) \leq 0. \end{cases}$$

3.5. THE CLASS $\mathcal{K}_\alpha(\varphi)$

Our next theorem gives the Fekete-Szegö inequality for functions in the class $\mathcal{K}_\alpha(\varphi)$.

THEOREM 3.8. *Let φ be defined as in equation* (3.1) *and let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{K}_\alpha(\varphi)$. Then we have the following:*

(1) *If B_1, B_2 and μ satisfy the condition*

$$(2-\alpha)B_1^2\mu \leq (1-\alpha)B_1^2 + (B_2 - B_1),$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(2-\alpha)}\Big(B_2 + (1-\alpha)B_1^2 - (2-\alpha)\mu B_1^2\Big).$$

(2) *If B_1, B_2 and μ satisfy the condition*

$$(1-\alpha)B_1^2 + (B_2 - B_1) \leq (2-\alpha)B_1^2\mu \leq (1-\alpha)B_1^2 + (B_2 + B_1),$$

then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{(2-\alpha)}.$$

(3) *If B_1, B_2 and μ satisfy the condition*

$$(1-\alpha)B_1^2 + (B_2 + B_1) \leq (2-\alpha)B_1^2\mu,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(2-\alpha)}\Big(-B_2 - (1-\alpha)B_1^2 + (2-\alpha)\mu B_1^2\Big).$$

PROOF. Since $f \in \mathcal{K}_\alpha(\varphi)$, there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$\Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(\frac{zf'(z)}{f(z)}\Big)^{1-\alpha} = \varphi(w(z)). \tag{3.17}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots. \tag{3.18}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}\Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(\frac{zf'(z)}{f(z)}\Big)^{1-\alpha} &= 1 + a_2 z + ((2-\alpha)a_3 - (1-\alpha)a_2^2)z^2 \\ &\quad + ((3-2\alpha)a_4 - 3(1-\alpha)a_2 a_3 + (1-\alpha)a_2^3)z^3 + \cdots.\end{aligned} \tag{3.19}$$

Then from (3.17), (3.18) and (3.19), we get

$$a_2 = \frac{B_1 c_1}{2}. \tag{3.20}$$

$$a_3 = \frac{1}{4(2-\alpha)}\left[2B_1 c_2 - \Big((B_1 - B_2) - (1-\alpha)B_1^2\Big)c_1^2\right]. \tag{3.21}$$

Using (3.20) and (3.21) we get,

$$\begin{aligned}a_3 - \mu a_2^2 &= \frac{1}{4(2-\alpha)}\Big(2c_2 B_1 - c_1^2 B_1 + c_1^2 B_2 + (1-\alpha)c_1^2 B_1^2 - \mu(2-\alpha)B_1^2 c_1^2\Big) \\ &= \frac{B_1}{2(2-\alpha)}\left(c_2 - \frac{c_1^2}{2}\Big(\Big(1 - \frac{B_2}{B_1}\Big) - (1-\alpha)B_1 + (2-\alpha)\mu B_1\Big)\right) \\ &= \frac{B_1}{2(2-\alpha)}(c_2 - \nu c_1^2),\end{aligned}$$

where $\nu = \frac{1}{2}\left(\frac{-B_2}{B_1} + 1 - (1-\alpha)B_1 + \mu(2-\alpha)B_1\right)$. Using Lemma 3.1 we get the desired result. ∎

REMARK 3.9. Bounds for the second and the third coefficients for $f \in \mathcal{K}_\alpha(\varphi)$ can be directly obtained from Theorem 3.8 as follows:

$$|a_2| \leq B_1,$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_2 + (1-\alpha)B_1^2}{2-\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1) \geq 0; \\ \dfrac{B_1}{2-\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1) \leq 0 \quad \text{or} \\ & (1-\alpha)B_1^2 + (B_2 + B_1) \geq 0; \\ \dfrac{-B_2 - (1-\alpha)B_1^2}{2-\alpha}, & (1-\alpha)B_1^2 + (B_2 + B_1) \leq 0. \end{cases}$$

3.6. THE CLASS $\mathcal{T}_\alpha(\varphi)$

Our next theorem gives the Fekete-Szegö inequality for functions in the class $\mathcal{T}_\alpha(\varphi)$.

THEOREM 3.10. *Let φ be defined as in* (3.1) *and let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{T}_\alpha(\varphi)$. Then we have the following:*

(1) If B_1, B_2 and μ satisfy the condition

$$2(6-5\alpha)B_1^2\mu \leq (1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 - B_1),$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{2(6-5\alpha)}\left(2B_2 + \frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1^2 - \frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1^2\right).$$

(2) If B_1, B_2 and μ satisfy the condition

$$(1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 - B_1) \leq 2(6-5\alpha)B_1^2\mu$$
$$\leq (1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 + B_1),$$

then

$$|a_3-\mu a_2^2|\leq \frac{B_1}{6-5\alpha}.$$

(3) If B_1, B_2 and μ satisfy the condition

$$(1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2+B_1)\leq 2(6-5\alpha)B_1^2\mu,$$

then

$$|a_3-\mu a_2^2|\leq \frac{1}{2(6-5\alpha)}\Big(-2B_2-\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1^2+\frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1^2\Big).$$

PROOF. Since $f\in\mathcal{T}_\alpha(\varphi)$, there is an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$, such that

$$\Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(1+\frac{zf''(z)}{f'(z)}\Big)^{1-\alpha}=\varphi(w(z)). \tag{3.22}$$

Define $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots,$$

then this implies

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0)=1$ and $p_1\in\mathcal{P}$. Then, since $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots. \tag{3.23}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}\Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(1+\frac{zf''(z)}{f'(z)}\Big)^{1-\alpha}&=1+(2-\alpha)a_2z+\big((6-5\alpha)a_3-\frac{1}{2}(1-\alpha)(8+\alpha)a_2^2\big)z^2\\&\quad+\big((12-11\alpha)a_4-(1-\alpha)(18+5\alpha)a_2a_3\\&\quad+\frac{1}{6}(1-\alpha)(\alpha^2+28\alpha+48)a_2^3\big)z^3+\cdots.\end{aligned} \tag{3.24}$$

Then from (3.22), (3.23) and (3.24), we get

$$a_2 = \frac{B_1 c_1}{2(2-\alpha)}. \tag{3.25}$$

$$a_3 = \frac{1}{8(6-5\alpha)}\left[4B_1c_2 - \Big(\frac{2(B_1-B_2)(2-\alpha)^2-(1-\alpha)(8+\alpha)B_1^2}{(2-\alpha)^2}\Big)c_1^2\right]. \tag{3.26}$$

Using (3.25) and (3.26) we get,

$$\begin{aligned}
a_3-\mu a_2^2 &= \frac{1}{8(6-5\alpha)}\Big(4c_2B_1-2c_1^2B_1+2c_1^2B_2+\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}c_1^2B_1^2 \\
&\quad -\mu\frac{2(6-5\alpha)}{(2-\alpha)^2}B_1^2c_1^2\Big) \\
&= \frac{B_1}{2(6-5\alpha)}\left(c_2-\frac{c_1^2}{4}\Big(2\Big(1-\frac{B_2}{B_1}\Big)-\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1+\frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1\Big)\right) \\
&= \frac{B_1}{2(6-5\alpha)}(c_2-\nu c_1^2),
\end{aligned}$$

where $\nu = \frac{1}{4}\left[2\Big(1-\frac{B_2}{B_1}\Big)-\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1+\frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1\right]$. Using Lemma 3.1 we get the desired result. ∎

REMARK 3.11. Bounds for the second and the third coefficients for f can be directly obtained from Theorem 3.10 as follows:

$$|a_2| \le \frac{B_1}{(2-\alpha)}, \quad \text{and}$$

$$|a_3| \le \begin{cases}
\dfrac{B_2+\frac{(1-\alpha)(8+\alpha)}{2(2-\alpha)^2}B_1^2}{6-5\alpha}, & (1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2-B_1)\ge 0; \\
\dfrac{B_1}{(6-5\alpha)}, & (1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2-B_1)\le 0 \quad \text{or} \\
 & (1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2+B_1)\ge 0; \\
\dfrac{-B_2-\frac{(1-\alpha)(8+\alpha)}{2(2-\alpha)^2}B_1^2}{6-5\alpha}, & (1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2+B_1)\le 0.
\end{cases}$$

The classes studied in this chapter have also been explored in the subsequent chapters for various other properties..

Chapter 4

Hankel Determinant of Certain Classes of Analytic Functions

4.1. Hankel Determinants

Recall that the qth Hankel determinant denoted by $H_q(n)$, for $q = 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$ of the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is given by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Let φ be a univalent function with the positive real part, $\varphi(0) = 1$ and $\varphi'(0) > 0$. In this chapter, we determine the bounds on the second Hankel determinant $H_2(2) := a_2a_4 - a_3^2$ for the functions f belonging to five very important subclasses of analytic functions satisfying $\alpha f'(z) + (1-\alpha)zf'(z)/f(z)$, $(f'(z))^\alpha(zf'(z)/f(z))^{(1-\alpha)}$, $(f'(z))^\alpha(1 + zf''(z)/f'(z))^{(1-\alpha)}$, $(f(z)/z)^\alpha(zf'(z)/f(z))^{(1-\alpha)}$ or $(f(z)/z)^\alpha(1+zf''(z)/f'(z))^{(1-\alpha)}$

The contents of this chapter appeared in K. Khatter, S. K. Lee and S. S. Kumar, Coefficient bounds for certain analytic functions, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 1, 455–490. and K. Khatter, V. Ravichandran and S. Sivaprasad Kumar, *Third Hankel Determinant for starlike and convex functions, The Journal of Analysis*, 2017, DOI 10.1007/s41478-017-0037-6 is subordinate to φ. Also, in the later section, we determine the bounds on the third Hankel determinant $H_3(1) := a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$ for the functions f in the classes $\mathcal{M}_\alpha$ and $\mathcal{L}_\alpha$ defined by

$$\mathcal{M}_\alpha := \left\{ f \in \mathcal{S} : \operatorname{Re}\left((f'(z))^\alpha \left(\frac{zf'(z)}{f(z)} \right)^{1-\alpha} \right) > 0 \right\},$$
$$\mathcal{L}_\alpha := \left\{ f \in \mathcal{S} : \operatorname{Re}\left((f'(z))^\alpha \left(1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} \right) > 0 \right\}.$$

Our results include some previously known results.

4.2. Second Hankel Determinant

Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a function with positive real part with

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \qquad B_1 > 0; B_1, B_2, B_3 \in \mathbb{R}. \tag{4.1}$$

4.2.1. The Class $\mathcal{V}_\alpha(\varphi)$. For $0 \le \alpha \le 1$, we define the class $\mathcal{V}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination

$$\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

Note that

$$\mathcal{S}^*(\varphi) = \mathcal{V}_0(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

is the class of Ma-Minda starlike functions and

$$\mathcal{R}(\varphi) = \mathcal{V}_1(\varphi) := \{ f \in \mathcal{A} : f'(z) \prec \varphi(z) \}$$

is a subclass of close-to-convex function. Thus our class provides a continuous passage from a subclass of starlike functions to the subclass of close-to-convex functions when α varies from 0 to 1.

For functions in the class $\mathcal{V}_\alpha(\varphi)$, we have the following estimate for the second Hankel determinant.

THEOREM 4.1. *Let the function $f \in \mathcal{V}_\alpha(\varphi)$ be given by $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$.*

(1) If B_1, B_2 and B_3 satisfy the conditions

$$\alpha(1-\alpha)B_1^2 + 2(1+\alpha)|B_2| \leq (1+\alpha)(\alpha^2+4\alpha+2)B_1,$$

and

$$\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4 - (1+\alpha)^4(3+\alpha)B_2^2 + (1+\alpha)^3(2+\alpha)^2 B_1 B_3 + \alpha(1-\alpha)(1+\alpha)^2 B_1^2 B_2\Big| \leq (1+\alpha)^4(3+\alpha)B_1^2,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(2+\alpha)^2}.$$

(2) If B_1, B_2 and B_3 satisfy the conditions

$$\alpha(1-\alpha)B_1^2 + 2(1+\alpha)|B_2| \geq (1+\alpha)(\alpha^2+4\alpha+2)B_1,$$

and

$$2\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4 - (1+\alpha)^4(3+\alpha)B_2^2 + (1+\alpha)^3(2+\alpha)^2 B_1 B_3 + \alpha(1-\alpha)(1+\alpha)^2 B_1^2 B_2\Big| \geq (1+\alpha)^2[\alpha(1-\alpha)B_1^3 + 2(1+\alpha)|B_2|B_1 + (1+\alpha)(2+\alpha)^2 B_1^2],$$

or the conditions

$$\alpha(1-\alpha)B_1^2 + 2(1+\alpha)|B_2| \leq (1+\alpha)(\alpha^2+4\alpha+2)B_1,$$

and

$$\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4 - (1+\alpha)^4(3+\alpha)B_2^2 + (1+\alpha)^3(\alpha+2)^2 B_1 B_3 + \alpha(1-\alpha)(1+\alpha)^2 B_1^2 B_2\Big| \geq (1+\alpha)^4(3+\alpha)B_1^2,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4 - (1+\alpha)^4$$

$$(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|.$$

(3) If B_1, B_2 and B_3 satisfy the conditions

$$\alpha(1-\alpha)B_1^2+2(1+\alpha)|B_2|\ >(1+\alpha)(\alpha^2+4\alpha+2)B_1,$$

and

$$2\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3$$
$$+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|\leq(1+\alpha)^2[\alpha(1-\alpha)B_1^3+2(1+\alpha)|B_2|B_1$$
$$+(1+\alpha)(2+\alpha)^2B_1^2],$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\leq\frac{B_1^2M}{4(2+\alpha)^2(3+\alpha)N},$$

where,

$$M=4(3+\alpha)\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2$$
$$+(1+\alpha)^3(2+\alpha)^2B_1B_3+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|$$
$$+\Big[2(1+\alpha)|B_2|-(1+\alpha)(\alpha^2+4\alpha+2)B_1+\alpha(1-\alpha)B_1^2\Big]^2$$

and

$$N=\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3$$
$$+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|-\alpha(1-\alpha)(1+\alpha)^2B_1^3-(1+\alpha)^3\big(2B_1|B_2|+B_1^2\big).$$

Theorem 4.1 is proved by expressing the coefficients of the function f in terms of the coefficients of a function with the positive real part. Recall that the class $\mathcal{P}$ of functions with the positive real part consists of all analytic functions $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$ with $\operatorname{Re}p(z)>0$ for $z\in\mathbb{D}$. Let Ω be the class of all analytic functions $w:\mathbb{D}\to\mathbb{D}$ of the form $w(z)=w_1z+w_2z^2+\cdots$ satisfying $w(0)=0$ and $|w(z)|\leq1$. In order to prove our result, we need the following lemmas:

LEMMA 4.2. [17] *If the function*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{4.2}$$

is in $\mathcal{P}$, then the following sharp estimate holds: $|c_n| \leq 2 \quad (n = 1, 2, 3, \cdots)$.

LEMMA 4.3. [27] *If the function given by* (4.2) *is in $\mathcal{P}$, then,*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{4.3}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y, \tag{4.4}$$

for some x, y with $|x| \leq 1$ and $|y| \leq 1$.

LEMMA 4.4. [51] *Let $p \in \mathcal{P}$ be given by* (4.2). *Then,*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0; \\ 2, & \text{if } 0 \leq \nu \leq 1; \\ 4\nu - 2, & \text{if } \nu \geq 1. \end{cases}$$

PROOF OF THEOREM 4.1. Since $f \in \mathcal{V}_\alpha(\varphi)$, there exists an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$\alpha f'(z) + (1 - \alpha)\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{4.5}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots .$$

Then

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots . \tag{4.6}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}\alpha f'(z)+(1-\alpha)z\frac{f'(z)}{f(z)} &= 1+a_2(1+\alpha)z+\Big((2+\alpha)a_3-(1-\alpha)a_2^2\Big)z^2 \\ &\quad+\Big((3+\alpha)a_4-(1-\alpha)(3a_2a_3-a_2^3)\Big)z^3+\cdots.\end{aligned} \tag{4.7}$$

Then from (4.5), (4.6) and (4.7), we get

$$\begin{aligned}a_2 &= \frac{B_1c_1}{2(1+\alpha)}.\\ a_3 &= \frac{1}{4(2+\alpha)}\left[2B_1c_2+\Big(\frac{(1-\alpha)}{(1+\alpha)^2}B_1^2+(B_2-B_1)\Big)c_1^2\right].\\ a_4 &= \frac{1}{8(3+\alpha)}\Big[4B_1c_3+\Big(B_1-\frac{3(1-\alpha)}{(1+\alpha)(2+\alpha)}B_1^2+\frac{(1-\alpha)(1-4\alpha)}{(1+\alpha)^3(2+\alpha)}B_1^3-2B_2\\ &\quad+\frac{3(1-\alpha)}{(1+\alpha)(2+\alpha)}B_1B_2+B_3\Big)c_1^3+(-4B_1+\frac{6(1-\alpha)}{(1+\alpha)(2+\alpha)}B_1^2+4B_2)c_1c_2\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}a_2a_4-a_3^2 &= \frac{1}{16}B_1\Big[\Big\{\frac{(\alpha-1)(3\alpha^2+5\alpha+1)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3-\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1(B_2-B_1)\\ &\quad-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3+\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(B_1-2B_2)\Big\}c_1^4\\ &\quad+\Big\{\frac{4}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(B_2-B_1)+\frac{2\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2\Big\}c_2c_1^2\\ &\quad-\frac{4}{(2+\alpha)^2}B_1c_2^2+\frac{4}{(1+\alpha)(3+\alpha)}B_1c_1c_3\Big].\end{aligned}$$

Since the function $p(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, without loss of generality, we can assume that $c_1=c>0$. Substituting the values of c_2 and c_3 from (4.3) and (4.4) in the above expression, we get

$$\begin{aligned}|a_2a_4-a_3^2| &= \frac{1}{16}B_1\Big|\Big\{\frac{(\alpha-1)(3\alpha^2+5\alpha+1)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3+\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1B_2\\ &\quad-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3\Big\}c^4+\Big\{\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}B_2\\ &\quad+\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2\Big\}c^2(4-c^2)x-\Big\{\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2\\ &\quad+\frac{4}{(2+\alpha)^2}\Big\}B_1(4-c^2)x^2+\frac{2}{(1+\alpha)(3+\alpha)}B_1c(4-c^2)(1-|x|^2)y\Big|.\end{aligned}$$

Replacing $|x|$ by μ and by making use of the triangle inequality and the fact that $|y| \leq 1$ in the above expression, we get

$$
\begin{aligned}
|a_2a_4 - a_3^2| \leq \frac{1}{16}B_1\Bigg[\Bigg|\Bigg\{ & \frac{(\alpha-1)(3\alpha^2+5\alpha+1)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3 - \frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1} + \frac{1}{(1+\alpha)(3+\alpha)}B_3 \\
& + \frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1B_2\Bigg\}\Bigg|c^4 + \frac{2c}{(1+\alpha)(3+\alpha)}B_1(4-c^2) \\
& + \left\{\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2 + \frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}|B_2|\right\}c^2(4-c^2)\,\mu \\
& + \left\{\frac{c^2}{(1+\alpha)(2+\alpha)^2(3+\alpha)} - \frac{2c}{(1+\alpha)(3+\alpha)} + \frac{4}{(2+\alpha)^2}\right\}B_1(4-c^2)\mu^2\Bigg] \\
= F(c,\mu). & \qquad (4.8)
\end{aligned}
$$

We shall now maximize $F(c, \mu)$ for $(c, \mu) \in [0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (4.8) partially with respect to the parameter μ, we get

$$
\begin{aligned}
\frac{\partial F}{\partial \mu} = \frac{1}{16}B_1\Big[& \frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}c^2(4-c^2)B_1^2 + \frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)} \\
& c^2(4-c^2)|B_2| + 2\mu B_1(4-c^2)\Big(\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2 - \frac{2}{(1+\alpha)(3+\alpha)}c \\
& + \frac{4}{(2+\alpha)^2}\Big)\Big].
\end{aligned}
$$

For $0 < \mu < 1$, and for any fixed $c \in [0, 2]$, we observe that $\partial F/\partial\mu > 0$. Thus $F(c, \mu)$ is an increasing function of μ, and for $c \in [0, 2]$, $F(c, \mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max F(c, \mu) = F(c, 1) = G(c). \qquad (4.9)$$

The equations (4.8) and (4.9), upon a little simplification, yield

$$
\begin{aligned}
G(c) = \frac{1}{16}B_1\Bigg[\Bigg\{\Bigg| & \frac{(\alpha-1)(3\alpha^2+5\alpha+1)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3 - \frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1} + \frac{1}{(1+\alpha)(3+\alpha)}B_3 \\
& + \frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1B_2\Bigg| - \frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2 \\
& - \frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}\big(2|B_2| + B_1\big)\Bigg\}c^4 + 4\Bigg\{\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2 \\
& + \frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}|B_2| - \frac{\alpha^2+4\alpha+2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}B_1\Bigg\}c^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{16}{(2+\alpha)^2}B_1\Big]\\
&=\frac{B_1}{16}(Pc^4+Qc^2+R), \qquad (4.10)
\end{aligned}
$$

where

$$
\begin{aligned}
P=&(1+\alpha)^{-4}(2+\alpha)^{-2}(3+\alpha)^{-1}\bigg(\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^3-(1+\alpha)^4(3+\alpha)\frac{B_2^2}{B_1}\\
&+(1+\alpha)^3(2+\alpha)^2B_3+\alpha(1-\alpha)(1+\alpha)^2B_1B_2\Big|-\alpha(1-\alpha)(1+\alpha)^2B_1^2\\
&-(1+\alpha)^3(2|B_2|+B_1)\bigg), \qquad (4.11)\\
Q=&4(1+\alpha)^{-2}(2+\alpha)^{-2}(3+\alpha)^{-1}\Big(\alpha(1-\alpha)B_1^2+2(1+\alpha)|B_2|\\
&-(1+\alpha)(\alpha^2+4\alpha+2)B_1\Big), \qquad (4.12)
\end{aligned}
$$

and

$$
R=16(2+\alpha)^{-2}B_1. \qquad (4.13)
$$

We know that

$$
\max_{0\le t\le 4}(Pt^2+Qt+R)=\begin{cases} R, & Q\le 0,\ P\le -\frac{Q}{4};\\ 16P+4Q+R, & Q\ge 0,\ P\ge -\frac{Q}{8} \text{ or } Q\le 0,\ P\ge -\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\le -\frac{Q}{8}. \end{cases} \qquad (4.14)
$$

Thus, we have, from (4.10),

$$
|a_2a_4-a_3^2|\le \frac{B_1}{16}\begin{cases} R, & Q\le 0,\ P\le -\frac{Q}{4};\\ 16P+4Q+R, & Q\ge 0,\ P\ge -\frac{Q}{8} \text{ or } Q\le 0,\ P\ge -\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\le -\frac{Q}{8}. \end{cases}
$$

where P, Q, R are given by (4.11), (4.12) and (4.13), respectively. A simple computation will give the results stated in the theorem. ∎

REMARK 4.5. When $\alpha = 1$ and $\varphi = (1+z)/(1-z)$, Theorem 4.1 reduces to [34, Theorem 3.1]. When $\alpha = 0$, Theorem 4.1 reduces to [45, Theorem 1]. When $\varphi(z) = (1+(1-2\gamma)z)/(1-z), (0<\gamma<1)$, $\sqrt{1+z}$, $1+2/\pi^2(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$ and $((1+z)/(1-z))^\beta$, $0<\beta\le 1$, the class $\mathcal{S}^*(\varphi)$ becomes the class $\mathcal{S}^*(\gamma)$ of starlike functions of order γ, the class $\mathcal{S}_L^*$ of lemniscate starlike functions, the class $\mathcal{S}_p^*$ of parabolic starlike functions and the class $\mathcal{S}_\beta^*$ of strongly starlike functions of order β, respectively.

In particular, we get the following corollary:

COROLLARY 4.6. [45, Theorem 1]

(1) If $f \in \mathcal{S}^*(\gamma)$, then $|a_2a_4 - a_3^2| \leq (1-\gamma)^2$.
(2) If $f \in \mathcal{S}_L^*$, then $|a_2a_4 - a_3^2| \leq 1/16 = 0.0625$.
(3) If $f \in \mathcal{S}_p^*$, then $|a_2a_4 - a_3^2| \leq 16/\pi^4 \approx 0.164255$.
(4) If $f \in \mathcal{S}_\beta^*$, then $|a_2a_4 - a_3^2| \leq \beta^2$.

4.2.2. The Class $\mathcal{M}_\alpha(\varphi)$. Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic function given by (4.1). For $0 \leq \alpha \leq 1$, the class $\mathcal{M}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination

$$(f'(z))^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} \prec \varphi(z).$$

We see that $\mathcal{M}_0(\varphi) = \mathcal{S}^*(\varphi)$ is the class of Ma-Minda starlike functions and

$$\mathcal{M}_1(\varphi) = \mathcal{R}(\varphi) := \{f \in \mathcal{A} : f'(z) \prec \varphi(z)\}$$

is a subclass of close-to-convex function. Thus this class also provides a passage from a subclass of starlike functions to the subclass of close-to-convex functions when α varies from 0 to 1. Also, for different functions of φ we get different subclasses of starlike functions as stated earlier.

THEOREM 4.7. *Let the function $f \in \mathcal{M}_\alpha(\varphi)$ be given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then,*

(1) If B_1, B_2 and B_3 satisfy the conditions

$$2|B_2| \leq ((1+\alpha)(3+\alpha) - 1)B_1,$$

and

$$|-(2+\alpha)^2(3+\alpha)B_1^4-12(1+\alpha)^3(3+\alpha)B_2^2+12(1+\alpha)^2(2+\alpha)^2B_1B_3|$$
$$-12(1+\alpha)^3(3+\alpha)B_1^2 \le 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2| \le \frac{B_1^2}{(2+\alpha)^2}.$$

(2) *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2| \ge ((1+\alpha)(3+\alpha)-1)B_1,$$

and

$$|-(1-\alpha)(2+\alpha)^2(3+\alpha)B_1^4-12(1+\alpha)^3(3+\alpha)B_2^2+12(1+\alpha)^2(2+\alpha)^2$$
$$B_1B_3|-12(1+\alpha)^2|B_2|B_1-6(1+\alpha)^2((1+\alpha)(3+\alpha)+1)B_1^2 \ge 0,$$

or the conditions

$$2|B_2| \le ((1+\alpha)(3+\alpha)-1)B_1,$$

and

$$|-(2+\alpha)^2(3+\alpha)B_1^4-12(1+\alpha)^3(3+\alpha)B_2^2+12(1+\alpha)^2(2+\alpha)^2B_1B_3|$$
$$-12(1+\alpha)^3(3+\alpha)B_1^2 \ge 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2| \le \frac{1}{12(1+\alpha)^3(2+\alpha)^2(3+\alpha)}|-12(1+\alpha)^3(3+\alpha)B_2^2$$
$$-(2+\alpha)^2(3+\alpha)B_1^4+12(1+\alpha)^2(2+\alpha)^2B_1B_3|.$$

(3) *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2| > ((1+\alpha)(3+\alpha)-1)B_1,$$

and

$$|-(1-\alpha)(2+\alpha)^2(3+\alpha)B_1^4 - 12(1+\alpha)^3(3+\alpha)B_2^2 + 12(1+\alpha)^2$$
$$(2+\alpha)^2 B_1 B_3| - 12(1+\alpha)^2|B_2|B_1 - 6(1+\alpha)^2((1+\alpha)(3+\alpha)+1)B_1^2 \leq 0,$$

then

$$|a_2 a_4 - a_3^2| \leq \frac{B_1^2 M}{((2+\alpha)^2(3+\alpha))N},$$

where,

$$\begin{aligned} M = |&-(1-\alpha)(2+\alpha)^2(3+\alpha)^2 B_1^4 - 12(1+\alpha)^3(3+\alpha)^2 B_2^2 \\ &+ 12(1+\alpha)^2(2+\alpha)^2(3+\alpha)B_1B_3| - 12(1+\alpha)(2+\alpha)^2 B_1|B_2| \\ &- 12(1+\alpha)B_2^2 - 3(1+\alpha)(2+\alpha)^4 B_1^2, \end{aligned}$$

and

$$N = |-(1-\alpha)(2+\alpha)^2(3+\alpha)B_1^4 - 12(1+\alpha)^3(3+\alpha)B_2^2 + 12(1+\alpha)^2$$
$$(2+\alpha)^2 B_1 B_3| - 12(1+\alpha)^2 B_1^2 - 24(1+\alpha)^2 B_1|B_2|.$$

PROOF. Since $f \in \mathcal{M}_\alpha(\varphi)$, there exists an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$(f'(z))^\alpha \Big(\frac{zf'(z)}{f(z)}\Big)^{1-\alpha} = \varphi(w(z)). \tag{4.15}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

then this implies

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots. \tag{4.16}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}(f'(z))^{\alpha}\Big(\frac{zf'(z)}{f(z)}\Big)^{1-\alpha} &= 1 + a_2(1+\alpha)z + \frac{1}{2}((2+\alpha)(2a_3 - (1-\alpha)a_2^2)z^2 \\ &\quad + \frac{1}{6}(3+\alpha)(6a_4 - 6(1-\alpha)a_2a_3 + (1-\alpha)(2-\alpha)a_2^3))z^3 + \cdots.\end{aligned} \tag{4.17}$$

Then from (4.15), (4.16) and (4.17), we get

$$\begin{aligned}a_2 &= \frac{B_1 c_1}{2(1+\alpha)}.\\ a_3 &= \frac{1}{2(2+\alpha)}\left[B_1 c_2 - \Big(\frac{2(1+\alpha)^2(B_1 - B_2) - (1-\alpha)(2+\alpha)B_1^2}{4(1+\alpha)^2}\Big)c_1^2\right].\\ a_4 &= \frac{1}{2(3+\alpha)}\Big[B_1 c_3 + \Big(\frac{B_1}{4} - \frac{(1-\alpha)(3+\alpha)}{4(1+\alpha)(2+\alpha)}B_1^2 + \frac{(1-\alpha)(1-2\alpha)(3+\alpha)}{24(1+\alpha)^3}B_1^3\\ &\quad - \frac{B_2}{2} + \frac{(1-\alpha)(3+\alpha)}{4(1+\alpha)(2+\alpha)}B_1B_2 + \frac{B_3}{4}\Big)c_1^3 + \Big(-B_1 + \frac{(1-\alpha)(3+\alpha)}{2(1+\alpha)(2+\alpha)}B_1^2\\ &\quad + B_2\Big)c_1c_2\Big].\end{aligned}$$

Thus,

$$\begin{aligned}a_2a_4 - a_3^2 &= \frac{1}{16}B_1\Big[\Big\{\frac{-(1-\alpha)}{12(1+\alpha)^3}B_1^3 - \frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1} + \frac{1}{(1+\alpha)^3(3+\alpha)}B_3\\ &\quad + \frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(B_1 - 2B_2)\Big\}c_1^4 + \frac{4}{(1+\alpha)(2+\alpha)^2(3+\alpha)}\\ &\quad (B_2 - B_1)c_2c_1^2 - \frac{4}{(2+\alpha)^2}B_1c_2^2 + \frac{4}{(1+\alpha)(3+\alpha)}B_1c_1c_3\Big].\end{aligned}$$

Since the function $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ is in the class $\mathcal{P}$ for any $p \in \mathcal{P}$, without loss of generality, we can assume that $c_1 = c > 0$. Substituting the values of c_2 and c_3 from (4.3) and (4.4) in the above expression, we get

$$|a_2a_4 - a_3^2| = \frac{1}{16}B_1\Big|\Big\{\frac{(\alpha-1)}{12(1+\alpha)^3}B_1^3 - \frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1} + \frac{1}{(1+\alpha)(3+\alpha)}B_3\Big\}c^4$$

$$+\left\{\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}B_2\right\}c^2(4-c^2)x-\left\{\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2\right.$$
$$\left.+\frac{4}{(2+\alpha)^2}\right\}B_1(4-c^2)x^2+\frac{2}{(1+\alpha)(3+\alpha)}B_1c(4-c^2)(1-|x|^2)y\Big|.$$

Replacing $|x|$ by μ and by making use of the triangle inequality and the fact that $|y|\leq 1$ in the above expression, we get

$$\begin{aligned}|a_2a_4-a_3^2| &\leq \frac{1}{16}B_1\Bigg[\Bigg|\left\{\frac{(\alpha-1)}{12(1+\alpha)^3}B_1^3-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3\right\}\Bigg|c^4\\ &\quad+\frac{2c}{(1+\alpha)(3+\alpha)}B_1(4-c^2)+\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}|B_2|c^2(4-c^2)\,\mu\\ &\quad+\left\{\frac{c^2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}-\frac{2c}{(1+\alpha)(3+\alpha)}+\frac{4}{(2+\alpha)^2}\right\}\\ &\quad B_1(4-c^2)\mu^2\Bigg]\\ &= F(c,\mu). \end{aligned}\tag{4.18}$$

We shall now maximize $F(c,\mu)$ for (c,μ) in $[0,2]\times[0,1]$. On differentiating $F(c,\mu)$ in (4.18) partially with respect to the parameter μ, we get

$$\frac{\partial F}{\partial\mu}=\frac{1}{16}B_1\Big[\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2(4-c^2)|B_2|+2\mu B_1(4-c^2)$$
$$\Big\{\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2-\frac{2}{(1+\alpha)(3+\alpha)}c+\frac{4}{(2+\alpha)^2}\Big\}\Big].$$

For a fixed $c\in[0,2]$ and for $0<\mu<1$, we observe that $\partial F/\partial\mu>0$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c\in[0,2]$, the function $F(c,\mu)$ has a maximum value at $\mu=1$. Thus, we have

$$\max F(c,\mu)=F(c,1)=G(c). \tag{4.19}$$

The equations (4.18) and (4.19), upon a little simplification, yield

$$G(c)=\frac{B_1}{16}\Big[c^4\Big\{\Big|-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}-\frac{(1-\alpha)}{12(1+\alpha)^3}B_1^3+\frac{1}{(1+\alpha)(3+\alpha)}B_3\Big|$$
$$-\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}\big(2|B_2|+B_1\big)\Big\}+c^2\Big\{\frac{4}{(1+\alpha)(2+\alpha)^2(3+\alpha)}$$

$$\big(B_1+2|B_2|\big)-\frac{4}{(2+\alpha)^2}B_1\Big\}+\frac{16}{(2+\alpha)^2}B_1\Big]$$
$$=\frac{B_1}{16}(Pc^4+Qc^2+R), \tag{4.20}$$

where,

$$P=\frac{1}{12}(1+\alpha)^{-3}(2+\alpha)^{-2}(3+\alpha)^{-1}\Big(\Big|12(1+\alpha)^3(3+\alpha)\frac{-B_2^2}{B_1}-(1-\alpha)(2+\alpha)^2$$
$$(3+\alpha)B_1^3+12(1+\alpha)^2(2+\alpha)^2B_3\Big|-12(1+\alpha)^2\big(B_1+2|B_2|\big)\Big), \tag{4.21}$$
$$Q=4(1+\alpha)^{-1}(2+\alpha)^{-2}(3+\alpha)^{-1}\Big(\big(B_1+2|B_2|\big)-(1+\alpha)(3+\alpha)B_1\Big), \tag{4.22}$$
$$R=16(2+\alpha)^{-2}B_1. \tag{4.23}$$

Thus using (4.14) and (4.20) we get,

$$|a_2a_4-a_3^2|\leq\frac{B_1}{16}\begin{cases}R, & Q\leq 0,\ P\leq-\frac{Q}{4};\\ 16P+4Q+R, & Q\geq 0,\ P\geq-\frac{Q}{8}\text{ or } Q\leq 0\ ,\ P\geq-\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\leq-\frac{Q}{8},\end{cases}$$

where P,Q,R are given by (4.21), (4.22) and (4.23), respectively. ∎

REMARK 4.8. When $\alpha=1$ and $\varphi=(1+z)/(1-z)$, Theorem 4.7 reduces to [34, Theorem 3.1]. When $\alpha=0$, Theorem 4.7 reduces to [45, Theorem 1]. Note that, when $\varphi(z)=(1+(1-2\gamma)z)/(1-z),(0<\gamma<1)$, $\sqrt{1+z}$, $1+2/\pi^2(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$ and $((1+z)/(1-z))^\beta$, $0<\beta\leq 1$, the class $\mathcal{S}^*(\varphi)$ becomes the class $\mathcal{S}^*(\gamma)$ of starlike functions of order γ, the class $\mathcal{S}^*_L$ of lemniscate starlike functions, the class $\mathcal{S}^*_p$ of parabolic starlike functions and the class $\mathcal{S}^*_\beta$ of strongly starlike functions of order β, respectively. Therefore, Corollary 4.6 follows as a particular case.

4.2.3. The Class $\mathcal{L}_\alpha(\varphi)$. Let the analytic function $\varphi:\mathbb{D}\to\mathbb{C}$ be given by (4.1). For $0\leq\alpha\leq 1$, the class $\mathcal{L}_\alpha(\varphi)$ consists of functions $f\in\mathcal{A}$ satisfying the following subordination

$$(f'(z))^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\prec\varphi(z).$$

We see that $\mathcal{L}_0(\varphi) = \mathcal{K}(\varphi)$ is the unified class of Ma-Minda convex functions and

$$\mathcal{L}_1(\varphi) = \mathbb{R}(\varphi) := \{f \in \mathcal{A} : f'(z) \prec \varphi(z)\}$$

is a subclass of close-to-convex function. Thus this class also provides a continuous passage from a subclass of convex functions to the subclass of close-to-convex functions when α varies from 0 to 1.

THEOREM 4.9. *Let the function $f \in \mathcal{L}_\alpha(\varphi)$ be given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then,*

(1) If B_1, B_2 and B_3 satisfy the conditions

$$2|B_2|(9\alpha^2 - 20\alpha + 12) + B_1^2(1-\alpha)(6+5\alpha) \leq (-9\alpha^2 + 4\alpha + 12)B_1,$$

and

$$|(1-\alpha)(2\alpha^2 - 5\alpha - 6)B_1^4 - 8(3-2\alpha)B_2^2 + 9(2-\alpha)^2B_1B_3$$
$$+ (1-\alpha)(6+5\alpha)B_1^2B_2| - 8(3-2\alpha)B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{9(2-\alpha)^2}.$$

(2) If B_1, B_2 and B_3 satisfy the conditions

$$2|B_2|(9\alpha^2 - 20\alpha + 12) + B_1^2(1-\alpha)(6+5\alpha) \geq (-9\alpha^2 + 4\alpha + 12)B_1,$$

and

$$2|(1-\alpha)(2\alpha^2 - 5\alpha - 6)B_1^4 - 8(3-2\alpha)B_2^2 + 9(2-\alpha)^2B_1B_3$$
$$+ (1-\alpha)(6+5\alpha)B_1^2B_2| - 2(9\alpha^2 - 20\alpha + 12)|B_2|B_1 - (1-\alpha)(6+5\alpha)B_1^3$$
$$- 9(2-\alpha)^2B_1^2 \geq 0,$$

or the conditions

$$2|B_2|(9\alpha^2 - 20\alpha + 12) + B_1^2(1-\alpha)(6+5\alpha) \leq (-9\alpha^2 + 4\alpha + 12)B_1,$$

and

$$|(1-\alpha)(2\alpha^2-5\alpha-6)B_1^4-8(3-2\alpha)B_2^2+9(2-\alpha)^2B_1B_3$$
$$+(1-\alpha)(6+5\alpha)B_1^2B_2|-8(3-2\alpha)B_1^2\geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\leq\frac{1}{72(2-\alpha)^2(3-2\alpha)}|-8(3-2\alpha)B_2^2+(1-\alpha)(2\alpha^2-5\alpha-6)B_1^4$$
$$+9(2-\alpha)^2B_1B_3+(1-\alpha)(6+5\alpha)B_1^2B_2|.$$

(3) If B_1, B_2 and B_3 satisfy the conditions

$$2|B_2|(9\alpha^2-20\alpha+12)+B_1^2(1-\alpha)(6+5\alpha)>(-9\alpha^2+4\alpha+12)B_1,$$

and

$$2|(1-\alpha)(2\alpha^2-5\alpha-6)B_1^4-8(3-2\alpha)B_2^2+9(2-\alpha)^2B_1B_3$$
$$+(1-\alpha)(6+5\alpha)B_1^2B_2|-2(9\alpha^2-20\alpha+12)|B_2|B_1-(1-\alpha)(6+5\alpha)B_1^3$$
$$-9(2-\alpha)^2B_1^2\leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\leq\frac{B_1^2M}{(1152(2-\alpha)^2(3-2\alpha))N},$$

where,

$$M=4(|32(1-\alpha)(3-2\alpha)(2\alpha^2-5\alpha-6)B_1^4-256(3-2\alpha)^2B_2^2$$
$$+288(2-\alpha)^2(3-2\alpha)B_1B_3+32(1-\alpha)(6+5\alpha)(3-2\alpha)B_1^2B_2|$$
$$-18(1-\alpha)(2-\alpha)^2(6+5\alpha)B_1^3-36(2-\alpha)^2(12-20\alpha+9\alpha^2)B_1|B_2|$$
$$-81(2-\alpha)^4B_1^2-(1-\alpha)^2(6+5\alpha)^2B_1^4-4(12-20\alpha+9\alpha^2)^2B_2^2$$
$$-4(1-\alpha)(6+5\alpha)(12-20\alpha+9\alpha^2)B_1^2|B_2|)$$

and

$$N=|(1-\alpha)(2\alpha^2-5\alpha-6)B_1^4-8(3-2\alpha)B_2^2+9(2-\alpha)^2B_1B_3$$

$$+ (1-\alpha)(6+5\alpha)B_1^2 B_2| - (1-\alpha)(6+5\alpha)B_1^3 - 2(9\alpha^2 - 20\alpha + 12)B_1|B_2|$$
$$- (9\alpha^2 - 20\alpha + 12)B_1^2.$$

PROOF. Since $f \in \mathcal{L}_\alpha(\varphi)$, there exists an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$(f'(z))^\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = \varphi(w(z)). \tag{4.24}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

which then implies

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots. \tag{4.25}$$

Also, the Taylor series expansion of f gives

$$(f'(z))^\alpha \Big(1 + \frac{zf''(z)}{f'(z)}\Big)^{1-\alpha} = 1 + 2a_2 z + (3(2-\alpha)a_3 - 4(1-\alpha)a_2^2)z^2 \tag{4.26}$$
$$+ (4(3-2\alpha)a_4 - 18(1-\alpha)a_2 a_3 + 8(1-\alpha)a_2^3)z^3 + \cdots.$$

Then from (4.24), (4.25) and (4.26), we get

$$a_2 = \frac{B_1 c_1}{4}.$$
$$a_3 = \frac{1}{6(2+\alpha)}\left[B_1 c_2 - \Big(\frac{(B_1 - B_2) - (1-\alpha)B_1^2}{2}\Big)c_1^2\right].$$
$$a_4 = \frac{1}{8(3-2\alpha)}\Big[B_1 c_3 + \Big(\frac{B_1}{4} - \frac{3(1-\alpha)}{4(2-\alpha)}B_1^2 + \frac{(1-\alpha)(1-2\alpha)}{4(2-\alpha)}B_1^3 - \frac{B_2}{2}$$
$$+ \frac{3(1-\alpha)}{4(2-\alpha)}B_1 B_2 + \frac{B_3}{4}\Big)c_1^3 + \Big(-B_1 + \frac{3(1-\alpha)}{2(2-\alpha)}B_1^2 + B_2\Big)c_1 c_2\Big].$$

Thus,

$$\begin{aligned}a_2a_4-a_3^2=&\frac{1}{1152}B_1\Big[\Big\{\frac{(1-\alpha)(2\alpha^2-5\alpha-6)}{(2-\alpha)^2(3-2\alpha)}B_1^3-\frac{8}{(2-\alpha)^2}\frac{B_2^2}{B_1}+\frac{9}{(3-2\alpha)}B_3\\&+\frac{(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}(B_1-2B_2)+\frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1(B_2-B_1)\Big\}c_1^4\\&+\Big\{\frac{4(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}(B_2-B_1)+\frac{2(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2\Big\}c_2c_1^2\\&-\frac{32}{(2-\alpha)^2}B_1c_2^2+\frac{36}{(3-2\alpha)}B_1c_1c_3\Big].\end{aligned}$$

Since the function $p(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, without loss of generality, we can assume that $c_1=c>0$. Substituting the values of c_2 and c_3 from (4.3) and (4.4) in the above expression, we get

$$\begin{aligned}|a_2a_4-a_3^2|=&\frac{1}{1152}B_1\Big|\Big\{\frac{(1-\alpha)(2\alpha^2-5\alpha-6)}{(2-\alpha)^2(3-2\alpha)}B_1^3+\frac{(1-\alpha)(5\alpha+6)}{(2-\alpha)^2(3-2\alpha)}B_1B_2\\&-\frac{8}{(2-\alpha)^2}\frac{B_2^2}{B_1}+\frac{8}{(3-2\alpha)}B_3\Big\}c^4+\Big\{\frac{2(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}B_2\\&+\frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2\Big\}c^2(4-c^2)x-\Big\{\frac{(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}c^2\\&+\frac{32}{(2-\alpha)^2}\Big\}B_1(4-c^2)x^2+\frac{18}{(3-2\alpha)}B_1c(4-c^2)(1-|x|^2)y\Big|.\end{aligned}$$

Replacing $|x|$ by μ and by making use of the triangle inequality and the fact that $|y|\le 1$ in the above expression, we get

$$\begin{aligned}|a_2a_4-a_3^2|\le&\frac{1}{1152}B_1\Big[\Big|\Big\{\frac{(1-\alpha)(2\alpha^2-5\alpha-6)}{(2-\alpha)^2(3-2\alpha)}B_1^3-\frac{8}{(2-\alpha)^2}\frac{B_2^2}{B_1}+\frac{8}{(3-2\alpha)}B_3\\&+\frac{(1-\alpha)(5\alpha+6)}{(2-\alpha)^2(3-2\alpha)}B_1B_2\Big\}\Big|c^4+\frac{18c}{(3-2\alpha)}B_1(4-c^2)\\&+\Big\{\frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2+\frac{2(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}|B_2|\Big\}c^2(4-c^2)\,\mu\\&+\Big\{\frac{(9\alpha^2-20\alpha+12)c^2}{(2-\alpha)^2(3-2\alpha)}-\frac{18c}{(3-2\alpha)}+\frac{32}{(2-\alpha)^2}\Big\}B_1(4-c^2)\mu^2\Big]\\=&F(c,\mu).\end{aligned}\tag{4.27}$$

We shall now maximize $F(c,\mu)$ for (c,μ) in $[0,2]\times[0,1]$. On differentiating the function $F(c,\mu)$ in (4.27) partially with respect to the parameter μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{1}{1152}B_1\Big[\frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}c^2(4-c^2)B_1^2 + \frac{2(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}c^2(4-c^2)|B_2| + 2\mu B_1(4-c^2)\Big(\frac{(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}c^2 - \frac{18}{(3-2\alpha)}c + \frac{32}{(2-\alpha)^2}\Big)\Big]. \tag{4.28}$$

For any fixed $c\in[0,2]$ and $0<\mu<1$, we observe that $\partial F/\partial\mu > 0$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c\in[0,2]$, the function $F(c,\mu)$ has a maximum value at $\mu=1$. Thus, we have

$$\max F(c,\mu) = F(c,1) = G(c). \tag{4.29}$$

The equations (4.27) and (4.29), upon a little simplification, yield

$$\begin{aligned} G(c) = \frac{B_1}{1152}\Big[&c^4\Big\{\Big| -\frac{8}{(2-\alpha)^2}\frac{B_2}{B_1} + \frac{(1-\alpha)(2\alpha^2-5\alpha-6)}{(2-\alpha)^2(3-2\alpha)}B_1^3 + \frac{9}{(3-2\alpha)}B_3 \\ &+ \frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1B_2\Big| - \frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2 - \frac{(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)} \\ &(B_1+2|B_2|)\Big\} + 4c^2\Big\{\frac{2(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}|B_2| + \frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2 \\ &+ \frac{(9\alpha^2-4\alpha-12)}{(2-\alpha)^2(3-2\alpha)}B_1\Big\} + \frac{128}{(2-\alpha)^2}B_1\Big] \\ = \frac{B_1}{1152}&(Pc^4+Qc^2+R), \end{aligned} \tag{4.30}$$

where,

$$\begin{aligned} P = (2-\alpha)^{-2}(3-2\alpha)^{-1}\Big(&\Big| -8(3-2\alpha)\frac{B_2^2}{B_1} + (1-\alpha)(2\alpha^2-5\alpha-6)B_1^3 + 9(2-\alpha)^2B_3 \\ &+ (1-\alpha)(6+5\alpha)B_1B_2\Big| - 2(9\alpha^2-20\alpha+12)|B_2| - (1-\alpha)(6+5\alpha)B_1^2 \\ &- (9\alpha^2-20\alpha+12)B_1\Big), \end{aligned} \tag{4.31}$$

$$Q = (2-\alpha)^{-2}(3-2\alpha)^{-1}\Big(4(1-\alpha)(6+5\alpha)B_1^2 + 4(9\alpha^2-4\alpha-12)(B_1+2|B_2|)\Big), \tag{4.32}$$

$$R = 128(2-\alpha)^{-2}B_1. \tag{4.33}$$

Thus using (4.14) and (4.30) we get,

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{1152}\begin{cases} R, & Q \leq 0,\, P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0,\, P \geq -\frac{Q}{8} \text{ or } Q \leq 0\,,\, P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0,\, P \leq -\frac{Q}{8}. \end{cases}$$

where P, Q, R are given by (4.31), (4.32) and (4.33), respectively. ∎

REMARK 4.10. When $\alpha = 1$ and $\varphi = (1+z)/(1-z)$, Theorem 4.9 reduces to [34, Theorem 3.1]. When $\alpha = 0$, Theorem 4.9 reduces to [45, Theorem 2].

4.2.4. The Class $\mathcal{K}_\alpha(\varphi)$. Let the function $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and is given by (4.1). For $0 \leq \alpha \leq 1$, the class $\mathcal{K}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination

$$\left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} \prec \varphi(z).$$

We see that $\mathcal{K}_0(\varphi) = \mathcal{S}^*(\varphi)$ is the Ma-Minda unified class of starlike functions.

THEOREM 4.11. *Let the function $f \in \mathcal{K}_\alpha(\varphi)$ be given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then,*

(1) If B_1, B_2 and B_3 satisfy the conditions

$$2(1-\alpha)^2|B_2| + (1-\alpha)\alpha B_1^2 \leq (2-\alpha^2)B_1,$$

and

$$|-(1-\alpha)B_1^4 - (3-2\alpha)B_2^2 + (2-\alpha)^2B_1B_3 + \alpha(1-\alpha)B_1^2B_2| \\ -(3-2\alpha)B_1^2 \leq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(2-\alpha)^2}.$$

(2) If B_1, B_2 and B_3 satisfy the conditions

$$2(1-\alpha)^2|B_2| + (1-\alpha)\alpha B_1^2 \geq (2-\alpha^2)B_1,$$

and

$$\begin{aligned}&|-2(1-\alpha)B_1^4-2(3-2\alpha)B_2^2+2(2-\alpha)^2B_1B_3+2\alpha(1-\alpha)B_1^2B_2|\\&-2(1-\alpha)^2|B_2|B_1-(1-\alpha)\alpha B_1^3-(2-\alpha)^2B_1^2\geq 0,\end{aligned}$$

or the conditions

$$2(1-\alpha)^2|B_2|+(1-\alpha)\alpha B_1^2\leq(2-\alpha^2)B_1,$$

and

$$\begin{aligned}&|-(1-\alpha)B_1^4-(3-2\alpha)B_2^2+(2-\alpha)^2B_1B_3+\alpha(1-\alpha)B_1^2B_2|\\&-(3-2\alpha)B_1^2\geq 0,\end{aligned}$$

then

$$\begin{aligned}|a_2a_4-a_3^2|\leq\frac{1}{(2-\alpha)^2(3-2\alpha)}&|-(3-2\alpha)B_2^2-(1-\alpha)B_1^4\\&+(2-\alpha)^2B_1B_3+\alpha(1-\alpha)B_1^2B_2|.\end{aligned}$$

(3) If B_1, B_2 and B_3 satisfy the conditions

$$2(1-\alpha)^2|B_2|+(1-\alpha)\alpha B_1^2>(2-\alpha^2)B_1,$$

and

$$\begin{aligned}&|-2(1-\alpha)B_1^4-2(3-2\alpha)B_2^2+2(2-\alpha)^2B_1B_3+2\alpha(1-\alpha)B_1^2B_2|\\&-2(1-\alpha)^2|B_2|B_1-(1-\alpha)\alpha B_1^3-(2-\alpha)^2B_1^2\leq 0,\end{aligned}$$

then

$$|a_2a_4-a_3^2|\leq\frac{B_1^2M}{((2-\alpha)^2(3-2\alpha))N},$$

where,

$$\begin{aligned}M=&|-(1-\alpha)(3-2\alpha)B_1^4-(3-2\alpha)^2B_2^2+(2-\alpha)^2(3-2\alpha)B_1B_3\\&+\alpha(1-\alpha)(3-2\alpha)B_1^2B_2|-\frac{\alpha}{2}(1-\alpha)(2-\alpha)^2B_1^3-(1-\alpha)^2(2-\alpha)^2B_1|B_2|\end{aligned}$$

$$-\frac{(2-\alpha)^4}{4}B_1^2-\frac{\alpha^2}{4}(1-\alpha)^2B_1^4-(1-\alpha)^4B_2^2-\alpha(1-\alpha)^3B_1^2|B_2|$$

and

$$\begin{aligned}N=&|-(1-\alpha)B_1^4-(3-2\alpha)B_2^2+(2-\alpha)^2B_1B_3+\alpha(1-\alpha)B_1^2B_2|\\&-\alpha(1-\alpha)B_1^3-2(1-\alpha)^2B_1|B_2|-(1-\alpha)^2B_1^2.\end{aligned}$$

PROOF. Since $f\in\mathcal{K}_\alpha(\varphi)$, there exists an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$, satisfying

$$\left(\frac{f(z)}{z}\right)^\alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}=\varphi(w(z)). \tag{4.34}$$

Define $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots,$$

which implies

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0)=1$ and $p_1\in\mathcal{P}$. Then, since $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots. \tag{4.35}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}\left(\frac{f(z)}{z}\right)^\alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}&=1+a_2z+((2-\alpha)a_3-(1-\alpha)a_2^2)z^2\\&\quad+((3-2\alpha)a_4-3(1-\alpha)a_2a_3+(1-\alpha)a_2^3)z^3+\cdots.\end{aligned} \tag{4.36}$$

Then from (4.34), (4.35) and (4.36), we get

$$\begin{aligned}a_2&=\frac{B_1c_1}{2}.\\a_3&=\frac{1}{4(2-\alpha)}\left[2B_1c_2-\left((B_1-B_2)-(1-\alpha)B_1^2\right)c_1^2\right].\end{aligned}$$

$$a_4 = \frac{1}{8(3-2\alpha)}\Big[4B_1c_3 + \Big(B_1 - \frac{3(1-\alpha)}{(2-\alpha)}B_1^2 + \frac{(1-\alpha)(1-2\alpha)}{(2-\alpha)}B_1^3 - 2B_2 + \frac{3(1-\alpha)}{(2-\alpha)}B_1B_2 + B_3\Big)c_1^3 + \Big(-4B_1 + \frac{6(1-\alpha)}{(2-\alpha)}B_1^2 + 4B_2\Big)c_1c_2\Big].$$

Thus,

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{1}{16}B_1\Big[\Big\{ & -\frac{(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^3 - \frac{1}{(2-\alpha)^2}\frac{B_2^2}{B_1} + \frac{1}{(3-2\alpha)}B_3 \\ & + \frac{(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)}(B_1 - 2B_2) + \frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1(B_2 - B_1)\Big\}c_1^4 \\ & + \Big\{\frac{4(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)}(B_2 - B_1) + \frac{2\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2\Big\}c_2c_1^2 \\ & - \frac{4}{(2-\alpha)^2}B_1c_2^2 + \frac{4}{(3-2\alpha)}B_1c_1c_3\Big]. \end{aligned}$$

Proceeding similarly as in the proof of Theorem 4.1, we would see that $|a_2a_4 - a_3^2|$ will be bounded by

$$\begin{aligned} G(c) = \frac{B_1}{16}\Big[c^4\Big\{\Big| & -\frac{1}{(2-\alpha)^2}\frac{B_2^2}{B_1} - \frac{(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^3 + \frac{1}{(3-2\alpha)}B_3 \\ & + \frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1B_2\Big| - \frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2 - \frac{(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)} \\ & \big(B_1 + 2|B_2|\big)\Big\} + 4c^2\Big\{\frac{2(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)}|B_2| + \frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2 \\ & - \frac{(2-\alpha^2)}{(2-\alpha)^2(3-2\alpha)}B_1\Big\} + \frac{16}{(2-\alpha)^2}B_1\Big] \\ = {} & \frac{B_1}{16}(Pc^4 + Qc^2 + R), \end{aligned} \tag{4.37}$$

where,

$$\begin{aligned} P = {} & (2-\alpha)^{-2}(3-2\alpha)^{-1}\Big(\Big| -(3-2\alpha)\frac{B_2^2}{B_1} - (1-\alpha)B_1^3 + (2-\alpha)^2B_3 \\ & + \alpha(1-\alpha)B_1B_2\Big| - 2(1-\alpha)^2|B_2| - \alpha(1-\alpha)B_1^2 + (1-\alpha)^2B_1\Big), \end{aligned} \tag{4.38}$$

$$Q = 4(2-\alpha)^{-2}(3-2\alpha)^{-1}\Big(2(1-\alpha)^2|B_2| + \alpha(1-\alpha)B_1^2 - (2-\alpha^2)B_1\Big), \tag{4.39}$$

$$R = 16(2-\alpha)^{-2}B_1. \tag{4.40}$$

Thus using (4.14) and (4.37) we get,

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{16}\begin{cases} R, & Q \leq 0,\, P \leq -\frac{Q}{4};\\ 16P+4Q+R, & Q \geq 0,\, P \geq -\frac{Q}{8} \text{ or } Q \leq 0\,,\, P \geq -\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q > 0,\, P \leq -\frac{Q}{8}.\end{cases}$$

where P, Q, R are given by (4.38), (4.39) and (4.40), respectively. ∎

REMARK 4.12. When $\alpha = 0$, Theorem 4.11 reduces to [45, Theorem 2]. Then Corollary 4.6 comes as a particular case.

4.2.5. The Class $\mathcal{T}_\alpha(\varphi)$. Let the analytic function $\varphi : \mathbb{D} \to \mathbb{C}$ be given by (4.1). For $0 \leq \alpha \leq 1$, the class $\mathcal{T}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination

$$\left(\frac{f(z)}{z}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z).$$

We see that,

$$\mathcal{T}_0(\varphi) = \mathcal{K}(\varphi) := \left\{f \in \mathcal{A} : 1 + \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$$

is the generalised Ma-Minda class of convex functions.

THEOREM 4.13. *Let the function $f \in \mathcal{T}_\alpha(\varphi)$ be given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then,*

(1) If B_1, B_2 and B_3 satisfy the conditions

$$4(1-\alpha)(2-\alpha)|6-7\alpha||B_2| + 2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2 \leq ((2-\alpha)^2(12-11\alpha) - 2(1-\alpha)(2-\alpha)|6-7\alpha|)B_1,$$

and

$$|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4 - 12(2-\alpha)^4(12-11\alpha)B_2^2 + 12(6-5\alpha)^2(2-\alpha)^3B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2| - 12(2-\alpha)^4(12-11\alpha)B_1^2 \leq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(6-5\alpha)^2}.$$

(2) If B_1, B_2 and B_3 satisfy the conditions

$$4(1-\alpha)(2-\alpha)|6-7\alpha||B_2| + 2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2 \geq ((2-\alpha)^2(12-11\alpha) \\ -2(1-\alpha)(2-\alpha)|6-7\alpha|)B_1,$$

and

$$|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4 - 12(2-\alpha)^4(12-11\alpha)B_2^2 \\ +12(6-5\alpha)^2(2-\alpha)^3B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2| \\ -24(1-\alpha)(2-\alpha)^3|6-7\alpha||B_2|B_1 - 12(1-\alpha)(2-\alpha)^2|7\alpha^2-8\alpha-6|B_1^3 \\ -6(2-\alpha)^3(2(1-\alpha)|6-7\alpha|+(2-\alpha)(12-11\alpha))B_1^2 \geq 0,$$

or the conditions

$$4(1-\alpha)(2-\alpha)|6-7\alpha||B_2| + 2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2 \leq ((2-\alpha)^2(12-11\alpha) \\ -2(1-\alpha)(2-\alpha)|6-7\alpha|)B_1,$$

and

$$|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4 - 12(2-\alpha)^4(12-11\alpha)B_2^2 \\ +12(6-5\alpha)^2(2-\alpha)^3B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2| \\ -12(2-\alpha)^4(12-11\alpha)B_1^2 \geq 0,$$

then

$$|a_2a_4 - a_3^2| \leq \frac{1}{12(2-\alpha)^4(6-5\alpha)^2(12-11\alpha)}\Big|-12(2-\alpha)^4(12-11\alpha)B_2^2 \\ +(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4 \\ +12(2-\alpha)^3(6-5\alpha)^2B_1B_3 - 24(1-\alpha)(2-\alpha)^2 \\ (7\alpha^2-8\alpha-6)B_1^2B_2\Big|.$$

(3) If B_1, B_2 and B_3 satisfy the conditions

$$4(1-\alpha)(2-\alpha)|6-7\alpha||B_2|+2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2>((2-\alpha)^2(12-11\alpha)$$
$$-2(1-\alpha)(2-\alpha)|6-7\alpha|)B_1,$$

and

$$|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4-12(2-\alpha)^4(12-11\alpha)B_2^2$$
$$+12(6-5\alpha)^2(2-\alpha)^3B_1B_3-24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2|$$
$$-24(1-\alpha)(2-\alpha)^3|6-7\alpha||B_2|B_1-12(1-\alpha)(2-\alpha)^2|7\alpha^2-8\alpha-6|B_1^3$$
$$-6(2-\alpha)^3(2(1-\alpha)|6-7\alpha|+(2-\alpha)(12-11\alpha))B_1^2\leq 0,$$

then

$$|a_2a_4-a_3^2|\leq\frac{B_1^2M}{(2(6-5\alpha)^2(12-11\alpha))N},$$

where,

$$M=|2(1-\alpha)(12-11\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4$$
$$-24(2-\alpha)^4(12-11\alpha)^2B_2^2+24(2-\alpha)^3(12-11\alpha)(6-5\alpha)^2B_1B_3$$
$$-48(1-\alpha)(2-\alpha)^2(12-11\alpha)(7\alpha^2-8\alpha-6)B_1^2B_2|-24(1-\alpha)(2-\alpha)$$
$$|7\alpha^2-8\alpha-6|\Big((2-\alpha)(12-11\alpha)+2(1-\alpha)|6-7\alpha|\Big)B_1^3$$
$$-48(1-\alpha)(2-\alpha)^2|6-7\alpha|\Big(2(1-\alpha)|6-7\alpha|+(2-\alpha)(12-11\alpha)\Big)B_1|B_2|$$
$$-6(2-\alpha)^2\Big(4(1-\alpha)|6-7\alpha|(2-\alpha)(12-11\alpha)+(2-\alpha)^2(12-11\alpha)^2$$
$$+4(1-\alpha)^2(6-7\alpha)^2\Big)B_1^2-24(1-\alpha)^2(7\alpha^2-8\alpha-6)^2B_1^4$$
$$-96(1-\alpha)^2(6-7\alpha)^2(2-\alpha)^2B_2^2-96(1-\alpha)^2(2-\alpha)|6-7\alpha|$$
$$|7\alpha^2-8\alpha-6|B_1^2|B_2|$$

and

$$N=|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4-12(2-\alpha)^4(12-11\alpha)B_2^2$$
$$+12(2-\alpha)^3(6-5\alpha)^2B_1B_3-24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2|$$

$$- 24(1-\alpha)(2-\alpha)^2|7\alpha^2 - 8\alpha - 6|B_1^3 - 48(1-\alpha)(2-\alpha)^3|6-7\alpha|B_1|B_2|$$
$$- 24(1-\alpha)(2-\alpha)^3|6-7\alpha|B_1^2.$$

PROOF. Since $f \in \mathcal{T}_\alpha(\varphi)$, there exists an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$\Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(1+\frac{zf''(z)}{f'(z)}\Big)^{1-\alpha} = \varphi(w(z)). \tag{4.41}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

then this implies

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots. \tag{4.42}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}\Big(\frac{f(z)}{z}\Big)^{\alpha}\Big(1+\frac{zf''(z)}{f'(z)}\Big)^{1-\alpha} &= 1 + (2-\alpha)a_2 z + \big((6-5\alpha)a_3 - \frac{1}{2}(1-\alpha)(8+\alpha)a_2^2\big)z^2\\ &+ \big((12-11\alpha)a_4 - (1-\alpha)(18+5\alpha)a_2a_3\\ &+ \frac{1}{6}(1-\alpha)(\alpha^2+28\alpha+48)a_2^3\big)z^3 + \cdots.\end{aligned} \tag{4.43}$$

Then from (4.41), (4.42) and (4.43), we get

$$a_2 = \frac{B_1 c_1}{2(2-\alpha)}.$$

$$a_3 = \frac{1}{8(6-5\alpha)}\left[4B_1 c_2 - \Big(\frac{2(B_1-B_2)(2-\alpha)^2 - (1-\alpha)(8+\alpha)B_1^2}{(2-\alpha)^2}\Big)c_1^2\right].$$

$$a_4 = \frac{1}{8(12-11\alpha)}\Big[4B_1 c_3 + \Big(B_1 - \frac{(1-\alpha)(18+5\alpha)}{(2-\alpha)(6-5\alpha)}B_1^2 - 2B_2 + B_3$$

$$- \frac{(1-\alpha)(10\alpha^3+25\alpha^2+186\alpha-144)}{6(2-\alpha)^3(6-5\alpha)}B_1^3 + \frac{(1-\alpha)(18+5\alpha)}{(2-\alpha)(6-5\alpha)}B_1B_2\Big)c_1^3$$
$$+ \Big(-4B_1 + \frac{2(1-\alpha)(18+5\alpha)}{(2-\alpha)(6-5\alpha)}B_1^2 + 4B_2\Big)c_1c_2\Big].$$

Thus,

$$a_2a_4 - a_3^2 = \frac{1}{8}B_1\Big[\Big\{\frac{(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)}{24(2-\alpha)^4(6-5\alpha)^2(12-11\alpha)}B_1^3 - \frac{1}{2(6-5\alpha)^2}\frac{B_2^2}{B_1}$$
$$+ \frac{1}{2(2-\alpha)(12-11\alpha)}B_3 + \frac{(1-\alpha)(6-7\alpha)}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)}(B_1-4B_2)$$
$$+ \frac{(1-\alpha)(7\alpha^2-8\alpha-6)}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)}B_1(B_1-B_2)\Big\}c_1^4$$
$$+ \Big\{\frac{4(1-\alpha)(6-7\alpha)}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)}(B_2-B_1) - \frac{2(1-\alpha)(7\alpha^2-8\alpha-6)}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)}$$
$$B_1^2\Big\}c_2c_1^2 - \frac{2}{(6-5\alpha)^2}B_1c_2^2 + \frac{2}{(2-\alpha)(12-11\alpha)}B_1c_1c_3\Big].$$

Again, proceeding as in the proof of Theorem 4.1, we see that $|a_2a_4-a_3^2|$ is bounded by

$$G(c) = \frac{B_1}{8}\Big[c^4\Big\{\Big| - \frac{1}{2(6-5\alpha)^2}\frac{B_2^2}{B_1} + \frac{(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)}{24(2-\alpha)^4(6-5\alpha)^2(12-11\alpha)}B_1^3$$
$$+ \frac{1}{2(2-\alpha)(12-11\alpha)}B_3 - \frac{(1-\alpha)(7\alpha^2-8\alpha-6)}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)}B_1B_2\Big|$$
$$- \frac{(1-\alpha)|7\alpha^2-8\alpha-6|}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)}B_1^2 - \frac{(1-\alpha)|6-7\alpha|}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)}(B_1+2|B_2|)\Big\}$$
$$+ 4c^2\Big\{\frac{(1-\alpha)|6-7\alpha|}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)}(B_1+2|B_2|) + \frac{(1-\alpha)|7\alpha^2-8\alpha-6|}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)}$$
$$B_1^2 - \frac{1}{2(6-5\alpha)^2}B_1\Big\} + \frac{8B_1}{(6-5\alpha)^2}\Big]$$
$$= \frac{B_1}{8}(Pc^4+Qc^2+R), \tag{4.44}$$

where,

$$P = \frac{1}{24}(2-\alpha)^{-4}(6-5\alpha)^{-2}(12-11\alpha)^{-1}\Big(\Big| - 12(2-\alpha)^4(12-11\alpha)\frac{B_2^2}{B_1}$$
$$+ (1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^3 + 12(2-\alpha)^3(6-5\alpha)^2B_3$$
$$- 24(1-\alpha)(2-a)^2(7\alpha^2-8\alpha-6)B_1B_2\Big| - 48(1-\alpha)(2-\alpha)^3|6-7\alpha||B_2|$$

$$-24(1-\alpha)(2-\alpha)^2|7\alpha^2-8\alpha-6|B_1^2-24(1-\alpha)(2-\alpha)^3|6-7\alpha|B_1\Big), \quad (4.45)$$

$$Q=2(2-\alpha)^{-2}(6-5\alpha)^{-2}(12-11\alpha)^{-1}\Big(2(1-\alpha)(2-\alpha)|6-7\alpha|(2|B_2|+B_1)$$
$$+2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2-(2-\alpha)^2(12-11\alpha)B_1\Big), \quad (4.46)$$

$$R=8(6-5\alpha)^{-2}B_1. \quad (4.47)$$

Thus using (4.14) and (4.44) we get,

$$|a_2a_4-a_3^2|\leq\frac{B_1}{8}\begin{cases} R, & Q\leq 0,\, P\leq -\frac{Q}{4};\\ 16P+4Q+R, & Q\geq 0,\, P\geq -\frac{Q}{8} \text{ or } Q\leq 0\,,\, P\geq -\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q>0,\, P\leq -\frac{Q}{8}.\end{cases}$$

where P, Q, R are given by (4.45), (4.46) and (4.47), respectively. ∎

REMARK 4.14. When $\alpha=0$, Theorem 4.13 reduces to [45, Theorem 2].

4.3. THIRD HANKEL DETERMINANT

We know that the qth Hankel determinant (denoted by $H_q(n)$) for $q=1,2,\dots$ and $n=1,2,3,\dots$ of the function f is the determinant of the $q\times q$ matrix given by $H_q(n):=\det(a_{n+i+j-2})$. Here $a_{n+i+j-2}$ denotes the entry for the i^{th} row and j^{th} column of the matrix. Thus, the third Hankel determinant is given by the expression $H_3(1):=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2)$. Here, we have computed the bounds for the third Hankel determinant for two very fascinating classes of analytic functions.

4.3.1. The Class $\mathcal{M}_\alpha$. The first theorem gives the coefficient bounds for the first five coefficients for the functions in the class $\mathcal{M}_\alpha$ which is the class of all normalised analytic univalent functions f in $\mathcal{S}$ satisfying

$$\operatorname{Re}\left((f'(z))^\alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\right)>0.$$

Note that

$$\mathcal{S}^*=\mathcal{M}_0:=\left\{f\in\mathcal{S}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0\right\}$$

and

$$\mathcal{R} = \mathcal{M}_1 := \{f \in \mathcal{S} : \operatorname{Re}\left(f'(z)\right) > 0\}$$

are the renowned classes of starlike and a subclass of close-to-convex functions, respectively. Thus as α varies from 0 to 1, our class $\mathcal{M}_\alpha$ furnishes a continuous passage from the class of starlike functions to a class of functions whose derivative has a positive real part. This class is a subclass of close-to-convex functions. The first theorem in this section gives the bounds for he first five coefficient estimates for the function $f \in \mathcal{M}_\alpha$.

THEOREM 4.15. *If the function $f \in \mathcal{M}_\alpha$, then the coefficients a_n $(n = 2, 3, 4, \cdots)$ of f satisfy*

$$\begin{aligned}
|a_2| \leq& \frac{2}{(1+\alpha)},\\
|a_3| \leq& \frac{2(3+\alpha)}{(2+\alpha)(1+\alpha)^2},\\
|a_4| \leq& \frac{2\left(36+19\alpha+11\alpha^2+5\alpha^3+\alpha^4\right)}{3(1+\alpha)^3(2+\alpha)(3+\alpha)},
\end{aligned}$$

and

$$|a_5| \leq \frac{2\left(360+433\alpha+437\alpha^2+331\alpha^3+137\alpha^4+28\alpha^5+2\alpha^6\right)}{3(1+\alpha)^4(2+\alpha)^2(3+\alpha)(4+\alpha)}.$$

PROOF. Since $f \in \mathcal{M}_\alpha$, there exists an analytic function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$ such that

$$(f'(z))^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} = p(z). \tag{4.48}$$

The Taylor series expansion of the function f gives

$$\begin{aligned}
(f'(z))^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} =& 1 + a_2(1+\alpha)z + \frac{1}{2}((2+\alpha)(2a_3 - (1-\alpha)a_2^2)z^2 \\
&+ \frac{1}{6}(3+\alpha)(6a_4 - 6(1-\alpha)a_2 a_3 + (1-\alpha)(2-\alpha)a_2^3))z^3 + \cdots .
\end{aligned} \tag{4.49}$$

Then using (4.48), (4.49) and the expansion for the function p, the coefficients $a_2 - a_5$ can be expressed as a function of the coefficients c_i of $p \in \mathcal{P}$:

$$a_2 = \frac{c_1}{(1+\alpha)}, \tag{4.50}$$

$$a_3 = \frac{1}{2(2+\alpha)(1+\alpha)^2}\big(2(1+\alpha)^2 c_2 + (1-\alpha)(2+\alpha)c_1^2\big), \tag{4.51}$$

$$a_4 = \frac{1}{6(1+\alpha)^3(2+\alpha)(3+\alpha)}\big((1-\alpha)(2+\alpha)(3+\alpha)(1-2\alpha)c_1^3 \tag{4.52}$$
$$+ 6(1+\alpha)^3(2+\alpha)c_3 + 6(1+\alpha)^2(1-\alpha)(3+\alpha)c_1c_2\big),$$

and

$$a_5 = \frac{1}{24(1+\alpha)^4(2+\alpha)^2(3+\alpha)(4+\alpha)}\big(24(1+\alpha)^3(2+\alpha)^2(1-\alpha)(4+\alpha)c_1c_3$$
$$+ 24(1+\alpha)^4(3+\alpha)(2+\alpha)^2c_4 + 12(1+\alpha)^4(3+\alpha)(1-\alpha)(4+\alpha)c_2^2$$
$$+ 12(1+\alpha)^2(1-\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(1-2\alpha)c_1^2c_2$$
$$+ (2+\alpha)^2(1-\alpha)(3+\alpha)(4+\alpha)(1-2\alpha)(1-3\alpha)c_1^4\big). \tag{4.53}$$

Consequently, using the triangle inequality and the fact that $|c_k| \leq 2$ $(k = 1, 2, 3, \cdots)$, we arrive at the desired bounds for a_2, a_3, a_4 and a_5. ∎

We now prove some results which will be required to estimate the third Hankel determinant $H_3(1)$ for functions in the class $\mathcal{M}_\alpha$.

THEOREM 4.16. *Let* $\alpha_0 = 0.267554 \in [0,1]$ *be the root of*

$$(18-\alpha-4\alpha^2-\alpha^3)(7\alpha+4\alpha^2+\alpha^3)^{1/2} = (1+\alpha)(6+3\alpha+\alpha^2)(6+9\alpha+4\alpha^2+\alpha^3)^{1/2}.$$

For the function $f \in \mathcal{M}_\alpha$*, the following coefficient bounds hold:*

(1) When $0 \leq \alpha \leq \alpha_0$*, then*
$$|a_2a_3 - a_4| \leq \frac{2(18-\alpha-4\alpha^2-\alpha^3)}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}.$$

(2) When $\alpha_0 \leq \alpha \leq 1$*, then*
$$|a_2a_3 - a_4| \leq \frac{2(6+3\alpha+\alpha^2)\sqrt{6+9\alpha+4\alpha^2+\alpha^3}}{3(1+\alpha)(2+\alpha)(3+\alpha)\sqrt{7\alpha+4\alpha^2+\alpha^3}}.$$

PROOF. Using the expressions for a_2, a_3 and a_4 from (4.50), (4.51) and (4.52), we see that

$$\begin{aligned}|a_2a_3 - a_4| =& \frac{1}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}\big|(3(1+\alpha)^2(2+\alpha)c_3 + 3\alpha(1+\alpha)(3+\alpha)c_1c_2 \\ & - (1-\alpha)(2+\alpha)(3+\alpha)c_1^3)\big|.\end{aligned}$$

Substituting the values for c_2 and c_3 from Lemma 4.3 in the above expression, we have

$$\begin{aligned}|a_2a_3 - a_4| =& \frac{1}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}\big|(18-\alpha-4\alpha^2-\alpha^3)c_1^3 - 12(4-c^2)(1+\alpha)c_1x \\ & + 3(4-c^2)(1+\alpha)^2(2+\alpha)c_1x^2 - 6(4-c^2)(1+\alpha)^2(2+\alpha)(1-|x|^2)y\big|. \\ \leq& \frac{1}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}\big((18-\alpha-4\alpha^2-\alpha^3)c_1^3 + 12(4-c^2)(1+\alpha)c_1|x| \\ & + 3(4-c^2)(1+\alpha)^2(2+\alpha)c_1|x|^2 + 6(4-c^2)(1+\alpha)^2(2+\alpha)(1-|x|^2)|y|\big).\end{aligned}$$

Choosing $c_1 = c \in [0,2]$, replacing $|x|$ by μ and using the fact that $|y| \leq 1$ in the above inequality, we get

$$\begin{aligned}|a_2a_3 - a_4| \leq& \frac{1}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}\big((18-\alpha-4\alpha^2-\alpha^3)c^3 + 12(4-c^2)(1+\alpha)c\mu \\ & + 3(4-c^2)(1+\alpha)^2(2+\alpha)c\mu^2 + 6(4-c^2)(1+\alpha)^2(2+\alpha)(1-\mu^2)\big). \\ =& F(c,\mu).\end{aligned}$$

We shall now maximize the function $F(c,\mu)$ for (c,μ) in $[0,2]\times[0,1]$. On differentiating $F(c,\mu)$ partially with respect to the parameter μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{(4-c^2)}{2(2+\alpha)(3+\alpha)}\big(2c + (2+\alpha)\mu(c-2)\big).$$

Then $\partial F/\partial \mu = 0$ for $\mu_0 = (2c)/((2-c)(1+\alpha)(2+\alpha)) \in [0,1]$ when $c \in [0,1]$. As observed from the graph of the function $F(c,\mu)$, when $c \in [0,1]$, maximum value of $F(c,\mu)$ exists at μ_0 and for $c \in [1,2]$, maximum exists at $\mu = 1$. Thus, we maximize the function $G(c)$ given by

$$G(c)=\begin{cases} G_1(c), & 0\leq c\leq 1;\\ G_2(c), & 1\leq c\leq 2,\end{cases}$$

where

$$G_1(c)=\frac{24(1+\alpha)^2(2+\alpha)^2+6c^2\alpha(3+\alpha)(4+3\alpha+\alpha^2)+c^3(3+\alpha)(-16+3\alpha^2+\alpha^3)}{12(1+\alpha)^2(2+\alpha)(3+\alpha)},$$

and

$$G_2(c)=\frac{12c(4-c^2)(1+\alpha)+3c(4-c^2)(1+\alpha)^2(2+\alpha)+c^3(18-\alpha-4\alpha^2-\alpha^3)}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}.$$

Firstly, we observe that $G_1'(c)=-c\big(4\alpha(1+\alpha)(2+\alpha)+c(-16+3\alpha^2+\alpha^3)\big)/(4(1+\alpha)^2(2+\alpha)^2)$. When $\alpha\in[\alpha^*,1]$, $G_1(c)$ is an increasing function of c as $G_1'(c)>0$ for all values of $c\in[0,1]$ and when $\alpha\in[\alpha^*,1]$, $G_1(c)$ is a decreasing function of c as $G_1'(c)<0$ for all $c\in[0,1]$. Thus, for $\alpha\in[0,\alpha^*]$, maximum is attained at $c=1$ and for $\alpha\in[\alpha^*,1]$, maximum is at $c=0$, and is given by

$$\max_{0\leq c\leq 1} G_1(c)=\begin{cases}\dfrac{144+232\alpha+225\alpha^2+102\alpha^3+17\alpha^4}{12(1+\alpha)^2(2+\alpha)^2(3+\alpha)}, & 0\leq\alpha\leq\alpha^*,\\ \dfrac{2(2+\alpha)^2}{(2+\alpha)^2(3+\alpha)}, & \alpha^*\leq\alpha\leq 1.\end{cases} \tag{4.54}$$

Here α^* is the root of the equation $144+232\alpha+225\alpha^2+102\alpha^3+17\alpha^4=24(1+\alpha)^2(2+\alpha)^2$.

We now maximize $G_2(c)$. It is seen that $G_2'(c)=\big(-\alpha(7+4\alpha+\alpha^2)c^2+(1+\alpha)(6+3\alpha+\alpha^2)\big)/\big((1+\alpha)^2(2+\alpha)(3+\alpha)\big)$. When $\alpha\in[0,\alpha']$, $G_2'(c)>0$ for all $c\in[1,2]$, thereby implying that $G_2(c)$ is an increasing function of c. For $\alpha\in[\alpha',1]$, it can be seen that

$$\max_{1\leq c\leq 2} G_2(c)=\max_{1\leq c\leq 2}\{G_2(c_0),G_2(1),G_2(2)\}=G_2(c_0),$$

where $c_0=((6+9\alpha+4\alpha^2+\alpha^3)/\alpha(7+4\alpha+\alpha^2))^{1/2}$ is the positive root of $G_2'(c)=0$. Thus, it is seen that

$$\max_{1\leq c\leq 2} G_2(c) = \begin{cases} \dfrac{2(18-\alpha-4\alpha^2-\alpha^3)}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}, & 0 \leq \alpha \leq \alpha', \\ \dfrac{2(6+3\alpha+\alpha^2)\sqrt{6+9\alpha+4\alpha^2+\alpha^3}}{3(1+\alpha)(2+\alpha)(3+\alpha)\sqrt{\alpha(7+4\alpha+\alpha^2)}}, & \alpha' \leq \alpha \leq 1. \end{cases} \tag{4.55}$$

where α' is the root of $(18-\alpha-4\alpha^2-\alpha^3)(\alpha(7+4\alpha+\alpha^2))^{1/2} = (1+\alpha)(6+3\alpha+\alpha^2)(6+9\alpha+4\alpha^2+\alpha^3)^{1/2}$. The absolute maximum value of $G(c)$ over the interval $c \in [0,2]$ is given by

$$\begin{aligned} \max_{0\leq c\leq 2} G(c) &= \max_{0\leq c\leq 2}\{G_1(c), G_2(c)\} \\ &= \begin{cases} \dfrac{2(18-\alpha-4\alpha^2-\alpha^3)}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}, & 0 \leq \alpha \leq \alpha_0, \\ \dfrac{2(6+3\alpha+\alpha^2)\sqrt{6+9\alpha+4\alpha^2+\alpha^3}}{3(1+\alpha)(2+\alpha)(3+\alpha)\sqrt{\alpha(7+4\alpha+\alpha^2)}}, & \alpha_0 \leq \alpha \leq 1. \end{cases} \end{aligned} \tag{4.56}$$

where α_0 is the root of $(18-\alpha-4\alpha^2-\alpha^3)(\alpha(7+4\alpha+\alpha^2))^{1/2} = (1+\alpha)(6+3\alpha+\alpha^2)(6+9\alpha+4\alpha^2+\alpha^3)^{1/2}$. ∎

For the third Hankel determinant, we thus have the following corollary:

COROLLARY 4.17. *If $f \in \mathcal{M}_\alpha$, then the third Hankel determinant $H_3(1)$ satisfies*

$$|H_3(1)| \leq \begin{cases} R, & 0 \leq \alpha \leq \alpha_0, \\ S, & \alpha_0 \leq \alpha \leq 1. \end{cases}$$

where

$$\begin{aligned} R =& \frac{4}{9(1+\alpha)^5(2+\alpha)^3(3+\alpha)^2(4+\alpha)}\big(10368+22815\alpha+27229\alpha^2+22644\alpha^3 \\ &+12505\alpha^4+4190\alpha^5+739\alpha^6+32\alpha^7-9\alpha^8-\alpha^9\big), \\ S =& \frac{4}{9(1+\alpha)^4(2+\alpha)^3(3+\alpha)^2(4+\alpha)(\alpha(7+4\alpha+\alpha^2))^{1/2}}\{(2+\alpha)(4+\alpha) \\ &(6+9\alpha+4\alpha^2+\alpha^3)^{1/2}(216+222\alpha+159\alpha^2+82\alpha^3+32\alpha^4+8\alpha^5+\alpha^6) \\ &+3(3+\alpha)(\alpha(7+4\alpha+\alpha^2))^{1/2}(576+1063\alpha+1109\alpha^2+655\alpha^3+209\alpha^4 \\ &+34\alpha^5+2\alpha^6)\}, \end{aligned}$$

and $\alpha_0 = 0.267554$ *is the root of the following equation:*

$$(18-\alpha-4\alpha^2-\alpha^3)(7\alpha+4\alpha^2+\alpha^3)^{1/2} = (1+\alpha)(6+3\alpha+\alpha^2)(6+9\alpha+4\alpha^2+\alpha^3)^{1/2}.$$

PROOF. Since $f \in \mathcal{A}$, $a_1 = 1$, so that we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{4.57}$$

By substituting $B_i = 2$ $(i = 1, 2, 3, \cdots)$ and $\mu = 1$ in [36, Theorem 2.11], we get the following bound for the expression $|a_3 - a_2^2|$ for $f \in \mathcal{M}_\alpha$:

$$|a_3 - a_2^2| \leq 2/(2+\alpha).$$

Similarly, [36, Theorem 2.9] gives the following bound for $f \in \mathcal{M}_\alpha$:

$$|a_2a_4 - a_3^2| \leq 4/(2+\alpha)^2.$$

Using these two bounds, the bound for the expression $|a_4 - a_2a_3|$ from Theorem 4.16 and the bounds for $|a_k|$ $(k = 1, 2, 3, \cdots)$ from Theorem 4.15 in the equation (4.57), the desired estimates for the thrid Hankel determinant follows. ∎

REMARK 4.18. For $\alpha = 0$, Corollary 4.17 reduces to $H_3(1) \leq 16$ for starlike functions [10].

4.3.2. The Class $\mathcal{L}_\alpha$. Our next theorem gives bounds for the first five coefficients for functions in the class $\mathcal{L}_\alpha$ which is the class of all normalised analytic functions $f \in \mathcal{S}$ satisfying

$$\operatorname{Re}\left((f'(z))^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right) > 0.$$

We see that,

$$\mathcal{K} = \mathcal{L}_0 := \left\{f \in \mathcal{A} : \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0\right\}$$

and

$$\mathcal{R} = \mathcal{L}_1 := \{f \in \mathcal{A} : \operatorname{Re}(f'(z)) > 0\}$$

are the classes of convex functions and a subclass of close-to-convex functions respectively. Thus as α varies from 0 to 1, our class $\mathcal{L}_\alpha$ provides a continuous passage from the class of convex functions to a subclass of close-to-convex functions. The following theorem gives the bounds for the first five coefficient estimates for $f \in \mathcal{L}_\alpha$

THEOREM 4.19. *If the function $f \in \mathcal{L}_\alpha$, then*

$$\begin{aligned}
|a_2| \leq& 1,\\
|a_3| \leq& \frac{2}{3(2-\alpha)}(3-2\alpha),\\
|a_4| \leq& \frac{1}{2(2-\alpha)(3-2\alpha)}\big((8-7\alpha)+4(1-\alpha)|1-2\alpha|\big),
\end{aligned}$$

and

$$\begin{aligned}
|a_5| \leq& \frac{1}{10(2-\alpha)^2(3-2\alpha)(4-3\alpha)}\big(4(56-101\alpha+54\alpha^2-8\alpha^3)\\
&+16(1-\alpha)|1-2\alpha|((12-7\alpha)+|4-13\alpha+6\alpha^2|)\big).
\end{aligned}$$

PROOF. Since the function $f \in \mathcal{L}_\alpha$, there exists an analytic function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$, such that

$$(f'(z))^\alpha \left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = p(z). \tag{4.58}$$

The Taylor series expansion of the function f gives

$$\begin{aligned}
(f'(z))^\alpha \Big(1+\frac{zf''(z)}{f'(z)}\Big)^{1-\alpha} =& 1+2a_2z+(3(2-\alpha)a_3-4(1-\alpha)a_2^2)z^2 \qquad (4.59)\\
&+(4(3-2\alpha)a_4-18(1-\alpha)a_2a_3+8(1-\alpha)a_2^3)z^3+\cdots.
\end{aligned}$$

Then using (4.58), (4.59) and the expansion for the function p, we express a_n in terms of the coefficients c_i of $p \in \mathcal{P}$:

$$a_2 = \frac{c_1}{2}, \tag{4.60}$$

$$a_3 = \frac{1}{3(2-\alpha)}\big(c_2+(1-\alpha)c_1^2\big), \tag{4.61}$$

$$a_4 = \frac{1}{4(2-\alpha)(3-2\alpha)}\big((1-\alpha)(1-2\alpha)c_1^3+(2-\alpha)c_3+3(1-\alpha)c_1c_2\big), \tag{4.62}$$

and

$$a_5 = \frac{1}{10(2-\alpha)^2(3-2\alpha)(4-3\alpha)}\big(8(2-\alpha)^2(1-\alpha)c_1c_3 + 2(2-\alpha)^2(3-2\alpha)c_4 \tag{4.63}$$
$$+ (1-\alpha)(4+\alpha)(3-2\alpha)c_2^2 + 2(1-\alpha)(1-2\alpha)(12-7\alpha)c_1^2c_2$$
$$+ (1-\alpha)(1-2\alpha)(4-13\alpha+6\alpha^2)c_1^4\big).$$

Therefore, by making use of the triangle inequality and the fact that $|c_k| \leq 2$ $(k = 1,2,3,\cdots)$, for $p \in \mathcal{P}$, we get the desired bounds for a_2, a_3, a_4 and a_5. ∎

Next, we prove certain results which will be required later to estimate the third hankel determinant $H_3(1)$ for the class $\mathcal{L}_\alpha$. To begin with, first we find an upper bound for $|a_2a_3 - a_4|$ for the function $f \in \mathcal{L}_\alpha$.

THEOREM 4.20. *Let $\alpha_0 = 0.852183 \in [0,1]$ be the root of the equation $(24-19\alpha)^{3/2} = 9\sqrt{3}(2-\alpha)^{3/2}(3-2\alpha)^{1/2}$. If $f \in \mathcal{L}_\alpha$, then*

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{(24-19\alpha)^{3/2}}{18\sqrt{3}(2-\alpha)(3-2\alpha)\sqrt{6-7\alpha+2\alpha^2}}, & 0 \leq \alpha < \alpha_0; \\ \frac{1}{2(3-2\alpha)}, & \alpha_0 \leq \alpha \leq 1. \end{cases}$$

PROOF. By making use of the equations (4.60), (4.61) and (4.62), we get

$$a_2a_3 - a_4 = \frac{1}{12(2-\alpha)(3-2\alpha)}\big(-3(2-\alpha)c_3 - (3-5\alpha)c_1c_2 + (1-\alpha)(3-2\alpha)c_1^3\big).$$

Substituting the values for c_2 and c_3 from Lemma 4.3 in the above expression, we have

$$|a_2a_3 - a_4| = \frac{1}{48(2-\alpha)(3-2\alpha)}\big|\alpha(9-8\alpha)c_1^3 - 2(4-c^2)(9-8\alpha)c_1x$$
$$+ 3(4-c^2)(2-\alpha)c_1x^2 - 6(4-c^2)(2-\alpha)(1-|x|^2)y\big|.$$
$$\leq \frac{1}{48(2-\alpha)(3-2\alpha)}\big(\alpha(9-8\alpha)c_1^3 + 2(4-c^2)(9-8\alpha)c_1|x|$$
$$+ 3(4-c^2)(2-\alpha)c_1|x|^2 + 6(4-c^2)(2-\alpha)(1-|x|^2)|y|\big).$$

Choosing $c_1 = c \in [0,2]$, replacing $|x|$ by μ and using the fact that $|y| \leq 1$ in the above inequality, we get

$$
\begin{aligned}
|a_2a_3 - a_4| \leq & \frac{1}{48(2-\alpha)(3-2\alpha)}\big(\alpha(9-8\alpha)c^3 + 2(4-c^2)(9-8\alpha)c\mu \\
& + 3(4-c^2)(2-\alpha)c\mu^2 + 6(4-c^2)(2-\alpha)(1-\mu^2)\big). \\
= & F(c,\mu).
\end{aligned}
$$

We shall now maximize the function $F(c,\mu)$ for $(c,\mu) \in [0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$
\frac{\partial F}{\partial \mu} = \frac{(4-c^2)}{48(2-\alpha)(3-2\alpha)}\big(2(9-8\alpha)c + 6\mu(2-\alpha)(c-2)\big).
$$

Then $\partial F/\partial\mu = 0$ for $\mu_0 = ((9-8\alpha)c)/(3(2-c)(2-\alpha)) \in [0,1]$ when $c \in [0, 0.8]$. As observed from the graph of the function $F(c,\mu)$, when $c \in [0,0.8]$, maximum of $F(c,\mu)$ occurs at μ_0 and for $c \in [0.8, 2]$, maximum occurs at $\mu = 1$. Thus, we have:

$$
\max_{0\leq\mu\leq 1} F(c,\mu) = G(c) = \begin{cases} G_1(c), & 0 \leq c \leq 0.8; \\ G_2(c), & 0.8 \leq c \leq 2, \end{cases}
$$

where

$$
G_1(c) = \frac{72(2-\alpha)^2 + 2c^2(3-5\alpha)(15-11\alpha) - c^3(9-8\alpha)(-9+2\alpha+3\alpha^2)}{144(2-\alpha)^2(3-2\alpha)},
$$

and

$$
G_2(c) = \frac{4c((24-19\alpha) - c^2(2-\alpha)(3-2\alpha))}{48(2-\alpha)(3-2\alpha)}.
$$

Note that $G_1'(c) = \big(4c(3-5\alpha)(15-11\alpha) - 3c^2(9-8\alpha)(3\alpha^2+2\alpha-9)\big)/(144(2-\alpha)^2(3-2\alpha))$. The function $G_1'(c) = 0$ implies $c = 0$ and $c_0 = (4(3-5\alpha)(15-11\alpha))/((9-8\alpha)(3\alpha^2+2\alpha-9))$. In order to find the maximum value for $G_1(c)$, we check the behaviour of $G_1(c)$ at the end points of the interval $[0,0.8]$ and at $c = c_0$. It can be observed that there exists some $\alpha^* \in [0, 0.8]$ such that for all values of $c \in [0, 0.8]$

and $\alpha \in [0,\alpha^*]$, $G_1'(c) > 0$, thereby implying that $G_1(c)$ is an increasing function of $c \in [0,0.8]$ and maximum occurs at $c = 0.8$. Similarly, using a similar argument, it is observed that when $\alpha \in [\alpha^*, 0.8]$, $G_1(c)$ decreases as $c \in [0,0.8]$ and hence, maximum occurs at $c = 0$ and is given as:

$$\max_{0\leq c\leq 0.8} G_1(c) = \begin{cases} G_1(0.8), & 0 \leq \alpha \leq \alpha^*; \\ \dfrac{1}{2(3-2\alpha)}, & \alpha^* \leq \alpha \leq 1. \end{cases} \tag{4.64}$$

Here α^* is the root of the equation $2(3-2\alpha)G_1(0.8) = 1$. We now maximize $G_2(c)$. It is seen that $G_2'(c) = \big((24-19\alpha) - 3c^2(2-\alpha)(3-2\alpha)\big)/\big(12(2-\alpha)(3-2\alpha)\big)$. On solving $G_2'(c) = 0$, the critical points as obtained are $c = \pm\sqrt{(24-19\alpha)}/\sqrt{3(2-\alpha)(3-2\alpha)}$. Since c cannot be negative, thus the only points of consideration in finding the maximum of $G_2(c)$ are the end points of the interval $[0.8, 2]$ and $c_0 = ((24-19\alpha)/3(2-\alpha)(3-2\alpha))^{1/2} \in [0.8, 2]$ for all $\alpha \in [0,1]$. It is observed that $G_2'(c) > 0$ for $c \in [0.8, c_0]$ and $G_2'(c) < 0$ when $c \in [c_0, 1]$, thereby implying that the function $G_2(c)$ increases first in the interval $[0.8, c_0]$ and then decreases in the interval $[c_0, 1]$. Hence the maximum occurs at $c = c_0$ and is given by:

$$\max_{0.8\leq c\leq 2} G_2(c) = \frac{1}{18\sqrt{3}}\left(\frac{(24-19\alpha)}{(2-\alpha)(3-2\alpha)}\right)^{3/2}. \tag{4.65}$$

In order to find the find the absolute maximum value of $G(c)$ over the interval $c \in [0,2]$, we compare the maximum values of $G_1(c)$ and $G_2(c)$ as obtained in (4.64) and (4.65) to get:

$$\max_{0\leq c\leq 2} G(c) = \begin{cases} \dfrac{1}{18\sqrt{3}}\left(\dfrac{(24-19\alpha)}{(2-\alpha)(3-2\alpha)}\right)^{3/2}, & 0 \leq \alpha \leq \alpha_0; \\ \dfrac{1}{2(3-2\alpha)}, & \alpha_0 \leq \alpha \leq 1. \end{cases} \tag{4.66}$$

where α_0 is the root of $(24-19\alpha)^{3/2} = 9\sqrt{3}(2-\alpha)^{3/2}(3-2\alpha)^{1/2}$. ∎

Thus, the following bound for the third Hankel determinant $H_3(1)$ for the function $f \in \mathcal{L}_\alpha$ comes as a corollary to Theorem 4.19 and 4.20.

COROLLARY 4.21. *If $f \in \mathcal{L}_\alpha$, then the third Hankel determinant $H_3(1)$ satisfies*

$$|H_3(1)| \leq \frac{1}{540(2-\alpha)^3}\left(P+Q+R\right)$$

where

$$P = \frac{5\sqrt{3}(24-19\alpha)^{3/2}(2-\alpha)\{8-7\alpha+4(1-\alpha)|1-2\alpha|\}}{(6-7\alpha+2\alpha^2)^{1/2}(3-2\alpha)^2},$$
$$Q = \frac{5\{(72-78\alpha+17\alpha)^2-32\alpha(3-2\alpha)|18-27\alpha+8\alpha^2|\}}{48-62\alpha+17\alpha^2-\alpha|18-27\alpha+8\alpha^2|},$$
$$R = \frac{144\{56-101\alpha+54\alpha^2-8\alpha^3+4(1-\alpha)|1-2\alpha|(12-7\alpha+|4-13\alpha+6\alpha^2|)\}}{(4-3\alpha)(3-2\alpha)}.$$

PROOF. By substituting $B_i = 2$ $(i = 1, 2, 3, \cdots)$ and $\mu = 1$ in [36, Theorem 2.15], we get the following bound for the expression $|a_3 - a_2^2|$ for $f \in \mathcal{L}_\alpha$:

$$|a_3 - a_2^2| \leq 2/(3(2-\alpha)).$$

Similarly, [36, Theorem 2.13] gives the following bound for $f \in \mathcal{L}_\alpha$:

$$|a_2a_4 - a_3^2| \leq \frac{32\alpha(3-2\alpha)|-18+27\alpha-8\alpha^2|-(72-78\alpha+17\alpha^2)^2}{72(2-\alpha)^2(3-2\alpha)\{\alpha|18-27\alpha+8\alpha^2|+(-48+62\alpha-17\alpha^2)\}}.$$

Using these two bounds, the bound for the expression $|a_4 - a_2a_3|$ from Theorem 4.20 and the bounds for $|a_k|$ $(k = 1, 2, 3, \cdots)$ from Theorem 4.19 in the equation (4.57), the desired estimates for the third Hankel determinant follows. ∎

REMARK 4.22. For $\alpha = 0$, Corollary 4.21 reduces to $H_3(1) \leq 1/8$ obtained in [10] for the function $f \in K$, the class of convex functions.

Chapter 5

Janowski Starlikeness and Convexity

This chapter deals mainly with the univalent functions having negative coefficients. Precisely, we consider the class $\mathcal{T}$ of analytic univalent functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{5.1}$$

These functions are indeed from the class $\mathcal{A}$ of all normalized functions analytic in $\mathbb{D}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and the class $\mathcal{S}$ of univalent functions.

5.1. Preliminaries

For $-1 \leq B < A \leq 1$, let $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$ be the subclasses of $\mathcal{S}$ consisting of Janowski starlike and Janowski convex functions respectively, defined analytically as:

$$\mathcal{S}^*[A,B] := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\mathcal{K}[A,B] := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

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When $A = 1-2\alpha$, $(0 \leq \alpha < 1)$ and $B = -1$, the above mentioned classes reduce to the classes of starlike functions of order α denoted by $\mathcal{S}^*(\alpha)$ and convex functions of order α denoted by $\mathcal{K}(\alpha)$ respectively. When $A = 0$ and $B = 0$, then $\mathcal{S}^*[0,0] =: \mathcal{S}^*$ and $\mathcal{K}[0,0] =: \mathcal{K}$ are the familiar classes of starlike and convex functions. A function $f \in \mathcal{S}$ is k*-uniformly convex* $(k \geq 0)$, if f maps every circular arc γ contained in $\mathbb{D}$ with center ζ, $|\zeta| \leq k$, onto a convex arc. This class of such functions introduced by Kanas and Wisniowska [35] is an extension of the class of uniformly convex functions introduced by Goodman [23]. They showed that f is k-uniformly convex [35, Theorem 2.2, p. 329] (see also [6] for details) if and only if f satisfies the inequality

$$k\left|\frac{zf''(z)}{f'(z)}\right| < \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right),$$

and presented the following theorem over the series of coefficients of $f \in \mathcal{T}$ to be k-uniformly convex.

THEOREM 5.1 ([35, Theorem 3.3, p. 334]). *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *satisfies the inequality* $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq 1/(k+2)$ $(k \geq 0)$, *then* f *is* k*-uniformly convex. The bound* $1/(k+2)$ *cannot be replaced by a larger number.*

Note that, Theorem 5.1 is an extension of [23, Theorem 6] to k-uniformly convex functions. It is well-known that a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $\sum_{n=2}^{\infty} n|a_n| \leq 1$ is necessarily univalent. This follows easily from the fact that derivative of such functions has positive real part. There are other coefficient conditions that are relevant.

A function $f \in \mathcal{A}$ is *parabolic starlike of order* α if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - 2\alpha + \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right).$$

Ali [4] gave the following sufficient condition over the series of coefficients for functions $f \in \mathcal{S}$ to be parabolic starlike of order α.

THEOREM 5.2 ([4, Theorem 3.1, p. 564]). *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *satisfies the inequality* $\sum_{n=2}^{\infty}(n-1)|a_n| \leq (1-\alpha)/(2-\alpha)$, *then* f *is parabolic starlike of order* α. *The bound* $(1-\alpha)/(2-\alpha)$ *cannot be replaced by a larger number.*

Ali *et al.* also investigated the condition on β so that the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$ implies either f is starlike or convex of some positive order. Our primary interest is the investigation of some similar sufficient coefficient conditions for functions to be in the classes $\mathcal{TS}^*[A,B] := \mathcal{T} \cap \mathcal{S}^*[A,B]$, and $\mathcal{TC}[A,B] := \mathcal{T} \cap \mathcal{K}[A,B]$. We obtain here certain necessary and sufficient conditions in terms of the series of the coefficients a_1, a_2, a_3, $\cdots$ for the functions in the class $\mathcal{T}$ to be in the classes $\mathcal{TS}^*[A,B]$ and $\mathcal{TC}[A,B]$. We also investigate the class $\mathcal{R}(A,B,\alpha)$ ($\alpha \in \mathbb{R}$) defined by

$$\mathcal{R}(A,B,\alpha) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right) \prec \frac{1+Az}{1+Bz} \right\} \tag{5.2}$$

We let $\mathcal{TR}(A,B,\alpha) := \mathcal{T} \cap \mathcal{R}(A,B,\alpha)$. When $\alpha = 0$, $\mathcal{R}(A,B,\alpha)$ is the class $\mathcal{S}^*[A,B]$. Finally, the reverse implications are investigated for functions to be in the above mentioned subclasses.

5.2. Coefficient Inequalities for Starlikeness and Convexity

In this section, we obtain some conditions over the coefficients of the function $f \in \mathcal{T}$ to belong the classes $\mathcal{TS}^*[A,B]$ and $\mathcal{TC}[A,B]$. We prove our results using the following lemma:

LEMMA 5.3. [9] *Let $-1 \leq B < A \leq 1$. A function $f \in \mathcal{TS}^*[A,B]$ if and only if it satisfies the following inequality:*

$$\sum_{n=2}^{\infty} \big((n-1)(1-B) + (A-B)\big)a_n \leq A - B. \tag{5.3}$$

and the function $f \in \mathcal{TC}[A,B]$ if and only if it satisfies the inequality

$$\sum_{n=2}^{\infty} n\big((n-1)(1-B) + (A-B)\big)a_n \leq A - B. \tag{5.4}$$

With the help of the preceding lemma, we now prove the sufficient condition for the function f to belong to the classes $\mathcal{TS}^*[A,B]$ and $\mathcal{TC}[A,B]$ respectively.

THEOREM 5.4. *Let $-1 \leq B < A \leq 1$. A function f of the form* (5.1) *belongs to $\mathcal{TS}^*[A,B]$ if it satisfies any one of the inequalities:*

(1) $\sum_{n=2}^{\infty} n(n-1)a_n \leq 2(A-B)/(1+A-2B)$;

(2) $\sum_{n=2}^{\infty}(n-1)a_n \leq (A-B)/(1+A-2B)$;

(3) $\sum_{n=2}^{\infty} na_n \leq 2(A-B)/(2-3B+A)$;

(4) $\sum_{n=2}^{\infty} n^2a_n \leq 4(A-B)/(1-2B+A)$.

The bounds are sharp.

PROOF. Let f satisfies (1). It can be easily seen that, for $n \geq 2$, the following inequality holds:

$$(n-1)(1-B)+(A-B) \leq \frac{1+A-2B}{2}n(n-1).$$

Consequently, the hypothesis yields

$$\sum_{n=2}^{\infty}\big((n-1)(1-B)+(A-B)\big)a_n \leq \frac{1+A-2B}{2}\sum_{n=2}^{\infty} n(n-1)a_n \leq A-B.$$

Therefore, by Lemma 5.3, $f \in \mathcal{TS}^*[A,B]$.

Let us now assume that f satisfies (2). Then since, for $n \geq 2$, the following inequality can be easily proved:

$$(n-1)(1-B)+(A-B) \leq (1+A-2B)(n-1)$$

Thus,

$$\sum_{n=2}^{\infty}\big((n-1)(1-B)+(A-B)\big)a_n \leq (1+A-2B)\sum_{n=2}^{\infty}(n-1)a_n \leq A-B.$$

Thus the result holds as a consequence of Lemma 5.3.

We next suppose that f satisfies (3). Then in order to show that f belongs to the class $\mathcal{TS}^*[A,B]$, we use Lemma 5.3 and the following inequality for $n \geq 2$:

$$(n-1)(1-B)+(A-B) \leq \frac{(2-3B+A)}{2}n$$

We, therefore, have the desired result by Lemma 5.3 as f satisfies

$$\sum_{n=2}^{\infty}\big((n-1)(1-B)+(A-B)\big)a_n \leq \frac{(2-3B+A)}{2}\sum_{n=2}^{\infty} na_n \leq A-B.$$

Finally, let (4) holds. Then, for $n \geq 2$, we have the following inequality:

$$(n-1)(1-B)+(A-B) \leq \frac{(1-2B+A)}{4} n^2.$$

Thus,

$$\sum_{n=2}^{\infty} \big((n-1)(1-B)+(A-B)\big) a_n \leq \frac{(1-2B+A)}{4} \sum_{n=2}^{\infty} n^2 a_n \leq A-B.$$

Hence, by equation (5.3) f belongs to the class $\mathcal{TS}^*[A,B]$ and this completes the proof of the theorem. ∎

REMARK 5.5. When $A = 1-\alpha$ and $B = 0$, clearly the class $\mathcal{TS}^*(A,B)$ reduces to the subclass $\mathcal{TS}^*_\alpha$ of $\mathcal{T}$, and hence we get the the following results:

(1) If the inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq 2(1-\alpha)/(2-\alpha)$ holds, then $f \in \mathcal{TS}^*_\alpha$.
(2) If the inequality $\sum_{n=2}^{\infty}(n-1)a_n \leq (1-\alpha)/(2-\alpha)$ holds, then $f \in \mathcal{TS}^*_\alpha$.
(3) If the inequality $\sum_{n=2}^{\infty} n a_n \leq 2(1-\alpha)/(3-\alpha)$ holds, then $f \in \mathcal{TS}^*_\alpha$.
(4) If the inequality $\sum_{n=2}^{\infty} n^2 a_n \leq 4(1-\alpha)/(2-\alpha)$ holds, then $f \in \mathcal{TS}^*_\alpha$.

The first two results and the last result obtained here are same as proved in [8, Theorem 2.1], [8, Corollary 2.3], and [8, Theorem 2.5] whereas the third coefficient inequality obtained above is an improvement of the already known coefficient bound as in [8, Theorem 2.5].

THEOREM 5.6. *Let $-1 \leq B < A \leq 1$. If the function $f \in \mathcal{T}$ satisfies any of the following inequalities:*

(1) $\sum_{n=2}^{\infty} n(n-1)a_n \leq (A-B)/(1+A-2B)$;
(2) $\sum_{n=2}^{\infty} n^2 a_n \leq (A-B)/(2-3B+A)$.

then $f \in \mathcal{TC}[A,B]$. The bounds obtained above is sharp.

PROOF. Let the function f satisfies the inequality (1). We see that the following inequality holds trivially for $n \geq 2$:

$$n\big((n-1)(1-B)+(A-B)\big) \leq (1+A-2B)n(n-1).$$

Therefore, the above inequality leads to

$$\sum_{n=2}^{\infty} n\big((n-1)(1-B)+(A-B)\big)a_n \leq (1+A-2B)\sum_{n=2}^{\infty} n(n-1)a_n \leq A-B$$

Thus by Lemma 5.3, $f \in \mathcal{TC}[A,B]$. For proving the second part of the theorem, we again make use of the Lemma 5.3 and the following inequality for $n \geq 2$:

$$n\big((n-1)(1-B)+(A-B)\big) \leq (2-3B+A)n^2.$$

We, therefore see that

$$\sum_{n=2}^{\infty} n\big((n-1)(1-B)+(A-B)\big)a_n \leq (2-3B+A)\sum_{n=2}^{\infty} n^2 a_n \leq A-B$$

which immediately proves the result using the equation (5.4). ∎

REMARK 5.7. When $A = 1-\alpha$ and $B = 0$, clearly the class $\mathcal{TC}(A,B)$ reduces to the subclass $\mathcal{TC}_\alpha$ of $\mathcal{T}$, and hence we get the the following coefficient inequalities:

(1) If the inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq (1-\alpha)/(2-\alpha)$ holds, then $f \in \mathcal{TC}_\alpha$.
(2) If the inequality $\sum_{n=2}^{\infty} n^2 a_n \leq (1-\alpha)/(3-\alpha)$ holds, then $f \in \mathcal{TC}_\alpha$.

The second coefficient inequality obtained above is an improvement of the already known coefficient inequality as in [8, Theorem 2.5] and the first one is same as obtained in [8, Theorem 2.1].

Our next theorem aims at finding some necessary conditions for the functions belonging to the class $\mathcal{TC}[A,B]$.

THEOREM 5.8. *If $f \in \mathcal{TC}[A,B]$, then the following holds:*

(1) The inequality $\sum_{n=2}^{\infty} na_n \leq (A-B)/(1+A-2B)$ holds and the bound is sharp.
(2) The inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq (A-B)/(1-B)$ holds.
(3) The inequality $\sum_{n=2}^{\infty}(n-1)a_n \leq (A-B)/2(1+A-2B)$ holds and the bound is sharp.

(4) *The inequality* $\sum_{n=2}^{\infty} n^2 a_n \leq 2(A-B)/(1+A-2B)$ *holds and the bound is sharp.*

PROOF. (1) Since $f \in \mathcal{TC}[A,B]$, then by Lemma 5.3, the coefficients of the function f satisfy:

$$\sum_{n=2}^{\infty} n\big((n-1)(1-B)+(A-B)\big)a_n \leq A-B. \tag{5.5}$$

It can be seen that for $n \geq 2$, the following inequality holds,

$$(1+A-2B)n \leq n((n-1)(1-B)+(A-B)). \tag{5.6}$$

Thus, making use of equations (5.5) and (5.6), we get:

$$\sum_{n=2}^{\infty} na_n \leq \sum_{n=2}^{\infty} \frac{n\big((n-1)(1-B)+(A-B)\big)}{(1+A-2B)} a_n \leq \frac{A-B}{(1+A-2B)}.$$

(2) Lemma 5.3 along with the inequality

$$n(n-1)(1-B) \leq n\big((n-1)(1-B)+(A-B)\big), \qquad n \geq 2$$

immediately yields

$$\sum_{n=2}^{\infty} n(n-1)a_n \leq \sum_{n=2}^{\infty} \frac{n\big((n-1)(1-B)+(A-B)\big)}{(1-B)} a_n \leq \frac{A-B}{(1-B)}.$$

(3) For $n \geq 2$, the following inequality holds true:

$$2(n-1)(1+A-2B) \leq n\big((n-1)(1-B)+(A-B)\big). \tag{5.7}$$

Now, Lemma 5.3 together with (5.7) clearly gives

$$\sum_{n=2}^{\infty} (n-1)a_n \leq \sum_{n=2}^{\infty} \frac{n\big((n-1)(1-B)+(A-B)\big)}{2(1+A-2B)} a_n \leq \frac{(A-B)}{2(1+A-2B)}.$$

(4) The following inequality holds for $n \geq 2$:

$$n^2(1+A-2B) \leq 2n\big((n-1)(1-B)+(A-B)\big)$$

Thus, using the above inequality along with Lemma 5.3 we get:

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty} \frac{2n\big((n-1)(1-B)+(A-B)\big)}{(1+A-2B)} a_n \leq \frac{2(A-B)}{(1+A-2B)}.$$

This completes the proof of the theorem. ∎

COROLLARY 5.9. *If $f \in \mathcal{TS}^*[A,B]$, then:*

(1) the inequality $\sum_{n=2}^{\infty} a_n \leq (A-B)/(1+A-2B)$ holds and the bound is sharp.
(2) the inequality $\sum_{n=2}^{\infty}(n-1)a_n \leq (A-B)/(1-B)$ holds.
(3) the inequality $\sum_{n=2}^{\infty} na_n \leq 2(A-B)/(1+A-2B)$ holds and the bound is sharp.

PROOF. The results follow from Theorem 5.8 and the Alexander relation between the classes $\mathcal{TS}^*[A,B]$ and $\mathcal{TC}[A,B]$. It can be directly proved by using Lemma 5.3 by using the inequalities $(1+A-2B) \leq (n-1)(1-B)+(A-B)$, $(1-B)(n-1) \leq (n-1)(1-B)+(A-B)$ and $(1+A-2B)n \leq 2((n-1)(1-B)+(A-B))$ respectively for $n \geq 2$. ∎

REMARK 5.10. For $A = 1-2\alpha$ and $B = -1$, the above results reduce to [8, Theorem 2.1,2.5,4.4,4.5].

5.3. THE SUBCLASS $\mathcal{TR}(A,B,\alpha)$

Recall that the class $\mathcal{R}(A,B,\alpha)$ ($\alpha \in \mathbb{R}$) is defined by

$$\mathcal{R}(A,B,\alpha) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) \prec \frac{1+Az}{1+Bz} \right\} \tag{5.8}$$

and $\mathcal{TR}(A,B,\alpha) := \mathcal{T} \cap \mathcal{R}(A,B,\alpha)$. The class $\mathcal{R}(\beta,\alpha) = \mathcal{R}(1-2\beta,-1,\alpha)$ was studied earlier in [8, 50]. Note that $\mathcal{R}(A,B,0) = \mathcal{S}^*[A,B]$ and the following lemma extends Lemma 5.3 and provides a necessary and sufficient condition for function f to belong to the class $\mathcal{TR}(A,B,\alpha)$.

LEMMA 5.11. *Let $\alpha \in \mathbb{R}$ and $-1 \leq B < A \leq 1$. Let f be of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. Then $f \in \mathcal{TR}(A,B,\alpha)$ if and only if f satisfies the following* *coefficient inequality:*

$$\sum_{n=2}^{\infty}(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1)a_n \leq (A-B). \tag{5.9}$$

PROOF. Let $f \in \mathcal{R}(A,B,\alpha)$. Then

$$\left(\frac{zf'(z)}{f(z)}\Big(\alpha\frac{zf''(z)}{f'(z)}+1\Big)\right) = \frac{1+Aw(z)}{1+Bw(z)}, \quad (-1 \leq B < A \leq 1) \tag{5.10}$$

where $w(z)$ is the Schwartz function satisfying $w(0)=0$, $|w(z)|<1$, $z \in \mathbb{D}$. That is

$$w(z) = \frac{zf'(z)+\alpha z^2 f''(z)-f(z)}{Af(z)-Bzf'(z)-B\alpha z^2 f''(z)}, \qquad w(0)=0$$

and

$$\begin{aligned}|w(z)| &= \Big|\frac{zf'(z)+\alpha z^2 f''(z)-f(z)}{Af(z)-Bzf'(z)-B\alpha z^2 f''(z)}\Big| \\ &= \Big|\frac{\sum_{n=2}^{\infty} a_n z^n\big(-n-\alpha n(n-1)+1\big)}{(A-B)z+\sum_{n=2}^{\infty} a_n z^n\big(-A+Bn+B\alpha n(n-1)\big)}\Big| < 1.\end{aligned}$$

Thus, this implies

$$\text{Re}\left\{\frac{\sum_{n=2}^{\infty} a_n z^n\big(-n-\alpha n(n-1)+1\big)}{(A-B)z+\sum_{n=2}^{\infty} a_n z^n\big(-A+Bn+B\alpha n(n-1)\big)}\right\} < 1$$

On further solving we get,

$$\sum_{n=2}^{\infty}\big(n^2\alpha(1-B)+n(1-\alpha+B\alpha-B)+A-1\big)a_n r^n < (A-B)r,$$

that is

$$\sum_{n=2}^{\infty}\big(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1\big)a_n r^n < (A-B)r,$$

Letting $r \to 1$, we get

$$\sum_{n=2}^{\infty}\big(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1\big)a_n < A-B.$$

Conversely, let (5.9) holds. We now have to show that $f \in \mathcal{R}(A,B,\alpha)$. For this, we prove that (5.10) holds and therefore, it is sufficient to show that

$$\Big(\frac{\alpha z^2 f''(z) + zf'(z)}{f(z)}\Big) = \frac{1+Aw(z)}{1+Bw(z)},$$

Equivalently, we can show

$$|\alpha z^2 f''(z) + zf'(z) - f(z)| - |Af(z) - B(\alpha z^2 f''(z) + zf'(z))| \leq 0.$$

Consider

$$\begin{aligned}
&\Big|\alpha z^2 f''(z) + zf'(z) - f(z)| - |Af(z) - B(\alpha z^2 f''(z) + zf'(z))\Big| \\
&= \Big| -\alpha \sum_{n=2}^{\infty} n(n-1)a_n z^n - \sum_{n=2}^{\infty} na_n z^n + \sum_{n=2}^{\infty} a_n z^n \Big| - \Big|(A-B)z - A\sum_{n=2}^{\infty} a_n z^n \\
&\quad + \alpha B \sum_{n=2}^{\infty} n(n-1)a_n z^n + B\sum_{n=2}^{\infty} na_n z^n\Big| \\
&= \Big|\sum_{n=2}^{\infty} a_n z^n\big(-\alpha n(n-1) - (n-1)\big)\Big| - \Big|(A-B)z + \sum_{n=2}^{\infty} a_n z^n\big(-A + \alpha Bn(n-1) \\
&\quad + Bn\big)\Big| \\
&= \sum_{n=2}^{\infty}\big(n^2\alpha + n - n\alpha - 1 + A - nB - n^2 B\alpha + nB\alpha\big)a_n - (A-B) \\
&= \sum_{n=2}^{\infty}\big(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1\big)a_n - (A-B) \leq 0.
\end{aligned}$$

which completes the proof of the lemma. ∎

The first theorem in this section gives a sufficient condition for the functions to belong to the classes $\mathcal{TR}(A,B,\alpha) \cap \mathcal{TS}^*[C,D]$ or $\mathcal{TR}(A,B,\alpha) \cap \mathcal{TC}[C,D]$ respectively.

THEOREM 5.12. *Let* $\alpha > 0$. *If* $f \in \mathcal{T}$ *satisfies* (5.9), *then the following results hold:*

(1) The function f is in the class $\mathcal{TS}^*[C,D]$ for

$$C \geq \frac{A - B + D(1-A) + 2\alpha D(1-B)}{(1-B)(1+2\alpha)}$$

.

(2) The function f is in the class $\mathcal{TC}[C,D]$ for

$$C \geq \frac{A-B+D(\alpha-A)+BD(1-\alpha)}{\alpha(1-B)}.$$

The bounds obtained are sharp.

PROOF. (1) In [83, Theorem 2], Silverman and Silvia proved that $\mathcal{S}^*[C,D] \subset \mathcal{S}^*[A,B]$ (or $\mathcal{K}[C,D] \subset \mathcal{K}[A,B]$) if and only if the following inequalities hold:

$$\frac{1-A}{1-B} \leq \frac{1-C}{1-D} \qquad and \qquad \frac{1+C}{1+D} \leq \frac{1+A}{1+B}.$$

In particular, when $B = D$, both of the above conditions reduce to $A \geq C$. Consequently, if $C \geq C_0 = (A-B+D(1-A)+2\alpha D(1-B))/((1-B)(1+2\alpha))$, then $\mathcal{TS}^*[C_0,D] \subset \mathcal{TS}^*[C,D]$. Hence, we only need to prove that $f \in \mathcal{TS}^*[C_0,D]$. Here we make use the following inequality for $n \geq 2$,

$$(n-1)(1-B)(1+2\alpha)+(A-B) \leq \alpha(1-B)n^2+(1-\alpha)(1-B)n+A-1 \quad (5.11)$$

Now, using (5.9) and (5.11), it readily follows that:

$$\begin{aligned}
&\sum_{n=2}^{\infty}((n-1)(1-D)+(C_0-D))a_n \\
&= \sum_{n=2}^{\infty}\Big((n-1)(1-D)+\frac{(A-B)(1-D)}{(1-B)(1+2\alpha)}\Big)a_n \\
&= \sum_{n=2}^{\infty}(1-D)\times\Big(\frac{(n-1)(1+2\alpha)(1-B)+(A-B)}{(1-B)(1+2\alpha)}\Big)a_n \\
&\leq \sum_{n=2}^{\infty}(1-D)\times\Big(\frac{n^2(1-B)\alpha+n(1-B)(1-\alpha)+A-1}{(1-B)(1+2\alpha)}\Big)a_n \\
&\leq \frac{(1-D)(A-B)}{(1-B)(1+2\alpha)} = C_0-D
\end{aligned}$$

Thus by Lemma 5.3, $f \in \mathcal{TS}^*[C_0,D]$.

(2) If $C \geq C_0 = (A-B+D(\alpha-A)+BD(1-\alpha))/\alpha(1-B)$, then $\mathcal{TC}[C_0,D] \subset \mathcal{TC}[C,D]$. Thus, it is enough to show that f belongs to $\mathcal{TC}[C_0,D]$. The following inequality holds for $n \geq 2$:

$$n\big((n-1)\alpha(1-B)+(A-B)\big) \leq n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1$$

Now, the above inequality together with (5.9) shows that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\big((n-1)(1-D)+(C_0-D)\big)a_n \\
&=\sum_{n=2}^{\infty} n\Big((n-1)(1-D)+\frac{(A-B)(1-D)}{(1-B)\alpha}\Big)a_n \\
&=\sum_{n=2}^{\infty}(1-D)n\Big(\frac{(n-1)\alpha(1-B)+(A-B)}{(1-B)\alpha}\Big)a_n \\
&\le \frac{(1-D)}{\alpha(1-B)}\sum_{n=2}^{\infty}(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1)a_n \\
&\le \frac{(1-D)(A-B)}{(1-B)\alpha}=C_0-D.
\end{aligned}$$

Thus by making use of Lemma 5.3, we get that the function f belongs to the class $\mathcal{TC}[C_0,D]$. ∎

The next theorem provides a sufficient coefficient inequality for the functions of the form (5.1) to belong to the class $\mathcal{TR}(A,B,\alpha)$.

THEOREM 5.13. *Let $\alpha\in\mathbb{R}$. If the function f defined by* (5.1) *satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-1)a_n \le \frac{2(A-B)}{2\alpha-2B\alpha-2B+A+1}, \tag{5.12}$$

then $f\in\mathcal{TR}(A,B,\alpha)$. The bound obtained is sharp.

PROOF. Since, for $n\ge 2$, the following inequality holds,

$$2\big(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1\big)\le(2\alpha(1-B)-2B+A+1)n(n-1),$$

and using this, we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\big(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1\big)a_n \\
&\le \frac{1}{2}\sum_{n=2}^{\infty}\big(2\alpha(1-B)-2B+A+1\big)n(n-1)a_n\le A-B.
\end{aligned}$$

Thus, by Lemma 5.11, $f\in\mathcal{TR}(A,B,\alpha)$. ∎

In our next result, we determine the condition on C so that $\mathcal{TC}[C,D] \subseteq \mathcal{TR}(A,B,\alpha)$.

THEOREM 5.14. *Let* $\alpha > 0$. *If* $C \leq (2A-2B+(1+2\alpha-3A+2B-2\alpha B)D)/(1-A-2\alpha(-1+B))$, *then* $\mathcal{TC}[C,D] \subseteq \mathcal{TR}(A,B,\alpha)$.

PROOF. For $C \leq C_0$, $\mathcal{TC}[C,D] \subset \mathcal{TC}[C_0,D]$. Thus it is enough to show that $\mathcal{TC}[C_0,D] \subseteq \mathcal{TR}(A,B,\alpha)$, where $C_0 = (2A-2B+(1+2\alpha-3A+2B-2\alpha B)D)/(1-A-2\alpha(-1+B))$. For $n \geq 2$, the following inequality holds:

$$2(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1) \leq A(3-n)+(n-1)(1+2\alpha) + 2B(-1+\alpha-n\alpha)$$

This yields,

$$\begin{aligned}
&\sum_{n=2}^{\infty}\big(n^2(1-B)\alpha+(1-B)(1-\alpha)n+A-1\big)a_n \\
&\leq \sum_{n=2}^{\infty}\frac{A(3-n)+(n-1)(1+2\alpha)+2B(-1+\alpha-n\alpha)}{2}a_n \\
&= \sum_{n=2}^{\infty}\frac{(n-1)(1-D)+(C_0-D)}{2(1-D)}\times\big(1-A-2\alpha(-1+B)\big)a_n \\
&\leq \frac{(C_0-D)}{2(1-D)}\times(1-A-2\alpha(-1+B))a_n \\
&= \frac{2(1-D)(A-B)}{2(1-D)(1-A-2\alpha(-1+B))}\times\big(1-A-2\alpha(-1+B)\big)a_n \\
&= A-B
\end{aligned}$$

Thus by Lemma 5.11 we get $f \in \mathcal{TR}(A,B,\alpha)$. ∎

Finally, we prove certain necessary conditions for the functions to belong to the class $\mathcal{R}(A,B,\alpha)$.

THEOREM 5.15. *Let* $-1 \leq B < A \leq 1$, *and* $\alpha \in \mathbb{R}$. *If* $f \in \mathcal{TR}(A,B,\alpha)$, *then*

(1) $\sum_{n=2}^{\infty} n(n-1)a_n \leq (A-B)/(\alpha(1-B))$, *where* $\alpha > 0$

(2) $\sum_{n=2}^{\infty}(n-1)a_n \leq \gamma$ *where*

$$\gamma = \begin{cases} \frac{A-B}{(1-B)(1-\alpha)}, & (1+3\alpha)B < 3\alpha + A; \\ \frac{A-B}{(1+A+2\alpha-2B-2\alpha B)}, & (1+3\alpha)B \geq 3\alpha + A. \end{cases}$$

The result is sharp when $(1+3\alpha)B > 3\alpha + A$ *.*

(3) $\sum_{n=2}^{\infty} n^2 a_n \leq \gamma$ *where*

$$\gamma = \begin{cases} \frac{A-B}{(1-B)\alpha}, & (A+1) > 2(\alpha + B - \alpha B); \\ \frac{4(A-B)}{(1+A+2\alpha-2B-2\alpha B)}, & (A+1) \leq 2(\alpha + B - \alpha B). \end{cases}$$

The result is sharp when $(A+1) < 2(\alpha + B - \alpha B)$

(4) $\sum_{n=2}^{\infty} n a_n \leq 2(A-B)/(1+2\alpha+A-2B-2\alpha B)$*. The result is sharp.*

PROOF. (1) Since $f \in \mathcal{TR}(A,B,\alpha)$, by Lemma 5.11 we have

$$\sum_{n=2}^{\infty}\big(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1\big)a_n \leq (A-B). \tag{5.13}$$

For $n \geq 2$, the following inequality holds:

$$\alpha(1-B)n(n-1) \leq \big(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1\big). \tag{5.14}$$

Then, equations (5.13) and (5.14) readily give

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)a_n &\leq \sum_{n=2}^{\infty} \frac{\big(n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1\big)}{\alpha(1-B)} a_n \\ &\leq \frac{(A-B)}{\alpha(1-B)}. \end{aligned}$$

(2) When $(1+3\alpha)B < 3\alpha + A$, then for $n \geq 2$,

$$(1-\alpha)(1-B)(n-1) \leq n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1. \tag{5.15}$$

Then, equations (5.15) and (5.13) give

$$\begin{aligned} \sum_{n=2}^{\infty}(n-1)a_n &\leq \sum_{n=2}^{\infty} \frac{n^2\alpha(1-B) + n(1-\alpha)(1-B) + A - 1}{(1-\alpha)(1-B)} a_n \\ &\leq \frac{(A-B)}{(1-\alpha)(1-B)}. \end{aligned}$$

When $(1+3\alpha)B \geq 3\alpha + A$, then for $n \geq 2$ the following inequality holds,

$$(1+A+2\alpha-2B-2\alpha B)(n-1) \leq n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1. \quad (5.16)$$

Using (5.13) and (5.16), we get

$$\begin{aligned}\sum_{n=2}^{\infty}(n-1)a_n &\leq \sum_{n=2}^{\infty}\left(\frac{n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1}{(1+A+2\alpha-2B-2\alpha B)}\right)a_n \\ &\leq \frac{(A-B)}{(1+A+2\alpha-2B-2\alpha B)}.\end{aligned}$$

(3) When $(A+1) > 2(\alpha+B-\alpha B)$, then the inequality,

$$\alpha(1-B)n^2 \leq n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1 \qquad n \geq 2, \quad (5.17)$$

together with the equation (5.13) give

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty}\frac{n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1}{\alpha(1-B)}a_n \leq \frac{(A-B)}{\alpha(1-B)}.$$

When $(A+1) \leq 2(\alpha+B-\alpha B)$, then for $n \geq 2$ the following inequality holds,

$$\frac{(1+A+2\alpha-2B-2\alpha B)n^2}{4} \leq n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1. \quad (5.18)$$

Using (5.13) and (5.18), we get

$$\begin{aligned}\sum_{n=2}^{\infty} n^2 a_n &\leq \sum_{n=2}^{\infty}\frac{4(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1)}{(1+A+2\alpha-2B-2\alpha B)}a_n \\ &\leq \frac{4(A-B)}{(1+A+2\alpha-2B-2\alpha B)}.\end{aligned}$$

(4) For $\alpha > 0$, the inequality

$$(1+A+2\alpha-2B-2\alpha B)n \leq 2\big(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1\big).$$

together with (5.13) shows that

$$\begin{aligned}\sum_{n=2}^{\infty} n a_n &\leq \sum_{n=2}^{\infty}\frac{2\big(n^2\alpha(1-B)+n(1-\alpha)(1-B)+A-1\big)}{(1+A+2\alpha-2B-2\alpha B)}a_n \\ &\leq \frac{2(A-B)}{(1+A+2\alpha-2B-2\alpha B)}.\end{aligned}$$

∎

REMARK 5.16. Replacing $C = 1 - 2\alpha$ and $D = -1$, our results reduce to the results obtained in [8] for the class $\mathcal{TR}(\alpha, \beta)$.

5.4. Coefficient Inequalities for Starlikeness

The functions f represented in the form:

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad \mu \in \mathbb{C}. \tag{5.19}$$

were studied in detail in [44]. Motivated by this, we determine the necessary and sufficient conditions for the functions given by (5.19) to be in the class $\mathcal{S}^*[A, B]$.

We need the following lemma to prove our results:

LEMMA 5.17. [44] *Suppose that* $f \in \mathcal{A}$ *has the representation* (5.19) *and the coefficients* b_n *satisfy the inequality*

$$\sum_{n=1}^{\infty} \big(n + |(A-B)\mu + Bn|\big)|b_n| \leq (A-B)\mu, \tag{5.20}$$

where $-1 \leq B \leq A \leq 1$. *Then* $f \in \mathcal{S}^*[A, B]$.

However, if $B > 0$ and $\mu \geq -B/(A-B)$, then inequality (5.20) reduces to:

$$\sum_{n=1}^{\infty} \big((1+B)n + (A-B)\mu\big)|b_n| \leq (A-B)\mu. \tag{5.21}$$

And if $B < 0$ and $\mu \leq -B/(A-B)$, then equation (5.20) reduces to:

$$\sum_{n=1}^{\infty} \big((1-B)n - (A-B)\mu\big)|b_n| \leq (A-B)\mu. \tag{5.22}$$

The following theorem provides sufficient condition over the series of coefficients inequality for the normalised analytic functions f with the representation (5.19) to be in the class $\mathcal{S}^*[A, B]$

THEOREM 5.18. *Let* $-1 \leq B < A \leq 1$ *and* $B > 0$ *and* $\mu \geq -B/(A-B)$. *Then if* $f \in \mathcal{A}$ *has the representation of the form* (5.19) *and* b_n *satisfies any one of the coefficient inequalities:*

(1) $\sum_{n=2}^{\infty} n(n-1)|b_n| \leq \dfrac{2\big((A-B)\mu - ((1+B)+(A-B)\mu)|b_1|\big)}{2(1+B)+(A-B)\mu}$;
(2) $\sum_{n=2}^{\infty}(n-1)|b_n| \leq \dfrac{(A-B)\mu - \big((1+B)+(A-B)\mu\big)|b_1|}{2(1+B)+(A-B)\mu}$;
(3) $\sum_{n=1}^{\infty} n|b_n| \leq \dfrac{(A-B)\mu}{(1+B)+(A-B)\mu}$;
(4) $\sum_{n=1}^{\infty} n^2|b_n| \leq \dfrac{(A-B)\mu}{(1+B)+(A-B)\mu}$.

Then $f \in \mathcal{S}^*[A,B]$.

PROOF. (1) For $n \geq 2$, the following inequality holds:

$$\big((1+B)n + (A-B)\mu\big) \leq \left((1+B) + \frac{(A-B)}{2}\mu\right) n(n-1). \tag{5.23}$$

Using the inequality (5.23), we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \big((1+B)n + (A-B)\mu\big)|b_n| \\
&= \big((1+B)+(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty} \big((1+B)n + (A-B)\mu\big)|b_n| \\
&\leq \big((1+B)+(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty} \left((1+B) + \frac{(A-B)}{2}\mu\right) n(n-1)|b_n| \\
&\leq \big((1+B)+(A-B)\mu\big)|b_1| \\
&\quad + \left(\frac{2(1+B)+(A-B)\mu}{2}\right) \times \left(\frac{2((A-B)\mu - (1+B)+(A-B)\mu|b_1|)}{2(1+B)+(A-B)\mu}\right) \\
&\leq (A-B)\mu
\end{aligned}$$

Thus by Lemma 5.17, we see that $f \in \mathcal{S}^*[A,B]$.

(2) For $n \geq 2$, the following inequality holds

$$\big((1+B)n + (A-B)\mu\big) \leq \big(2(1+B) + (A-B)\mu\big)(n-1). \tag{5.24}$$

Using equation (5.24) we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\big((1+B)n+(A-B)\mu\big)|b_n| \\
&= \big((1+B)+(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty}\big((1+B)n+(A-B)\mu\big)|b_n| \\
&\leq \big((1+B)+(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty}\big(2(1+B)+(A-B)\mu\big)(n-1)|b_n| \\
&\leq \big((1+B)+(A-B)\mu\big)|b_1| + \big(2(1+B)+(A-B)\mu\big) \\
&\quad \times\left(\frac{(A-B)\mu-((1+B)+(A-B)\mu)|b_1|}{2(1+B)+(A-B)\mu}\right) \leq (A-B)\mu
\end{aligned}$$

Thus by using Lemma 5.17, $f \in \mathcal{S}^*[A,B]$.

(3) For $n \geq 1$, the following inequality holds:

$$(1+B)n+(A-B)\mu \leq \big((1+B)+(A-B)\mu\big)n \tag{5.25}$$

Thus using the inequality (5.25), we see that

$$\begin{aligned}
\sum_{n=1}^{\infty}\big((1+B)n+(A-B)\mu\big)|b_n| &\leq \big((1+B)+(A-B)\mu\big)n|b_n| \\
&\leq (A-B)\mu.
\end{aligned}$$

Hence by Lemma 5.17, $f \in \mathcal{S}^*[A,B]$.

(4) For $n \geq 1$, the following inequality holds:

$$(1+B)n+(A-B)\mu \leq \big((1+B)+(A-B)\mu\big)n^2 \tag{5.26}$$

Thus using the inequality (5.26), we see that

$$\sum_{n=1}^{\infty}\big((1+B)n+(A-B)\mu\big)|b_n| \leq \big((1+B)+(A-B)\mu\big)n^2|b_n| \leq (A-B)\mu.$$

Thus by Lemma 5.17, $f \in \mathcal{S}^*[A,B]$. ∎

THEOREM 5.19. *Let* $-1 \leq B < A \leq 1$ *and* $B < 0$ *and* $\mu \leq -B/(A-B)$. *If* $f \in \mathcal{A}$ *has the form* (5.19) *and satisfies any one of the coefficient inequalities*

(1) $\sum_{n=2}^{\infty} n(n-1)|b_n| \leq \dfrac{(A-B)\mu-\big((1-B)-(A-B)\mu\big)|b_1|}{(1-B)}$

(2) $\sum_{n=2}^{\infty}(n-1)|b_n| \leq \dfrac{(A-B)\mu - \big((1-B)-(A-B)\mu|b_1|\big)}{2(1-B)}$

(3) $\sum_{n=1}^{\infty} n|b_n| \leq \dfrac{(A-B)\mu}{(1-B)}$

Then $f \in \mathcal{S}^*[A,B]$.

PROOF. (1) For proving the first part of the theorem, we observe that the following inequality holds for $n \geq 2$:

$$\big((1-B)n-(A-B)\mu\big) \leq (1+B)n(n-1). \tag{5.27}$$

Therefore, using (5.27) and (5.22) and the fact that $B<0$ and $\mu \leq -B/(A-B)$, we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}((1-B)n-(A-B)\mu)|b_n| \\
&= \big((1-B)-(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty}\big((1-B)n-(A-B)\mu\big)|b_n| \\
&\leq \big((1-B)-(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty}(1-B)n(n-1)|b_n| \\
&\leq \big((1-B)-(A-B)\mu\big)|b_1| + (1-B)\times \\
&\quad \left(\frac{(A-B)\mu-\big((1-B)-(A-B)\mu\big)|b_1|}{(1-B)}\right) \\
&\leq (A-B)\mu.
\end{aligned}$$

Thus, by Lemma 5.17, $f \in \mathcal{S}^*[A,B]$.

(2) For the second part, the following inequality can be proved easily for $n \geq 2$:

$$\big((1-B)n-(A-B)\mu\big) \leq 2(1+B)(n-1). \tag{5.28}$$

Therefore, using equation (5.28) and (5.22) we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\big((1-B)n-(A-B)\mu\big)|b_n| \\
&= \big((1-B)-(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty}\big((1-B)n-(A-B)\mu\big)|b_n|
\end{aligned}$$

$$
\begin{aligned}
&\le \big((1-B)-(A-B)\mu\big)|b_1| + \sum_{n=2}^{\infty} 2(1-B)(n-1)|b_n| \\
&\le \big((1-B)-(A-B)\big)|b_1| + 2(1-B)\times \\
&\quad \left(\frac{(A-B)\mu - \big((1-B)-(A-B)\mu|b_1|\big)}{2(1-B)}\right) \\
&\le (A-B)\mu.
\end{aligned}
$$

Thus by using Lemma 5.17 $f \in \mathcal{S}^*[A,B]$.

(3) Finally, we see that the following inequality holds for $n \geq 1$:

$$\big((1-B)n-(A-B)\mu\big) \le (1-B)n. \tag{5.29}$$

Therefore, using equations (5.29) and (5.22) we see that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\big((1-B)n-(A-B)\mu\big)|b_n| &\le \sum_{n=1}^{\infty}(1-B)n|b_n| \\
&\le (1-B)\times\left(\frac{(A-B)\mu}{(1-B)}\right) \\
&\le (A-B)\mu.
\end{aligned}
$$

Hence, by Lemma 5.17 $f \in \mathcal{S}^*[A,B]$. ∎

We now obtain certain properties of the functions of the form (5.19) in the class $S^*[A,B]$. We prove our results using the following lemma:

LEMMA 5.20. [44] *Every function $f \in S^*[A,B]$ $(-1 \le B < A \le 1)$ which has the form (5.19) with $0 < \mu < (1-B)/(A-B)$ satisfies the coefficient inequality*

$$\sum_{n=1}^{\infty}\big((1-B^2)n^2 - 2nB(A-B)\mu - (A-B)^2\mu^2\big)|b_n|^2 \le \mu^2(A-B)^2.$$

THEOREM 5.21. *If $f \in S^*[A,B]$, then the following holds:*

(1) The inequality

$$\sum_{n=1}^{\infty} n|b_n|^2 \le \frac{(A-B)^2\mu^2}{(1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2},$$

holds.

(2) The inequality

$$\sum_{n=1}^{\infty} n^2|b_n|^2 \leq \frac{(A-B)^2\mu^2}{(1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2}$$

holds.

PROOF. For $n \geq 1$, the following inequality holds

$$\big((1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2\big)n \leq (1-B^2)n^2-2nB(A-B)\mu-(A-B)^2\mu^2 \tag{5.30}$$

Therefore, using (5.30) and Lemma 5.20

$$\begin{aligned}\sum_{n=1}^{\infty} n|b_n|^2 &\leq \sum_{n=1}^{\infty} \frac{(1-B^2)n^2-2nB(A-B)\mu-(A-B)^2\mu^2}{\big((1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2\big)}|b_n|^2 \\ &\leq \frac{(A-B)^2\mu^2}{\big((1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2\big)},\end{aligned}$$

and hence the result. In order to prove the second part of the theorem, we see that the following inequality holds for all $n \geq 1$

$$\begin{aligned}\sum_{n=1}^{\infty} \big((1-B^2)-2B(A-B)\mu-(A-B)^2\mu^2\big)n^2 \leq (1-B^2)n^2-2nB(A-B)\mu \\ -(A-B)^2\mu^2\end{aligned} \tag{5.31}$$

Proceeding as in the proof of the first part, the result follows trivially using (5.31) and Lemma 5.20. ∎

Chapter 6

The Classes $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$

6.1. Introduction and Preliminaries

The concept of subordination plays a crucial role in the study of univalent functions and several important subclasses of univalent functions have been introduced and studied by using this. For instance, in 1985, Padmanabhan and Parvatham [67] used the concept of Hadamard product and subordination and introduced the class of functions f satisfying $z(k_a * f(z))'/(k_a * f(z)) \prec h$ where $k_a = z/(1-z)^\alpha$, $\alpha \in \mathbb{R}$, $f \in \mathcal{A}$ and h is a convex function. Later on, in the year 1989, Shanmugam [82] studied the class $\mathcal{S}^*_g(\omega)$ of all functions $f \in \mathcal{A}$ which satisfy $z(f*g)'/(f*g) \prec \omega$ where ω is a convex function, g is a fixed function in $\mathcal{A}$. Replacing g by the functions $z/(1-z)$ and $z/(1-z)^2$, we get the subclasses $\mathcal{S}^*(\omega)$ and $\mathcal{K}(\omega)$ respectively. Let φ be a univalent function with the positive real part satisfying $\varphi(0) = 1$ and $\varphi'(0) > 0$. Recall that for such a function φ, Ma and Minda [51] introduced the following two unified subclasses using the concept of subordination:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad \text{and} \quad \mathcal{K}(\varphi) = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

The importance of these classes comes from the fact that for different values of φ, these subclasses reduce to some renowned subclasses of univalent functions. For instance, when $\varphi(z) = (1+z)/(1-z)$, $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduce to the class $\mathcal{S}^*$ of starlike and $\mathcal{K}$ of convex functions respectively. For $-1 \leq B < A \leq 1$, when $\varphi(z) = (1+$

$Az)/(1+Bz)$, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduce to the classes $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$ respectively, which are the familiar classes consisting of Janowski starlike and convex functions. On replacing $A = 1-2\alpha$ and $B = -1$ where $0 \leq \alpha < 1$, the class $\mathcal{S}^*[A,B]$ reduces to the subclass $\mathcal{S}^*(\alpha)$, the class of the starlike functions of order α, whereas the class $\mathcal{K}[A,B]$ reduces to the subclass $\mathcal{K}(\alpha)$ of $\mathcal{S}$, the class of convex functions of order α. These classes were introduced and extensively studied by Robertson [78]. Similarly, when $\varphi = \sqrt{1+z}$, we get $\mathcal{S}^*_L = \mathcal{S}^*(\sqrt{1+z})$ which consists of the functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the domain bounded by the right half of the lemniscate of Bernoulli given by $|w^2-1| < 1$. Sokól and Stankiewicz [86, 87] introduced and studied this subclass.

Therefore, various authors investigated many attractive subclasses of the starlike and convex functions using the Ma - Minda classes of starlike and convex functions. In this chapter, we define two very fascinating subclasses of $\mathcal{S}^*$, namely $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$ respectively, where $0 \leq \alpha < 1$ and study these classes extensively. The classes are defined as follows:

$$\mathcal{S}^*_{\alpha,e} = \mathcal{S}^*(\alpha + (1-\alpha)e^z) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \alpha + (1-\alpha)e^z \right\},$$

and

$$\mathcal{SL}^*(\alpha) = \mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+z}) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \alpha + (1-\alpha)\sqrt{1+z} \right\}.$$

As a consequence of the Alexander's two way bridge relation between the class $\mathcal{S}^*$ of starlike and $\mathcal{K}$ of convex functions, which states that $f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}^*$, similar properties for the functions in $\mathcal{K}(\varphi_0)$ can be obtained from the corresponding properties for $\mathcal{S}^*(\varphi_0)$, where $\varphi_0 = \alpha + (1-\alpha)e^z$ or $\alpha + (1-\alpha)\sqrt{1+z}$.

We now discuss certain examples of functions in the classes $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$ which serve as an extremal function for many problems over the two respective subclasses. Define the function k_n $(n = 2, 3, 4, \cdots)$ by $k_n(0) = k_n'(0) - 1 = 0$ and

$$\frac{zk_n'(z)}{k_n(z)} = \varphi(z^{n-1})$$

where $\varphi = \alpha + (1-\alpha)e^z$. Then, clearly the function $k_n \in \mathcal{S}^*_{\alpha,e}$ where $n = 2, 3, \cdots$. In view of this, it can be seen that the function

$$k(z) = k_2(z) = z + (1-\alpha)z^2 + \frac{1}{4}(1-\alpha)(3-2\alpha)z^3 + \cdots, \tag{6.1}$$

serves as an extremal function for various problems for the class $\mathcal{S}^*_{\alpha,e}$. In a similar fashion, we find the extremal function for the class $\mathcal{SL}^*(\alpha)$ and it can be seen that the following function

$$h(z) = h_2(z) = z + (1-\alpha)z^2 + \frac{1}{16}(1-\alpha)(1-2\alpha)z^3 + \cdots \tag{6.2}$$

serves as an extremal function for various extremal problems for the class $\mathcal{SL}^*(\alpha)$.

The bound for the Fekete- Szegö inequality for the classes $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$ can be estimated as in [8, Theorem 1, p.38]. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}^*_{\alpha,e}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(1-\alpha)}{4}(1+2(1-2\mu)(1-\alpha)), & \mu \leq \dfrac{1}{4}\dfrac{(1-2\alpha)}{(1-\alpha)}; \\ \dfrac{(1-\alpha)}{2}, & \dfrac{1}{4}\dfrac{(1-2\alpha)}{(1-\alpha)} \leq \mu \leq \dfrac{1}{4}\dfrac{(5-2\alpha)}{(1-\alpha)}; \\ \dfrac{-(1-\alpha)}{4}(1+2(1-2\mu)(1-\alpha)), & \mu \geq \dfrac{1}{4}\dfrac{(5-2\alpha)}{(1-\alpha)}. \end{cases}$$

and if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{SL}^*(\alpha)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(1-\alpha)}{16}(2(1-\alpha)(1-2\mu)-1), & \mu \leq -\dfrac{1}{4}\dfrac{(3+2\alpha)}{(1-\alpha)}; \\ \dfrac{(1-\alpha)}{4}, & -\dfrac{1}{4}\dfrac{(3+2\alpha)}{(1-\alpha)} \leq \mu \leq \dfrac{1}{4}\dfrac{(5-2\alpha)}{(1-\alpha)}; \\ \dfrac{(1-\alpha)}{16}(1-2(1-\alpha)(1-2\mu)), & \mu \geq \dfrac{1}{4}\dfrac{(5-2\alpha)}{(1-\alpha)}. \end{cases}$$

When $\alpha = 0$, the above estimated bound for the class $\mathcal{S}^*_{\alpha,e}$ reduces to the bound for the Fekete- Szegö inequality for the subclass $\mathcal{S}^*_e$ as in [54, Section 2.2], whereas bound for the class $\mathcal{SL}^*(\alpha)$ reduces to the bound for the Fekete- Szegö inequality for the subclass

$\mathcal{SL}$. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{SL}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1}{16}(1-4\mu), & \mu \leq -\dfrac{3}{4}; \\ \dfrac{1}{4}, & -\dfrac{3}{4} \leq \mu \leq \dfrac{5}{4}; \\ -\dfrac{1}{16}(1-4\mu), & \mu \geq \dfrac{5}{4}. \end{cases}$$

The Fekete- Szegö inequality together with [8, Theorem 1, p.38] gives the sharp first four coefficient bounds for both the classes $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$, which are as follows:

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}^*_{\alpha,e}$, then

$$|a_2| \leq 1-\alpha, \qquad |a_3| \leq \begin{cases} (1-\alpha)(3-2\alpha)/4, & 0 < \alpha < 1/2; \\ (1-\alpha)(1-2\alpha)/4, & 1/2 \leq \alpha < 1, \end{cases}$$

and

$$|a_4| \leq \frac{(1-\alpha)}{36}(17 - 21\alpha + 6\alpha^2).$$

These bounds are sharp. Also, when $\alpha = 0$, the above estimated bounds reduce to the coefficient bounds for the class $\mathcal{S}^*_e$ as in [54, Thoerem 2.3].

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{SL}^*(\alpha)$, then

$$|a_2| \leq (1-\alpha)/2, \qquad |a_3| \leq (1-\alpha)/4 \qquad \text{and} \qquad |a_4| \leq \frac{(1-\alpha)}{6}.$$

These bounds are sharp. When, $\alpha = 0$, these bounds reduce to the first four sharp coefficient bounds for the functions in the class $\mathcal{SL}$, i.e. if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{SL}$, then $|a_2| \leq 1/2$, $|a_3| \leq 1/4$ and $|a_4| \leq 1/6$.

6.2. Two Subclasses

In the following section, we obtain certain inclusion relations, radius problems and certain coefficient estimates for functions in the classes $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$.

LEMMA 6.1. *For $r \in (0,1)$, the function Φ_0, where $\Phi_0(z) = \alpha + (1-\alpha)e^z$ or $\alpha + (1-\alpha)\sqrt{1+z}$ satisfies the following:*

$$\min_{|z|=r} \operatorname{Re} \Phi_0(z) = \Phi_0(-r) = \min_{|z|=r} |\Phi_0(z)|$$

and

$$\max_{|z|=r} \operatorname{Re} \Phi_0(z) = \Phi_0(r) = \max_{|z|=r} |\Phi_0(z)|$$

PROOF. For $0 \leq \theta < 2\pi$, the function $\Psi_0(\theta) = \operatorname{Re} \Phi_0(re^{\imath\theta}) = \alpha + (1-\alpha)e^{r\cos\theta} \cos(r\sin\theta)$ its maximum is attained at $\theta = 0$ whereas the minimum ocurs at $\theta = \pi$ and. Thus,

$$\min_{|z|=r} \operatorname{Re} \Phi_0(z) = \alpha + (1-\alpha)e^{-r} = \Phi_0(-r)$$

and

$$\max_{|z|=r} \operatorname{Re} \Phi_0(z) = \alpha + (1-\alpha)e^{r} = \Phi_0(r).$$

The proof for the function $\Phi_0(z) = \alpha + (1-\alpha)\sqrt{1+z}$ is similar and therefore has been skipped. ∎

LEMMA 6.2. *For $\alpha + (1-\alpha)/e < a < \alpha + (1-\alpha)e$, let r_a be given by*

$$r_a = \begin{cases} (a-\alpha) - (1-\alpha)/e, & \alpha + (1-\alpha)/e < a \leq \alpha + (1-\alpha)(e+e^{-1})/2; \\ e(1-\alpha) - (a-\alpha), & \alpha + (1-\alpha)(e+e^{-1})/2 \leq a < \alpha + (1-\alpha)e. \end{cases}$$

and R_a be given by

$$R_a = \begin{cases} e(1-\alpha) - (a-\alpha), & \alpha + (1-\alpha)/e < a \leq \alpha + (1-\alpha)e/2; \\ \sqrt{z(\theta_a)}, & \alpha + (1-\alpha)e/2 < a < \alpha + (1-\alpha)e. \end{cases}$$

Then,

$$\{w : |w-a| < r_a\} \subset \left\{w : \left|\log\left(\frac{w-\alpha}{1-\alpha}\right)\right| < 1\right\} \subset \{w : |w-a| < R_a\}.$$

PROOF. Let $\varphi_0(z) = \alpha + (1-\alpha)e^z$. Then any boundary point on the curve $\varphi_0(\mathbb{D})$ is of the form:

$$\varphi_0(e^{\iota\theta}) = \alpha + (1-\alpha)\big(e^{\cos\theta}\cos(\sin\theta) + \iota e^{\cos\theta}\sin(\sin\theta)\big).$$

It can be observed that the curve $w = \varphi_0(e^{\iota\theta})$ is symmetric with respect to x-axis, thereby reducing the interval under consideration to $\theta \in [0,\pi]$. Also,computing the distance between any arbitrary point on the curve $w = \varphi_0(e^{\iota\theta})$ to the point $(a,0)$ and squaring, we arrive at

$$\begin{aligned} z(\theta) &= \big(a - (\alpha + (1-\alpha)e^{\cos\theta}\cos(\sin\theta))\big)^2 + \big(e^{2\cos\theta}\sin^2(\sin\theta)(1-\alpha)^2\big) \\ &= (\alpha - a)^2 + (1-\alpha)^2 e^{2\cos\theta} + 2(1-\alpha)e^{\cos\theta}\cos(\sin\theta)(\alpha - a) \end{aligned}$$

Now, we have the following two cases:

Case 1: Let us first assume that $\alpha + (1-\alpha)/e < a \le \alpha + (1-\alpha)e/2$. Then, it can be observed that $z(\theta)$ is a decreasing function of θ, where $0 \le \theta \le \pi$. We consequently have the following

$$r_a = \min_{\theta\in[0,\pi]} \sqrt{z(\theta)} = \sqrt{z(\pi)} = (a-\alpha) - \frac{1-\alpha}{e}.$$

Case 2: Next, we assume that $\alpha + (1-\alpha)e/2 < a \le \alpha + (1-\alpha)e$. Then, a simple calculation leads to

$$\begin{aligned} z'(\theta) &= -2(1-\alpha)^2 e^{2\cos\theta}\sin\theta + 2(1-\alpha)(a-\alpha)e^{\cos\theta}\sin(\sin\theta)\cos\theta \\ &\quad + 2(1-\alpha)(a-\alpha)\cos(\sin\theta)e^{\cos\theta}\sin\theta, \end{aligned}$$

and it is observed

$$z'(0) = z'(\theta_a) = z'(\pi) = 0.$$

Here $0 < \theta_a < \pi$ is the real root of the following equation $e^{\cos\theta}(1-\alpha)\sin\theta = (a-\alpha)\sin(\sin\theta + \theta)$. Note that, $\theta_{a_1} < \theta_{a_2}$ for $a_1 < a_2$. Moreover, it can be observed that the $z(\theta)$ is an increasing function for θ belonging to the interval $[0,\theta_a]$ and decreasing function for θ belonging to the interval $[\theta_a,\pi]$. A simple calculation further leads to:

$$z(\pi) - z(0) = 2(1-\alpha)\left(e - \frac{1}{e}\right)\left(a - \alpha - \frac{(1-\alpha)}{2}\left(e + \frac{1}{e}\right)\right). \tag{6.3}$$

Using equation (6.3), we finally arrive at the following two subcases:

Subcase 1: $\alpha + (1-\alpha)e/2 \le a \le \alpha + (1-\alpha)(e+e^{-1})/2$.

Here, we see that

$$\min\{z(0), z(\theta_a), z(\pi)\} = z(\pi).$$

Thus, the minimum value of $z(\theta)$ is attained at $\theta = \pi$ and hence,

$$r_a = \min\sqrt{z(\theta)} = (a-\alpha) - \frac{(1-\alpha)}{e}.$$

Subcase 2: $\alpha + (1-\alpha)(e+e^{-1})/2 \le a \le \alpha + (1-\alpha)e$.

In this case, it can be easily seen that:

$$\min\{z(0), z(\theta_a), z(\pi)\} = z(0)$$

and therefore,

$$r_a = \min\sqrt{z(\theta)} = e(1-\alpha) + (\alpha - a).$$

which completes the proof of the first half of the lemma.

In order to prove the second part of the lemma, we compute the distance between any arbitrary point on the curve $w = \varphi_0(e^{\iota\theta})$ and the point $(a, 0)$ and square it to get

$$\begin{aligned} z(\theta) &= \left(a - (\alpha + (1-\alpha)e^{\cos\theta}\cos(\sin\theta))\right)^2 + \left(e^{2\cos\theta}\sin^2(\sin\theta)(1-\alpha)^2\right) \\ &= (\alpha - a)^2 + (1-\alpha)^2 e^{2\cos\theta} + 2(1-\alpha)e^{\cos\theta}\cos(\sin\theta)(\alpha - a) \end{aligned}$$

Then, it can be easily deduced that the following two cases arise:

Case 1.: $\alpha + (1-\alpha)/e < a \le \alpha + (1-\alpha)e/2$

And in this case $z(\theta)$ is a decreasing function of $0 \le \theta \le \pi$. Therefore,

$$\max\{z(0), z(\theta_a), z(\pi)\} = z(0),$$

i.e., the maximum value of $z(\theta)$is attained $\theta = 0$ and hence,

$$R_a = \max \sqrt{z(\theta)} = e(1-\alpha) + (\alpha - a).$$

Case 2.: $\alpha + (1-\alpha)e/2 < a < \alpha + (1-\alpha)e$

In this case, the function $z(\theta)$ increases initially for $\theta \in [0, \theta_a]$ and decreases for $\theta \in [\theta_a, \pi]$ for the entire range of a, and thus

$$\max\{z(0), z(\theta_a), z(\pi)\} = z(\theta_a).$$

which implies

$$R_a = \max \sqrt{z(\theta)} = \sqrt{z(\theta_a)}.$$

Thus the proof is complete. ∎

LEMMA 6.3. *For $\alpha < a < \alpha + (1-\alpha)\sqrt{2}$, let r_a be given by*

$$r_a = \begin{cases} P, & \alpha < a \leq \frac{3\alpha + 2\sqrt{2}(1-\alpha)}{3}; \\ Q, & \frac{3\alpha+2\sqrt{2}(1-\alpha)}{3} \leq a < \alpha + (1-\alpha)\sqrt{2}. \end{cases}$$

where

$$P = \left(\sqrt{(1-a)(1+a-2\alpha)}(1-\alpha) + (1-a)(2\alpha - 1 - a)\right)^{1/2},$$

and

$$Q = \sqrt{2}(1-\alpha) - (a - \alpha).$$

and R_a be given by

$$R_a = \begin{cases} \sqrt{2}(1-\alpha) - (a-\alpha), & \alpha < a \leq \sqrt{2}\alpha + (1-\alpha)/\sqrt{2}; \\ a - \alpha, & \sqrt{2}\alpha + (1-\alpha)/\sqrt{2} < a < \alpha + (1-\alpha)\sqrt{2}. \end{cases}$$

Then,

$$\{w : |w-a| < r_a\} \subset \left\{w : \left|\left(\frac{w-\alpha}{1-\alpha}\right)^2 - 1\right| < 1\right\} \subset \{w : |w-a| < R_a\}. \qquad (6.4)$$

PROOF. Let $\varphi_0(z) = \alpha + (1-\alpha)\sqrt{1+z}$. The parametric equation of the right half of this subclass of lemniscate of Bernoulli is given as follows

$$x(t) = \alpha + (1-\alpha)\frac{\sqrt{2}\cos t}{1+\sin^2 t}, \qquad y(t) = (1-\alpha)\frac{\sqrt{2}\sin t\cos t}{1+\sin^2 t} \qquad \Big(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\Big).$$

Here, the interval under consideration is $-\pi/2 \leq t \leq \pi/2$. The square of the distance from any point on the lemniscate to the point $(a, 0)$ is given by

$$\begin{aligned} z(t) =& (a - x(t))^2 + (y(t))^2 \\ =& \Big(a - \big(\alpha + (1-\alpha)\frac{\sqrt{2}\cos t}{1+\sin^2 t}\big)\Big)^2 + 2(1-\alpha)^2\Big(\frac{\sin^2 t\cos^2 t}{1+\sin^2 t}\Big) \\ =& (a-\alpha)^2 + 2(1-\alpha)^2\frac{\cos^2 t}{1+\sin^2 t} - 2\sqrt{2}(1-\alpha)(a-\alpha)\frac{\cos t}{1+\sin^2 t}. \end{aligned} \tag{6.5}$$

Differentiating both sides with respect to t yields

$$z'(t) = \frac{1}{(1+\sin^2 t)^2}\Big((1-\alpha)\sin t\big\{\sqrt{2}(a-\alpha)(4+2\cos^2 t) - 8(1-\alpha)\cos t\big\}\Big).$$

and

$$z'(0) = z'(t_a) = 0,$$

where $0 < t_a < \pi$ is the real root of the equation $4(1-\alpha)\cos t = \sqrt{2}(a-\alpha)(2 + \cos^2 t)$, which on simplifying gives

$$\cos t = \frac{\sqrt{2}}{(a-\alpha)}\big\{(1-\alpha) \pm \big((1-a^2) - 2\alpha(1-a)\big)^{1/2}\big\}.$$

Note that for $a > 1$, the numbers $\sqrt{2}\{(1-\alpha) \pm \big((1-a^2) - 2\alpha(1-a)\big)^{1/2}\}/(a-\alpha)$ are complex, and for $0 < a \leq 1$, the number $\sqrt{2}\{(1-\alpha) + \big((1-a^2) - 2\alpha(1-a)\big)^{1/2}\}/(a-\alpha) > 1$ and for $0 < a < 1$, the number $\sqrt{2}\{(1-\alpha) - \big((1-a^2) - 2\alpha(1-a)\big)^{1/2}\}/(a-\alpha)$ lies between -1 and 1 if and only if $\alpha < a < (3\alpha + 2\sqrt{2}(1-\alpha))/3$. We therefore arrive at the following two cases:

Case 1.: When $\alpha < a < \big(3\alpha + 2\sqrt{2}(1-\alpha)\big)/3$ and $t = t_0$ is given by

$$\cos t_0 = \frac{\sqrt{2}}{(a-\alpha)}\big\{(1-\alpha) - \big((1-a^2) - 2\alpha(1-a)\big)^{1/2}\big\} \tag{6.6}$$

Here, it is observed that

$$\min\{z(0), z(\pi/2), z(-\pi/2), z(t_0)\} = z(t_0)$$

Thus minimum of $z(t)$ occurs at $t = t_0$ and

$$\begin{aligned} r(a) =& \min \sqrt{z(t)} = \sqrt{z(t_0)} \\ =& \big(\sqrt{(1-a)(1+a-2\alpha)}(1-\alpha) + (1-a)(2\alpha-1-a)\big)^{1/2}. \end{aligned}$$

Case 2.: When $\big(3\alpha + 2\sqrt{2}(1-\alpha)\big)/3 < a < \alpha + (1-\alpha)\sqrt{2}$. Here

$$\min\{z(0), z(\pi/2), z(-\pi/2)\} = z(0),$$

and thus the minimum value of $z(t)$ is attained at $t = 0$, and thus

$$r(a) = \min \sqrt{z(t)} = \sqrt{z(0)} = \sqrt{2}(1-\alpha) - (a-\alpha)$$

and therefore, the first inclusion in (6.4) follows.

For proving the second inclusion in (6.4), we maximize $z(t)$ as in (6.5). It is observed as in the previous case that $z'(t) = 0$ for $t = 0$ and $\cos t_0 = \sqrt{2}\{(1-\alpha) - \big((1-a^2) - 2\alpha(1-a)\big)^{1/2}\}/(a-\alpha)$ and hence the following two cases arise:

Case 1.: When $\alpha < a \leq \dfrac{1 + \sqrt{2}\alpha(1-\sqrt{2}) + \alpha^2(1-\sqrt{3})}{\sqrt{2}(1-\alpha)}$ and $t = t_0$ is as in equation (6.6). It is seen that

$$\max\{z(0), z(\pi/2), z(-\pi/2), z(t_0)\} = z(0)$$

Thus $z(t)$ attains its maximum value at $t = 0$ and

$$R(a) = \max \sqrt{z(t)} = \sqrt{z(0)} = \sqrt{2}(1-\alpha) - (a-\alpha).$$

Case 2.: When $\dfrac{1 + \sqrt{2}\alpha(1-\sqrt{2}) + \alpha^2(1-\sqrt{3})}{\sqrt{2}(1-\alpha)} < a \leq \alpha + (1-\alpha)\sqrt{2}$.

$$\max\{z(0), z(\pi/2), z(-\pi/2)\} = z(\pi/2)$$

Thus $z(t)$ attains its maximum value at $t = \pi/2$ and

$$R(a) = \max \sqrt{z(t)} = \sqrt{z(\pi/2)} = (a - \alpha). \qquad \blacksquare$$

6.3. Some Inclusion Relations

The class of starlike functions of order β $(0 \leq \beta < 1)$ is characterised by the condition $\operatorname{Re}\left(zf'(z)/f(z)\right) > \beta$. We denote this class by $\mathcal{S}^*(\beta)$. The class $k - \mathcal{S}^*$, $(k > 0)$ of k-starlike functions was introduced by Kanas and Wisniowska [35]. This class is analytically defined by the following condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad z \in \mathbb{D}.$$

This class furnishes a continuous passage from the class of starlike functions to the class of parabolic starlike functions as k varies from 0 to 1. Uralegaddi *et al.* [91] studied a very interesting class $\mathcal{M}(\beta)$, which is defined as follows

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} < \beta \right\} \qquad z \in \mathbb{D}. \tag{6.7}$$

In this section, we discuss some inclusion relations between the classes $\mathcal{M}(\beta)$ $(\beta > 1)$, $k - \mathcal{S}^*$, $\mathcal{S}^*(\beta)$ and $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$.

THEOREM 6.4. *The class $\mathcal{S}^*_{\alpha,e}$ satisfies the following relationships:*

(1) $\mathcal{S}^*_{\alpha,e} \subset \mathcal{S}^*(\beta) \subset \mathcal{S}^*$ for $0 \leq \beta \leq \alpha + (1-\alpha)/e$.
(2) $\mathcal{S}^*_{\alpha,e} \subset \mathcal{M}(\beta)$ for $\beta \geq \alpha + (1-\alpha)e$.
(3) $k - \mathcal{S}^* \subset \mathcal{S}^*_{\alpha,e}$ for $k \geq (\alpha + (1-\alpha)e)/(e-1)(1-\alpha)$.

The constants obtained here are the best possible.

PROOF. Let the function $f \in \mathcal{S}^*_{\alpha,e}$. Then $zf'(z)/f(z) \prec \alpha + (1-\alpha)e^z$. Using Lemma 6.1, it can be easily seen that

$$\min_{|z|=1} \operatorname{Re}(\alpha + (1-\alpha)e^z) < \operatorname{Re} \frac{zf'(z)}{f(z)} < \max_{|z|=1} \operatorname{Re}(\alpha + (1-\alpha)e^z),$$

which immediately yields

$$\alpha+(1-\alpha)\frac{1}{e}\leq \operatorname{Re}\frac{zf'(z)}{f(z)}\leq \alpha+(1-\alpha)e \quad z\in\mathbb{D}.$$

From the definitions of $\mathcal{S}^*(\beta)$ and $\mathcal{M}(\beta)$, it is clear that $f\in\mathcal{S}^*(\alpha+(1-\alpha)/e)$ and $f\in\mathcal{M}(\alpha+(1-\alpha)e)$ which proves the first two parts of the theorem. For the third part, let the function $f\in k-\mathcal{S}^*$ and let us consider the following conic domain $\Gamma_k=\{w\in\mathbb{C}:\operatorname{Re}w>k|w-1|\}$. For $k>1$, the curve $\partial\Gamma_k$ is an ellipse $\gamma_k: x^2=k^2(x-1)^2+k^2y^2$ which may be rewritten as

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1,$$

where $x_0=k^2/(k^2-1)$, $y_0=0$, $a=k/(k^2-1)$ and $b=1/\sqrt{k^2-1}$. For the above defined ellipse γ_k to lie inside the domain $|\log((w-\alpha)/(1-\alpha))|\leq 1$, it is necessary that $x_0+a\leq\alpha+(1-\alpha)e$, which is equivalent to $k\geq(\alpha+(1-\alpha)e)/(e-1)(1-\alpha)$. Also, since $\Gamma_{k_1}\subset\Gamma_{k_2}$ for $k_1\geq k_2$, it follows that $k-\mathcal{S}^*\subset\mathcal{S}^*_{\alpha,e}$ for $k\geq(\alpha+(1-\alpha)e)/(e-1)(1-\alpha)$, thereby completing the proof. ∎

REMARK 6.5. When $\alpha=0$, Theorem 6.4 reduces to [54, Theorem 2.1].

THEOREM 6.6. *The class $\mathcal{SL}^*(\alpha)$ satisfies the following relationships:*

(1) $\mathcal{SL}^*(\alpha)\subset\mathcal{S}^*(\beta)\subset\mathcal{S}^*$ for $0\leq\beta\leq\alpha$.
(2) $\mathcal{SL}^*(\alpha)\subset\mathcal{M}(\beta)$ for $\beta\geq\alpha+(1-\alpha)\sqrt{2}$.
(3) $k-\mathcal{S}^*\subset\mathcal{SL}^*(\alpha)$ for $k\geq(\alpha+(1-\alpha)\sqrt{2})/(\sqrt{2}-1)(1-\alpha)$.

The constants obtained here are the best possible.

PROOF. Let $f\in\mathcal{SL}^*(\alpha)$. Then $zf'(z)/f(z)\prec\alpha+(1-\alpha)\sqrt{1+z}$. Using Lemma 6.1, it can be easily seen that

$$\min_{|z|=1}\operatorname{Re}(\alpha+(1-\alpha)\sqrt{1+z})<\operatorname{Re}\frac{zf'(z)}{f(z)}<\max_{|z|=1}\operatorname{Re}(\alpha+(1-\alpha)\sqrt{1+z}),$$

which immediately yields

$$\alpha\leq\operatorname{Re}\frac{zf'(z)}{f(z)}\leq\alpha+(1-\alpha)\sqrt{2} \quad z\in\mathbb{D}.$$

From the definitions of $\mathcal{S}^*(\beta)$ and $\mathcal{M}(\beta)$, it is clear that $f \in \mathcal{S}^*(\alpha)$ and $f \in \mathcal{M}(\alpha + (1-\alpha)\sqrt{2})$ which proves the first two parts of the theorem. For the third part, let $f \in k-\mathcal{S}^*$ and let us consider the following conic domain $\Gamma_k = \{w \in \mathbb{C} : \operatorname{Re} w > k|w-1|\}$. For $k > 1$, $\partial\Gamma_k$ is an ellipse $\gamma_k : x^2 = k^2(x-1)^2 + k^2y^2$ which may be rewritten as

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1,$$

where $x_0 = k^2/(k^2-1)$, $y_0 = 0$, $a = k/(k^2-1)$ and $b = 1/\sqrt{k^2-1}$. For the above defined ellipse γ_k to lie inside the domain $|((w-\alpha)/(1-\alpha))^2 - 1| \leq 1$, it is necessary that $x_0 + a \leq \alpha + (1-\alpha)\sqrt{2}$, which is equivalent to $k \geq (\alpha + (1-\alpha)e)/(e-1)(1-\alpha)$. Also, since $\Gamma_{k_1} \subset \Gamma_{k_2}$ for $k_1 \geq k_2$, it follows that $k-\mathcal{S}^* \subset \mathcal{S}^*_{\alpha,e}$ for $k \geq (\alpha + (1-\alpha)\sqrt{2})/(\sqrt{2}-1)(1-\alpha)$, which completes the proof of the theorem. ∎

When $\alpha = 0$, the class $\mathcal{SL}^*(\alpha)$ reduces to the class $\mathcal{SL}$ which consists of functions f of the form $z + \sum_{n=1}^{\infty} a_n z^n$ satisfying $zf'(z)/f(z) \prec \sqrt{1+z}$ and hence, we have the following corollary to Theorem 6.6:

COROLLARY 6.7. *The class $\mathcal{SL}$ satisfies the following relationships:*

(1) $\mathcal{SL} \subset \mathcal{S}^*$.
(2) $\mathcal{SL} \subset \mathcal{M}(\beta)$ for $\beta \geq \sqrt{2}$.
(3) $k-\mathcal{S}^* \subset \mathcal{SL}$ for $k \geq (\sqrt{2})/(\sqrt{2}-1)$.

The constants obtained here are the best possible.

Assume $-1 \leq B < A \leq 1$. For this range, we define the class $\mathcal{P}[A,B]$ consisting of all analytic functions p of the form $p(z) = 1 + c_1z + c_2z^2 + \cdots$ which satisfy $p(z) \prec (1+Az)/(1+Bz)$ where $z \in \mathbb{D}$. Note that, when $A = 1-2\alpha$ and $B = -1$, the class $\mathcal{P}[A,B]$ reduces to the class $\mathcal{P}(\alpha)$ $(0 \leq \alpha < 1)$. Whereas, replacing $A = B = 0$, the class $\mathcal{P}[A,B]$ becomes the renowned Carathéodory class $\mathcal{P}$. Our next result determines the conditions on the parameters A and B such that the class $\mathcal{S}^*[A,B]$ becomes a subclass of $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$ respectively. In order to prove our results, the following lemma will be needed:

LEMMA 6.8. *If the function* $p \in \mathcal{P}[A,B]$, *then*

$$\Big|p(z) - \frac{1-ABr^2}{1-B^2r^2}\Big| \leq \frac{(A-B)r}{1-B^2r^2} \quad (|z| = r < 1).$$

Moreover, if $p(z) \in \mathcal{P}(\alpha)$, *then*

$$\Big|p(z) - \frac{1+(1-2\alpha)r^2}{1-r^2}\Big| \leq \frac{2(1-\alpha)r}{1-r^2},$$

and

$$\Big|\frac{zp'(z)}{p(z)}\Big| \leq \frac{2r(1-\alpha)}{(1-r)(1+(1-2\alpha)r)} \quad (|z| = r < 1).$$

THEOREM 6.9. *Let* $-1 \leq B < A \leq 1$ *and either*

(1) $2(1-B^2)(\alpha e + (1-\alpha)) \leq 2(1-AB)e \leq (1-B^2)((e^2+1)(1-\alpha) + 2\alpha e)$ *and* $1 - B \leq (1-A)e/(1+\alpha(e-1))$; *or*
(2) $(2\alpha e + (1-\alpha)(1+e^2))(1-B^2) \leq 2(1-AB) \leq 2(1-B^2)(\alpha e + (1-\alpha)e^2)$ *and* $1 + A \leq (1+B)(\alpha + (1-\alpha)e)$;

Then $\mathcal{S}^*[A,B] \subset \mathcal{S}^*_{\alpha,e}$.

PROOF. Let the function $f \in \mathcal{S}^*[A,B]$ which implies $zf'(z)/f(z) \in \mathcal{P}[A,B]$. Next, using the above stated Lemma 6.8, we get,

$$\Big|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\Big| < \frac{A-B}{1-B^2}. \tag{6.8}$$

Assume that both the condition in part (1) hold and let $a = (1-AB)/(1-B^2)$. Multiplying the inequality $1 - B \leq (1-A)e/(1+\alpha(e-1))$ by the constant quantity $1 + B$ on both sides, we obtain

$$(A-B)\frac{e}{(1+\alpha(e-1))} \leq (1-AB)\frac{e}{(1+\alpha(e-1))} - (1-B^2).$$

On dividing by $(1-B^2)e/(1+\alpha(e-1)$, we get

$$\frac{A-B}{1-B^2} \leq \frac{1-AB}{1-B^2} - \frac{1+\alpha(e-1)}{e},$$

which is equivalent to

$$\frac{A-B}{1-B^2} \leq (a-\alpha) - \frac{1-\alpha}{e}.$$

Proceeding as above, the condition

$$2(1-B^2)\big(\alpha e + (1-\alpha)\big) \leq 2(1-AB)e \leq (1-B^2)\big((e^2+1)(1-\alpha)+2\alpha e\big)$$

is equivalent to

$$\alpha + (1-\alpha)\frac{1}{e} \leq a \leq \alpha + (1-\alpha)\frac{(e+e^{-1})}{2}.$$

From (6.8), it is clear that the values of $w = zf'(z)/f(z)$ lie in the following disk $|w - a| < r_a$, where $r_a = (a-\alpha) - (1-\alpha)/e$ and $\alpha + (1-\alpha)e^{-1} \leq a \leq \alpha + (1-\alpha)(e + e^{-1})/2$. Hence, $f \in \mathcal{S}^*_{\alpha,e}$ by Lemma 6.3. Following a similar argument we can see that $f \in \mathcal{S}^*_{\alpha,e}$ if the condition (2) holds and hence, the proof has been omitted. ∎

REMARK 6.10. When $\alpha = 0$, Theorem 6.9 reduces to [54, Theorem 2.2].

THEOREM 6.11. *Let* $-1 \leq B < A \leq 1$ *and either*

(1) $\big(2\sqrt{2}(1-\alpha)+3\alpha\big)(1-B^2) < 3(1-AB) \leq 3(\alpha+(1-\alpha)\sqrt{2})(1-B^2)$ *and* $1+A \leq (1+B)(\alpha+(1-\alpha)\sqrt{2})$*; or*

(2) $3\alpha(1-B^2) \leq 3(1-AB) < \big(2\sqrt{2}(1-\alpha)+3\alpha\big)(1-B^2)$ *and* $(A-B)^2 + (1-B^2)^2 \leq (1-B^2)(1-\alpha)\sqrt{\big((1-B^2)(1-2\alpha)+(1-AB)\big)B(A-B) + (1-AB)^2 + 2\alpha(1-B^2)B(A-B)}$.

Then the class $\mathcal{S}^*[A,B] \subset \mathcal{SL}^*(\alpha)$.

PROOF. Let the function $f \in \mathcal{S}^*[A,B]$ which implies $zf'(z)/f(z)$ is in $\mathcal{P}[A,B]$. Thus Lemma 6.8 gives

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}. \tag{6.9}$$

Let both the conditions in (1) hold and assume $a = (1-AB)/(1-B^2)$. On multiplying the inequality $1+A \leq (1+B)(\alpha+(1-\alpha)\sqrt{2})$ by $1-B$ on both sides and rewriting, we obtain

$$(A-B) \le \sqrt{2}(1-\alpha)(1-B^2) - (1-AB-\alpha(1-B^2)).$$

On dividing by $(1-B^2)$, we get

$$\frac{A-B}{1-B^2} \le \sqrt{2}(1-\alpha) - \Big(\frac{1-AB}{1-B^2} - \alpha\Big),$$

which is equivalent to

$$\frac{A-B}{1-B^2} \le \sqrt{2}(1-\alpha) - (a-\alpha).$$

Proceeding as above, the condition

$$\big(2\sqrt{2}(1-\alpha)+3\alpha\big)(1-B^2) < 3(1-AB) \le 3\big(\alpha+(1-\alpha)\sqrt{2}\big)(1-B^2)$$

is equivalent to

$$\frac{3\alpha+2\sqrt{2}(1-\alpha)}{3} \le a < \alpha+(1-\alpha)\sqrt{2}.$$

From equation (6.9), it follows that the values of $w = zf'(z)/f(z)$ lies in the disk $|w-a| < r_a$, where $r_a = \sqrt{2}(1-\alpha)-(a-\alpha)$ and $3\alpha+2\sqrt{2}(1-\alpha)/3 \le a < \alpha+(1-\alpha)\sqrt{2}$. Hence, $f \in \mathcal{S}^*_{\alpha,e}$ by Lemma 6.3. Following a similar argument we can see that $f \in \mathcal{S}^*_{\alpha,e}$ if the condition (2) holds and hence, the proof has been omitted. ∎

When $\alpha = 0$, we arrive at the following corollary for the class $\mathcal{SL}$:

COROLLARY 6.12. *Let* $-1 \le B < A \le 1$ *and either*

(1) $2\sqrt{2}(1-B^2) < 3(1-AB) \le 3\sqrt{2}(1-B^2)$ *and* $1+A \le \sqrt{2}(1+B)$*; or*
(2) $0 \le 3(1-AB) < 2\sqrt{2}(1-B^2)$ *and* $(A-B)^2+(1-B^2)^2 \le (1-AB)^2 + (1-B^2)\sqrt{(2-B^2-AB)B(A-B)}.$

Then $\mathcal{S}^*[A,B] \subset \mathcal{SL}$.

6.4. RADIUS PROBLEMS

In the first two theorems, we determine the sharp $\mathcal{S}^*(\beta)$ $(0<\beta<1)$, $\mathcal{M}(\beta)$ $(\beta>1)$ and $k-\mathcal{S}^*$ $(k\geq 0)$ radii for the functions in the class $\mathcal{S}^*_{\alpha,e}$ and $\mathcal{SL}^*(\alpha)$. By Theorem 6.4, it can obviously be seen that $\mathcal{R}_{\mathcal{S}^*(\beta)}(\mathcal{S}^*_{\alpha,e})=\mathcal{R}_{\mathcal{M}(\beta)}(\mathcal{S}^*_{\alpha,e})=1$ for $0\leq\beta\leq\alpha+(1-\alpha)/e$, and $\beta\geq\alpha+(1-\alpha)e$. Also Theorem 6.6 clearly tells us that $\mathcal{R}_{\mathcal{S}^*(\beta)}(\mathcal{SL}^*(\alpha))=\mathcal{R}_{\mathcal{M}(\beta)}(\mathcal{SL}^*(\alpha))=1$ in the domains $0\leq\beta\leq\alpha$, and $\beta\geq\alpha+(1-\alpha)\sqrt{2}$ respectively.

THEOREM 6.13. *Let $f\in\mathcal{S}^*_{\alpha,e}$. Then the following hold:*

(1) If $\alpha+(1-\alpha)/e\leq\beta<1$, then f is starlike of order β in $|z|<-\log\big((\beta-\alpha)/(1-\alpha)\big)$;
(2) If $1<\beta\leq\alpha+(1-\alpha)e$, then $f\in\mathcal{M}(\beta)$ in $|z|<\log\big((\beta-\alpha)/(1-\alpha)\big)$;
(3) If $k>0$, then f is k-starlike in $|z|<\log((1-\alpha)(k+1)/(k-\alpha(k+1)))$. In particular, f is parabolic starlike in $|z|<\log\big(2(1-\alpha)/(1-2\alpha)\big)$.

The results are sharp.

PROOF. Since $f\in\mathcal{S}^*_{\alpha,e}$ implies $zf'(z)/f(z)\prec\alpha+(1-\alpha)e^z$. Thus, by using Lemma 6.1, we see that

$$\alpha+(1-\alpha)e^{-r}\leq\operatorname{Re}\frac{zf'(z)}{f(z)}\leq\alpha+(1-\alpha)e^{r},\quad |z|=r<1$$

which clearly proves the first two parts of the theorem. The function k given by (6.1) proves that the constants obtained in the first two cases are best possible.

For the third part, we see that for the function f to be k-starlike in $|z|<r$, it must satisfy $\operatorname{Re}(\alpha+(1-\alpha)e^{w(z)})>k|\alpha+(1-\alpha)e^{w(z)}-1|$. Obviously $\operatorname{Re}(\alpha+(1-\alpha)e^{w(z)})>(\alpha+(1-\alpha)e^{-r})$ and $|\alpha+(1-\alpha)e^{w(z)}-1|<1-(\alpha+(1-\alpha)e^{-r})$. Thus, in order to prove our result, it is sufficient to prove that the inequality $(\alpha+(1-\alpha)e^{-r})>k(1-(\alpha+(1-\alpha)e^{-r}))$ holds. On solving the above inequality for r, we immediately obtain $r<\log\big((1-\alpha)(k+1)/(k-\alpha(k+1))\big)$. Sharpness follows by considering the function

k defined in (6.1) and for $z_0 = -\log\big((1-\alpha)(k+1)/(k-\alpha(k+1))\big)$,

$$\begin{aligned}\operatorname{Re}\frac{z_0h'(z_0)}{h(z_0)} &= \operatorname{Re}\big(\alpha+(1-\alpha)e^{z_0}\big)\\ &= \alpha + \frac{(k-\alpha(k+1))}{(k+1)}\\ &= \frac{k}{k+1} = k|1-(\alpha+(1-\alpha)e^{z_0})|\\ &= k\Big|1-\frac{z_0h'(z_0)}{h(z_0)}\Big|.\end{aligned}$$

This completes the proof of the theorem. ∎

REMARK 6.14. When $\alpha = 0$, Theorem 6.13 reduces to [54, Theorem 3.1].

THEOREM 6.15. *Let $f \in \mathcal{SL}^*(\alpha)$. Then the following hold:*

(1) If $\alpha \leq \beta < 1$, then the function f is starlike of order β for $|z| < 1-\big((\beta-\alpha)/(1-\alpha)\big)^2$;
(2) If $1 < \beta \leq \alpha+(1-\alpha)\sqrt{2}$, then the function $f \in \mathcal{M}(\beta)$ for $|z| < \big((\beta-\alpha)/(1-\alpha)\big)^2-1$;
(3) If $k > 0$, then the function f is k-starlike for $|z| < \big((1+2k)-2\alpha(1+k)\big)/\big((1-\alpha)^2(1+k)^2\big)$. In particular, the function f is parabolic starlike for $|z| < \big(3(1-2\alpha)\big)/4(1-\alpha)^2$.

The results are sharp.

PROOF. Since $f \in \mathcal{SL}^*(\alpha)$ implies $zf'(z)/f(z) \prec \alpha+(1-\alpha)\sqrt{1+z}$. Thus, by using Lemma 6.1, we see that

$$\alpha+(1-\alpha)\sqrt{1-r} \leq \operatorname{Re}\frac{zf'(z)}{f(z)} \leq \alpha+(1-\alpha)\sqrt{1+r}, \quad |z| = r < 1$$

which clearly proves the first two parts of the theorem. The constants are best possible which can be seen considering the function obtained in (6.2).

For the third part, we see that for the function f to be k-starlike in $|z| < r$, it must satisfy $\operatorname{Re}\big(\alpha+(1-\alpha)\sqrt{1+w(z)}\big) > k|\alpha+(1-\alpha)\sqrt{1+w(z)}-1|$. Obviously $\operatorname{Re}\big(\alpha+(1-\alpha)\sqrt{1+w(z)}\big) > \alpha+(1-\alpha)\sqrt{1-r}$ and $|\alpha+(1-\alpha)\sqrt{1+w(z)}-1| < 1-\big(\alpha+$

$(1-\alpha)\sqrt{1-r}\big)$. Thus, in order to prove our result, it is sufficient to prove that the inequality $\alpha + (1-\alpha)\sqrt{1-r} > k\big(1 - (\alpha + (1-\alpha)\sqrt{1-r})\big)$ holds. On solving the above inequality for r, we immediately obtain $r < \big((1+2k) - 2\alpha(1+k)\big)/(1-\alpha)^2(1+k)^2$. In view of the function $k(z)$ defined in (6.2) and for $z_0 = -\big((1+2k) - 2\alpha(1+k)\big)/(1-\alpha)^2(1+k)^2\big)$, we have

$$\begin{aligned}
\operatorname{Re}\frac{z_0 k'(z_0)}{k(z_0)} &= \operatorname{Re}\left(\alpha + (1-\alpha)\sqrt{1+z_0}\right)\\
&= \operatorname{Re}\left(\alpha + (1-\alpha)\sqrt{1 - \frac{1+2k-2\alpha(1+k)}{(1-\alpha)^2(1+k)^2}}\right) = \frac{k}{k+1}\\
&= k\big|1 - \big(\alpha + (1-\alpha)\sqrt{1+z}\big)\big|\\
&= k\left|1 - \frac{z_0 k'(z_0)}{k(z_0)}\right|,
\end{aligned}$$

which completes the proof. ∎

COROLLARY 6.16. *Let $f \in \mathcal{SL}$. Then the following hold:*

1. *If $0 \le \beta < 1$, then the function f is starlike of order β for $|z| < 1 - \beta^2$;*
2. *If $1 < \beta \le \sqrt{2}$, then the function $f \in \mathcal{M}(\beta)$ for $|z| < \beta^2 - 1$;*
3. *If $k > 0$, then the function f is k-starlike for $|z| < (1+2k)/(1+k)^2$. In particular, the function f is parabolic starlike for $|z| < 3/4$.*

The results are sharp.

Let us now discuss a few subclasses of $\mathcal{A}$. Let the class $\mathcal{W}$ consist of analytic functions f which satisfy $f(z)/z \in \mathcal{P}$. Let $\mathcal{F}_1$ be the class consisting of analytic functions $f \in \mathcal{A}$ which satisfy $f/g \in \mathcal{P}$ where $g \in \mathcal{W}$ and let $\mathcal{F}_2$ be the class of all analytic functions $f \in \mathcal{A}$ which satisfy the following inequality

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 \quad (z \in \mathbb{D}) \tag{6.10}$$

where $g \in \mathcal{W}$. The next theorem determines the sharp $\mathcal{S}^*_{\alpha,e}$-radius for the classes $\mathcal{S}^*[A,B]$, $\mathcal{W}$, $\mathcal{F}_1$ and $\mathcal{F}_2$.

THEOREM 6.17. *(1) Let $0 \leq B < A \leq 1$. Then the $\mathcal{S}^*_{\alpha,e}$-radius for the class $\mathcal{S}^*[A,B]$ is given by*

$$\mathcal{R}_{\mathcal{S}^*_{\alpha,e}}(\mathcal{S}^*[A,B]) = min\left\{1, \frac{(1-\alpha)(1-e)}{B-Ae-B\alpha(1-e)}\right\}.$$

(2) Let $-1 \leq B < A \leq 1$, with $B < 0$. Let

$$R_1 = \frac{(e-1)\sqrt{1-\alpha}}{\sqrt{B^2((1-\alpha)(1+e^2)+2e\alpha)-2ABe}}, \qquad R_2 = \frac{(e-1)(1-\alpha)}{e(A-B\alpha)-B(1-\alpha)}$$

and

$$R_3 = \frac{(1-\alpha)(e-1)}{A-B\alpha-Be(1-\alpha)}.$$

Then the $\mathcal{S}^*_{\alpha,e}$-radius for the class $\mathcal{S}^*[A,B]$ is given by

$$\mathcal{R}_{\mathcal{S}^*_{\alpha,e}}(\mathcal{S}^*[A,B]) = \begin{cases} R_2, & R_2 \leq R_1; \\ R_3, & R_2 > R_1. \end{cases}$$

(3) The $\mathcal{S}^*_{\alpha,e}$-radius for the class $\mathcal{W}$ is given by

$$\mathcal{R}_{\mathcal{S}^*_{\alpha,e}}(\mathcal{W}) = \frac{(e-1)(1-\alpha)}{e+\sqrt{e^2+(e-1)^2(1-\alpha)^2}}.$$

(4) The $\mathcal{S}^*_{\alpha,e}$-radius for the class $\mathcal{F}_1$ is

$$\mathcal{R}_{\mathcal{S}^*_{\alpha,e}}(\mathcal{F}_1) = \frac{(e-1)(1-\alpha)}{2e+\sqrt{4e^2+(e-1)^2(1-\alpha)^2}}.$$

(5) The $\mathcal{S}^*_{\alpha,e}$-radius for the class $\mathcal{F}_2$ is given by

$$\mathcal{R}_{\mathcal{S}^*_{\alpha,e}}(\mathcal{F}_2) = \frac{2(e-1)(1-\alpha)}{3e+\sqrt{9e^2+4(e-1)(1-\alpha)(e(2-\alpha)-(1-\alpha))}}.$$

PROOF. (1) Let $f \in \mathcal{S}^*[A,B]$. Then, by definition $zf'(z)/f(z)$ belongs to the class $\mathcal{P}[A,B]$. Therefore, using Lemma 6.8 we get:

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad |z| = r < 1.$$

Since $B \geq 0$, clearly $a = (1-ABr^2)/(1-B^2r^2) \leq 1$. Furthermore, using Lemma 6.3, the function f satisfies $|\log\left((zf'(z)/f(z)-\alpha)/(1-\alpha)\right)| \leq 1$

provided

$$\frac{(A-B)r}{1-B^2r^2} \leq \frac{1-ABr^2}{1-B^2r^2} - \alpha - \frac{1-\alpha}{e},$$

which after a little simplification implies:

$$r \leq \frac{(1-\alpha)(1-e)}{B-Ae-B\alpha(1-e)}.$$

The result is sharp for the function given by

$$f(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & B \neq 0; \\ ze^{Az}, & B = 0. \end{cases} \tag{6.11}$$

Clearly the function $f \in \mathcal{S}^*[A,B]$ and thus

$$\left| \log \frac{\frac{z_0 f'(z_0)}{f(z_0)} - \alpha}{1-\alpha} \right| = \left| \log \frac{\frac{1+Az_0}{1+Bz_0} - \alpha}{1-\alpha} \right|$$

Putting $z_0 = (1-\alpha)(1-e)/(B-Ae-B\alpha(1-e))$ in the above expression, we have

$$\left| \log \frac{\frac{z_0 f'(z_0)}{f(z_0)} - \alpha}{1-\alpha} \right| = |\log(1/e)| = 1,$$

which completes the proof of the first part of the theorem.

(2) Let $f \in \mathcal{S}^*[A,B]$. Then on using Lemma 6.8, we see that $w = zf'(z)/f(z)$ lies in the disk $|w-a| \leq R$, where

$$a := \frac{1-ABr^2}{1-B^2r^2} > 1 \qquad \text{and} \qquad R := \frac{(A-B)r}{1-B^2r^2}.$$

We next determine the numbers R_1, R_2 and R_3 in the following manner: $r \leq R_1$ if and only if $a \leq \alpha + (1-\alpha)(e+e^{-1})/2$, $r \leq R_2$ if and only if $R \leq a - \alpha - (1-\alpha)/e$ and $r \leq R_3$ if and only if $R \leq e(1-\alpha) + \alpha - a$.

Let us now suppose that $R_2 \leq R_1$. Since $r \leq R_1$ is equivalent to $a \leq \alpha + (1-\alpha)(e+e^{-1})/2$, for $0 \leq r \leq R_2$, it follows that $a \leq \alpha + (1-\alpha)(e+e^{-1})/2$. From Lemma 6.3, the $\mathcal{S}^*_{\alpha,e}$-radius satisfies the inequality $R \leq a - \alpha - \frac{(1-\alpha)}{e}$. This shows that $f \in \mathcal{S}^*_{\alpha,e}$ in $|z| \leq R_2$.

We next assume that $R_2 > R_1$. In this case, since $r \geq R_1$ if and only if $a \geq \alpha + (1-\alpha)(e+e^{-1})/2$, for $r = R_2$, we have $a \geq \alpha + (1-\alpha)(e+e^{-1})/2$. Lemma 6.3 shows that $f \in \mathcal{S}^*_{\alpha,e}$ in $|z| \leq r$ if $R \leq e(1-\alpha)+\alpha-a$, or equivalently, $r \leq R_3$.

(3) Next, let $f \in \mathcal{W}$. Then, by definition, the function $g(z) = f(z)/z \in \mathcal{P}$ and thereby using Lemma 6.8, we can easily deduce that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{2r}{1-r^2}. \tag{6.12}$$

Further, by making use of Lemma 6.3, the disk (6.12) lies inside the disk $\left|\log\left((w-\alpha)/(1-\alpha)\right)\right| \leq 1$ if

$$\frac{2r}{1-r^2} \leq 1-\alpha-\frac{(1-\alpha)}{e},$$

which yields $r \leq (e-1)(1-\alpha)/(e+\sqrt{e^2+(e-1)^2(1-\alpha)^2})$. The result is sharp for the function given by $f(z) = z(1+z)/(1-z)$ Clearly the function $f \in \mathcal{W}$ and thus

$$\left|\log\frac{\frac{z_0f'(z_0)}{f(z_0)}-\alpha}{1-\alpha}\right| = \left|\log\frac{\frac{1-2z_0-z_0^2}{1-z_0^2}-\alpha}{1-\alpha}\right|$$

Putting $z_0 = (e-1)(1-\alpha)/(e+\sqrt{e^2+(e-1)^2(1-\alpha)^2})$ in the above expression, we have

$$\left|\log\frac{\frac{z_0f'(z_0)}{f(z_0)}-\alpha}{1-\alpha}\right| = |\log(1/e)| = 1.$$

(4) Let $f \in \mathcal{F}_1$ and define $p, q : \mathbb{D} \to \mathbb{C}$ by $p(z) = g(z)/z$ and $q(z) = f(z)/g(z)$. Then $p, q \in \mathcal{P}$ and using Lemma 6.8, it follows that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zq'(z)}{q(z)}\right| \leq \frac{4r}{1-r^2} \quad (|z| = r).$$

Now, using Lemma 6.3, $f \in \mathcal{S}^*_{\alpha,e}$ provided $4r/1-r^2 \leq 1-\alpha-(1-\alpha)/e$. This immediately implies that $r \leq (e-1)(1-\alpha)/(2e+\sqrt{4e^2+(e-1)^2(1-\alpha)^2})$. The result is sharp for the function given by $f(z) = z(1+z)^2/(1-z)^2$. Clearly the function $f \in \mathcal{F}_1$ and thus

$$\left|\log \frac{\frac{z_0 f'(z_0)}{f(z_0)} - \alpha}{1-\alpha}\right| = \left|\log \frac{\frac{1-4z_0-z_0^2}{1-z_0^2} - \alpha}{1-\alpha}\right|$$

Putting $z_0 = (e-1)(1-\alpha)/(2e + \sqrt{4e^2 + (e-1)^2(1-\alpha)^2})$ in the above expression, we have

$$\left|\log \frac{\frac{z_0 f'(z_0)}{f(z_0)} - \alpha}{1-\alpha}\right| = |\log(1/e)| = 1.$$

(5) Let $f \in \mathcal{F}_2$ and define $p, q : \mathbb{D} \to \mathbb{C}$ by $p(z) = g(z)/z$ and $q(z) = g(z)/f(z)$. Since the inequality (6.10) is equivalent to $\operatorname{Re} g(z)/f(z) > 1/2$, therefore $p \in \mathcal{P}$ and $q \in \mathcal{P}(1/2)$. The following identity holds obviously:

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} - \frac{zq'(z)}{q(z)}$$

and applying Lemma 6.8 in this, we obtain

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{r(3+r)}{1-r^2}.$$

And finally, using Lemma 6.3, the function f satisfies $|\log((zf'(z)/f(z) - \alpha)/(1-\alpha))| \leq 1$ provided $r(3+r)/1-r^2 \leq (1-\alpha)(1-e^{-1})$, which on solving for r yields

$$r \leq \frac{2(e-1)(1-\alpha)}{3e + \sqrt{9e^2 + 4(e-1)(1-\alpha)\big(e(2-\alpha) - (1-\alpha)\big)}}.$$

The result is sharp for the function given by $f(z) = z(1+z)^2/(1-z)$ Clearly the function $f \in \mathcal{F}_1$ and thus

$$\left|\log \frac{\frac{z_0 f'(z_0)}{f(z_0)} - \alpha}{1-\alpha}\right| = \left|\log \frac{\frac{1-3z_0-2z_0^2}{1-z_0^2} - \alpha}{1-\alpha}\right|$$

Putting $z_0 = 2(e-1)(1-\alpha)/(3e+\sqrt{9e^2+4(e-1)(1-\alpha)(e(2-\alpha)-(1-\alpha))})$ in the above expression, we have

$$\left|\log \frac{\frac{z_0 f'(z_0)}{f(z_0)} - \alpha}{1-\alpha}\right| = |\log(1/e)| = 1. \quad \blacksquare$$

REMARK 6.18. When $\alpha = 0$, Theorem 6.17 reduces to [54, Theorem 3.3-3.7].

COROLLARY 6.19. *The $\mathcal{S}^*_{\alpha,e}$-radius for the class $\mathcal{K}$ is*

$$\mathcal{R}_{\mathcal{S}^*_{\alpha,e}}(\mathcal{K}) = \frac{(1-\alpha)(e-1)}{\alpha+(1-\alpha)e}.$$

PROOF. Marx Strohhäcker theorem states that $\mathcal{K} \subset \mathcal{S}^*(1/2)$. Also, it can be seen that $\mathcal{S}^*(1/2) = \mathcal{S}^*[0,-1]$. Therefore, using Theorem 6.17(2), the $\mathcal{S}^*_{\alpha,e}$-radius for the class $\mathcal{K}$ will be at least $(1-\alpha)(e-1)/(\alpha+(1-\alpha)e)$. $\blacksquare$

REMARK 6.20. When $\alpha = 0$, Corollary 6.19 reduces to [54, Corollary 3.1].

THEOREM 6.21. *(1) Let $-1 < B < A \leq 1$ and $B \leq 0$. Then the $\mathcal{SL}^*(\alpha)$- radius for the class $\mathcal{S}^*[A,B]$ is*

$$R_{\mathcal{SL}^*(\oslash)}(\mathcal{S}^*[A,B]) = \min\left\{1, \frac{(\sqrt{2}-1)(1-\alpha)}{(A-B\alpha)-\sqrt{2}B(1-\alpha)}\right\}.$$

(2) Let $0 < B < A \leq 1$, with $B < 0$. Let

$$R_1 = \left(\frac{2\sqrt{2}(1-\alpha)-3(1-\alpha)}{2\sqrt{2}B^2(1-\alpha)-3B(A-B\alpha)}\right)^{1/2},$$

and R_2 be the largest number as in

$$(A-B)^2r^2 + (1-B^2r^2)^2 - (1-ABr^2)^2 - (1-B^2r^2)\big(2\alpha(AB-B^2)r^2 + (1-\alpha)\sqrt{(AB-B^2)r^2(1-B^2r^2-ABr^2)}\big) \leq 0,$$

for all $0 \leq r \leq R_2$ and R_3 is given by

$$R_3 = \frac{(\sqrt{2}-1)(1-\alpha)}{(A-B\alpha)-\sqrt{2}B(1-\alpha)}.$$

Then the $\mathcal{SL}^*(\alpha)$-radius for the class $\mathcal{S}^*[A,B]$ is given by

$$\mathcal{R}_{\mathcal{SL}^*(\alpha)}(\mathcal{S}^*[A,B]) = \begin{cases} R_2, & R_2 \leq R_1; \\ R_3, & R_2 > R_1. \end{cases}$$

PROOF. (1) Let $f \in \mathcal{S}^*[A,B]$. Then, by definition $zf'(z)/f(z)$ belongs to the class $\mathcal{P}[A,B]$. Therefore, using Lemma 6.8 we get:

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad |z| = r < 1.$$

Since $B \leq 0$, clearly $a = (1-ABr^2)/(1-B^2r^2) \geq 1$. Furthermore, using Lemma 6.3, the function f satisfies

$$\left|\left(\frac{\frac{zf'(z)}{f(z)} - \alpha}{1-\alpha}\right)^2 - 1\right| < 1$$

provided

$$\frac{(A-B)r}{1-B^2r^2} < \sqrt{2}(1-\alpha) - \Big(\frac{1-ABr^2}{1-B^2r^2} - \alpha\Big),$$

that is

$$(\sqrt{2}(1-\alpha)B - A + \alpha B)Br^2 + (A-B)r - (\sqrt{2}-1)(1-\alpha) < 0$$

which after simplification yields:

$$r \leq \frac{(\sqrt{2}-1)(1-\alpha)}{(A-B\alpha) - \sqrt{2}B(1-\alpha)}.$$

The result is sharp for the function given by

$$f(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & B \neq 0; \\ ze^{Az}, & B = 0. \end{cases} \tag{6.13}$$

The function $f \in \mathcal{S}^*[A,B]$ and thus at the point
$z_0 = (\sqrt{2}-1)(1-\alpha)/(A-B\alpha) - \sqrt{2}B(1-\alpha)$, we have

$$\Big|\Big(\frac{\frac{z_0f'(z_0)}{f(z_0)} - \alpha}{1-\alpha}\Big)^2 - 1\Big| = \Big|\Big(\frac{\frac{1+Az_0}{1+Bz_0} - \alpha}{1-\alpha} - 1\Big)^2 - 1\Big| = |(\sqrt{2})^2 - 1| = 1$$

which completes the proof of the first part of the theorem.

(2) Let $f \in \mathcal{S}^*[A,B]$. Then on using Lemma 6.8, we see that $w = zf'(z)/f(z)$ lies in the disk $|w-a| \leq R$, where

$$a := \frac{1-ABr^2}{1-B^2r^2} > 1 \qquad \text{and} \qquad R := \frac{(A-B)r}{1-B^2r^2}.$$

We next determine the numbers R_1, R_2 and R_3 in the following manner: $r \leq R_1$ if and only if $a \leq (3\alpha + 2\sqrt{2}(1-\alpha))/3$, $r \leq R_2$ if and only if $R \leq \big((1-a)(2\alpha-1-a)+\sqrt{(1-a)(1+a-2\alpha)}(1-\alpha)\big)^{1/2}$ and $r \leq R_3$ if and only if $R \leq \sqrt{2}(1-\alpha)-(a-\alpha)$.

Let us now suppose that $R_2 \leq R_1$. Since $r \leq R_1$ is equivalent to $a \leq (3\alpha + 2\sqrt{2}(1-\alpha))/3$, for $0 \leq r \leq R_2$, it follows that $a \leq (3\alpha + 2\sqrt{2}(1-\alpha))/3$. From Lemma 6.3, the $\mathcal{SL}^*(\alpha)$-radius satisfies the inequality $R \leq \big((1-a)(2\alpha-1-a)+\sqrt{(1-a)(1+a-2\alpha)}(1-\alpha)\big)^{1/2}$. This shows that $f \in \mathcal{SL}^*(\alpha)$ in $|z| \leq R_2$.

Let us now consider the case where $R_2 > R_1$. Here, since $r \geq R_1$ if and only if $a \geq (3\alpha + 2\sqrt{2}(1-\alpha))/3$, for $r = R_2$, we have $a \geq (3\alpha + 2\sqrt{2}(1-\alpha))/3$. Lemma 6.3 shows that $f \in \mathcal{SL}^*(\alpha)$ in $|z| \leq r$ if $R \leq \sqrt{2}(1-\alpha)-(a-\alpha)$, or equivalently, $r \leq R_3$. ∎

When $\alpha = 0$, we get the following result for the functions in the class $\mathcal{SL}$ as a corollary to Theorem 6.21.

COROLLARY 6.22. *(1) Let $-1 < B < A \leq 1$ and $B \leq 0$. The $\mathcal{SL}$- radius for the class $\mathcal{S}^*[A,B]$ is given as follows*

$$R_{\mathcal{SL}}(\mathcal{S}^*[A,B]) = \min\left\{1, \frac{(\sqrt{2}-1)}{A-\sqrt{2}B}\right\}.$$

(2) Let $0 < B < A \leq 1$, with $B < 0$. Let

$$R_1 = \left(\frac{2\sqrt{2}-3}{2\sqrt{2}B^2-3AB}\right)^{1/2},$$

and R_2 be the largest number as in

$$(A-B)^2r^2+(1-B^2r^2)^2-(1-ABr^2)^2-(1-B^2r^2)\big(r^2 + \sqrt{(AB-B^2)r^2(1-B^2r^2-ABr^2)}\big)\leq 0,$$

for all $0\leq r\leq R_2$ and R_3 is given by

$$R_3=\frac{(\sqrt{2}-1)(1-\alpha)}{A-\sqrt{2}B}.$$

Then, the $\mathcal{SL}$-radius for the class $\mathcal{S}^*[A,B]$ is given by

$$\mathcal{R}_{\mathcal{SL}}(\mathcal{S}^*[A,B])=\begin{cases} R_2, & R_2\leq R_1;\\ R_3, & R_2>R_1.\end{cases}$$

6.5. Coefficient Estimates

THEOREM 6.23. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}^*_{\alpha,e}$, then*

$$\sum_{n=2}^{\infty}\big(n^2-(\alpha+(1-\alpha)e)^2\big)|a_n|^2\leq\big(\alpha+(1-\alpha)e\big)^2-1.$$

PROOF. Let the function $f\in\mathcal{S}^*_{\alpha,e}$. This implies that $zf'(z)/f(z)=\alpha+(1-\alpha)e^{\omega(z)}$, where ω is a function which is analytic in $\mathbb{D}$ and satisfies $\omega(0)=0$ and $|\omega(z)|\leq 1$ where $z\in\mathbb{D}$. It can be easily seen that $f^2(z)=(zf'(z))^2/(\alpha+(1-\alpha)e^{\omega(z)})^2$, and therefore we have,

$$\begin{aligned}2\pi\sum_{n=1}^{\infty}|a_n|^2r^{2n}&=\int_0^{2\pi}|f(re^{\iota\theta})|^2d\theta\\ &=\int_0^{2\pi}\frac{|re^{\iota\theta}f'(re^{\iota\theta})|^2}{(\alpha+(1-\alpha)e^{\omega(z)})^2}d\theta\\ &\geq\frac{1}{(\alpha+(1-\alpha)e)^2}\int_0^{2\pi}|re^{\iota\theta}f'(re^{\iota\theta})|^2d\theta\\ &=\frac{2\pi}{(\alpha+(1-\alpha)e)^2}\sum_{n=1}^{\infty}n^2|a_n|^2r^{2n}\end{aligned}$$

where $0 < r < 1$ and $a_1 = 1$. Thus,

$$\sum_{n=1}^{\infty} \left(n^2 - (\alpha + (1-\alpha)e)^2\right)|a_n|^2 r^{2n} \leq 0.$$

On letting $r \to 1^-$, we obtain the desired result. ∎

REMARK 6.24. When $\alpha = 0$, Theorem 6.23 reduces to [54, Theorem 2.5].

THEOREM 6.25. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{SL}^*(\alpha)$, then*

$$\sum_{n=2}^{\infty} \left(n^2 - (\alpha + (1-\alpha)\sqrt{2})^2\right)|a_n|^2 \leq (\alpha + (1-\alpha)\sqrt{2})^2 - 1. \tag{6.14}$$

PROOF. Let the funtion $f \in \mathcal{SL}^*(\alpha)$. This implies that $zf'(z)/f(z) = \alpha + (1-\alpha)\sqrt{1+\omega(z)}$, where ω is a Schwarz function in $\mathbb{D}$ satisfying $\omega(0) = 0$ and $|\omega(z)| \leq 1$ for all $z \in \mathbb{D}$. It can be easily seen that $f^2(z) = (zf'(z))^2/(\alpha + (1-\alpha)\sqrt{1+\omega(z)})^2$, and therefore we have,

$$\begin{aligned}
2\pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n} &= \int_0^{2\pi} |f(re^{\iota\theta})|^2 d\theta \\
&= \int_0^{2\pi} \frac{|re^{\iota\theta} f'(re^{\iota\theta})|^2}{(\alpha + (1-\alpha)\sqrt{1+\omega(z)})^2} d\theta \\
&\geq \frac{1}{(\alpha + (1-\alpha)\sqrt{2})^2} \int_0^{2\pi} |re^{\iota\theta} f'(re^{\iota\theta})|^2 d\theta \\
&= \frac{2\pi}{(\alpha + (1-\alpha)\sqrt{2})^2} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}
\end{aligned}$$

where $0 < r < 1$ and $a_1 = 1$. Thus,

$$\sum_{n=1}^{\infty} \left(n^2 - (\alpha + (1-\alpha)\sqrt{2})^2\right)|a_n|^2 r^{2n} \leq 0.$$

On letting $r \to 1^-$, we get the desired result. ∎

When $\alpha = 0$, Theorem 6.25 yields the following corollary.

COROLLARY 6.26. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{SL}$, then*

$$\sum_{n=2}^{\infty} (n^2 - 2)|a_n|^2 \leq 1.$$

Using Lemma 6.3, our next result determines certain condition over the n^{th} coefficient for a special type of function to be in the class $\mathcal{S}^*_{\alpha,e}$.

THEOREM 6.27. *The function given by* $f(z) = z + a_n z^n$ $(n = 2, 3, \cdots)$ *is in the class* $\mathcal{S}^*_{\alpha,e}$ *if and only if it satisfies* $|a_n| \leq (1-\alpha)(e-1)/(e(n-\alpha)-(1-\alpha))$.

PROOF. Since $\mathcal{S}^*_{\alpha,e} \subset \mathcal{S}^*$, implies that $|a_n| \leq 1/n$. In order to obtain the desired result, it can be observed that $w = zf'(z)/f(z) = (1 + na_n z^n - 1)/(1 + a_n z^n - 1)$ maps the open unit disk onto the following disk

$$\left| w - \frac{1 - n|a_n|^2}{1 - |a_n|^2} \right| < \frac{(n-1)|a_n|}{1 - |a_n|^2}. \tag{6.15}$$

Since, $(1 - n|a_n|^2)/(1 - |a_n|^2) \leq 1$, therefore by using Lemma 6.3, the disk (6.15) lies inside $|\log((w-\alpha)/(1-\alpha))| \leq 1$ if and only if

$$\frac{(n-1)|a_n|}{1 - |a_n|^2} \leq \frac{1 - n|a_n|^2}{1 - |a_n|^2} - \alpha - \frac{(1-\alpha)}{e},$$

which on a little simplification immediately yields $|a_n| \leq (1-\alpha)(e-1)/(e(n-\alpha) - (1-\alpha))$. ∎

REMARK 6.28. When $\alpha = 0$, Theorem 6.27 reduces to [54, Theorem 2.6(i)].

Chapter 7

Conclusion and Future Scope

In this final chapter of the thesis, we will conclude the research contributions of this thesis, as well as discuss the directions for future research. In the present work, several very interesting subclasses of univalent functions have been defined using the extremely important concept of subordination and many wonderful problems have been solved, including the various kinds of coefficient estimate problems, the radius problems, obtaining some inclusion relations etc. To begin with, it is a known fact that no bounds are known for the n^{th} coefficients of functions f satisfying $2zf'(z)/(f(z)-f(-z)) \prec \varphi(z)$, except for $n = 2, 3$. We denote the class of such functions by $\mathcal{S}_s^*(\varphi)$. Therefore, the sharp fourth coefficient bound has been estimated for this subclass. Note that, when $\varphi = e^z$, $\sqrt{1+z}$ and $\sqrt{2}-(\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}$, the class $\mathcal{S}_s^*(\varphi)$ reduces to the subclasses $S_s^*(e^z)$, $S_s^*(\sqrt{1+z})$ and $S_s^*\Big(\sqrt{2}-(\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}\Big)$ respectively. For these special subclasses of $\mathcal{S}_s^*(\varphi)$, sharp bounds for the first five coefficients have been obtained. The generalised n^{th} $(n \geq 5)$ coefficient bound for the class $\mathcal{S}_s^*(\varphi)$ is still an open problem as it involves complex computations. It can be attempted as a future task.

We also obtained Fekete Szegö coefficient functional for five important subclasses of analytic functions defined by us namely: $\mathcal{V}_\alpha(\varphi)$, $\mathcal{M}_\alpha(\varphi)$, $\mathcal{L}_\alpha(\varphi)$, $\mathcal{K}_\alpha(\varphi)$ and $\mathcal{T}_\alpha(\varphi)$. The computations here were quite complicated and the results so obtained yield some previously known results. We also obtained the second and the third coefficient bounds for the functions in the above mentioned subclasses as a corollary to our results. It is worth noting the fact that the classes considered here have been studied quite extensively in our thesis in the sense that we also obtained the bound for the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ for these subclasses. For $0 \leq \alpha \leq 1$, these classes reduce to many previously known classes such as the class $\mathcal{S}^*$ and $\mathcal{K}$, etc., thereby yielding many corollaries to our results. Another problem that we incorporated in our research is the third Hankel determinant bound. We studied a couple of quite interesting subclasses namely $\mathcal{M}_\alpha$ and $\mathcal{L}_\alpha$ respectively and obtained the first five coefficient's bounds for them and also the bound for the third Hankel determinant $H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$. For further investigation, the unification of the various classes considered can be done so that the results can be merged together. Higher order Hankel determinants can be investigated in the future. Furthermore, Toeplitz determinants can also be explored.

Motivated by the paper by Kanas and Wisniowska [35], wherein they have given certain sufficient conditions for the function f to be $k-$ uniformly convex, we have also obtained certain necessary and sufficient conditions in terms of the coefficients a_n for the function $f \in \mathcal{T}$ to be in certain subclasses of $\mathcal{T}$, namely $\mathcal{TS}^*[A,B]$, $\mathcal{TC}[A,B]$ and $\mathcal{R}(A,B,\alpha)$ ($\alpha \in \mathbb{R}$). Another extremely important subclass has been studied defined by

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad \mu \in \mathbb{C}.$$

and the necessary and sufficient conditions have been obtained for the functions in the above said class to belong to $\mathcal{S}^*[A,B]$.

Finally, in the concluding chapter of the thesis, the two prime subclasses $\mathcal{S}^*(\alpha + (1-\alpha)e^z)$ and $\mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+z})$ of $\mathcal{A}$, $0 \leq \alpha < 1$ have been studied extensively. Several enriching problems have been solved for these subclasses including the bounds for Fekete Szegö inequality, the bound for the first four coefficients, radius problems, various inclusion relations with the other important subclasses of analytic functions and many interesting coefficient inequalities. An interesting observation is that for $\alpha = 0$, all the results obtained yield the results for the classes $\mathcal{S}_e^*$ and $\mathcal{SL}$ respectively as corollaries to our results. For further study, we can obtain similar results by taking convex combination of univalent functions in place of $\alpha + (1 - \alpha)e^z$. Also, many other radii problems can be explored.

List of Publications

[1] K. Khatter, V. Ravichandran and S. Sivaprasad Kumar, Estimates for initial coefficients of certain starlike functions with respect to symmetric points, *Applied analysis in biological and physical sciences*, 385–395, Springer Proc. Math. Stat., 186, Springer, New Delhi.

[2] K. Khatter, S. K. Lee and S. S. Kumar, Coefficient Bounds for Certain Analytic Functions, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 1, 455–490. (SCIE/ IF: 0.720).

[3] K. Khatter, V. Ravichandran and S. Sivaprasad Kumar, Janowski starlikeness and convexity, Proceedings of the Jangjeon Mathematical Society, **20** (2017), No. 4, 623 – 639, Scopus.

[4] K. Khatter, V. Ravichandran and S. Sivaprasad Kumar, Third Hankel Determinant for subclasses of starlike and convex functions, The Journal of Analysis, Springer 2017, appeared online.

[5] K. Khatter, V. Ravichandran and S. Sivaprasad Kumar, Starlike functions associated with Exponential function and the Lemniscate of Bernoulli, Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemticas, appeared online, 2017.(SCIE/ IF: 0.690).

References

[1] L. V. Ahlfors, Complex Analysis, 3rd ed., Mc-Graw-Hill: New York, (1979)

[2] N. M. Alarifi, R. M. Ali and V. Ravichandran, Best bounds for the second Hankel determinant of the kth-root transform of analytic functions, Filomat, to appear.

[3] R. M. Ali, Coefficients of the inverse of strongly starlike functions, Bull. Malays. Math. Sci. Soc. (2) **26** (2003), no. 1, 63–71.

[4] R. M. Ali, Starlikeness associated with parabolic regions, Int. J. Math. Math. Sci. **2005**, no. 4, 561–570.

[5] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, The Fekete-Szegő coefficient functional for transforms of analytic functions, Bull. Iranian Math. Soc. **35** (2009), no. 2, 119–142, 276.

[6] R. M. Ali, V. Ravichandran, *Uniformly convex and uniformly starlike functions*, Math. Newsletter, Ramanujan Math. Soc. **21** (2011), no. 1, pp. 16–30.

[7] R. M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for p-valent functions, Appl. Math. Comput. **187** (2007), no. 1, 35–46.

[8] R. M. Ali, M. M. Nargesi and V. Ravichandran, Coefficient inequalities for starlikeness and convexity, Tamkang J. Math. **44** (2013), no. 2, 149–162.

[9] M. K. Aouf, Linear combinations of regular functions of order alpha with negative coefficients, Publ. Inst. Math. (Beograd) (N.S.) **47(61)** (1990), 61–67.

[10] K.O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, Inequality Theory and Applications, **6** (2010), 1–7.

[11] D. Bansal, Fekete-Szegö problem and upper bound of second Hankel determinant for a new class of analytic functions, Kyungpook Math. J. **54** (2014), no. 3, 443–452.

[12] L. Bieberbach, *Über einige Extremalprobleme im Gebiete der konformen Abbildung*, Math. Ann. **77** (1916), no. 2, 153–172.

[13] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* **154** (1985), no. 1-2, 137152

[14] N. E. Cho, B. Kowalczyk, and A. Lecko, Fekete-Szegö problem for close-to-convex functions with respect to a certain convex function dependent on a real parameter, Frontiers of Mathematics in China, appeared online. (DOI 10.1007/s11464-015-0510-y)

[15] J. H. Choi, Y. C. Kim and T. Sugawa, A general approach to the Fekete-Szegö problem, J. Math. Soc. Japan **59** (2007), no. 3, 707–727.

[16] R. N. Das and P. Singh, On subclasses of schlicht mapping, Indian J. Pure Appl. Math. **8** (1977), no. 8, 864–872.

[17] P. L. Duren, *Univalent Functions*, GTM 259, Springer-Verlag, New York, 1983.

[18] M. M. Elhosh, On the second Hankel determinant of univalent functions, Bull. Malays. Math. Soc. (2) **9** (1986), no. 1, 23–25.

[19] M. M. Elhosh, On the second Hankel determinant of close-to-convex functions, Bull. Malaysian Math. Soc. (2) **9** (1986), no. 2, 67–68.

[20]

[21] G. M. Goluzin, *On distortion theorems in the theory of conformal mappings, Mat. Sb.,* **43** *(1936), no. 1, 127–135.*

[22] G. M. Goluzin, Geometric theory of functions of a complex variable, Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, RI, 1969.

[23] A. W. Goodman, On uniformly convex functions, Ann. Polon. Math. **56** *(1991), no. 1, 87–92.*

[24] A. W. Goodman, Univalent Functions. Vol. I*, Mariner, Tampa, FL, 1983.*

[25] A. W. Goodman, Univalent Functions. Vol. II*, Mariner, Tampa, FL, 1983.*

[26] I. Graham and G. Kohr, Geometric function theory in one and higher dimensions, Monographs and Textbooks in Pure and Applied Mathematics, 255, Marcel Dekker, Inc., New York, 2003.

[27] U. Grenander and G. Szegö, Toeplitz Forms and Their Applications, California Monographs in Mathematical Sciences, Univ. California Press, Berkeley, 1958.

[28] M. Haji Mohd and M. Darus, Fekete-Szegő problems for quasi-subordination classes, Abstr. Appl. Anal. **2012***, Art. ID 192956, 14 pp.*

[29] S. A. Halim, A. Janteng and M. Darus, Classes with negative coefficients and starlike with respect to other points. II, Tamkang J. Math. **37** *(2006), no. 4, 345–354.*

[30] W. K. Hayman, On the second Hankel determinant of mean univalent functions, Proc. London Math. Soc. (3) **18** *(1968), 77–94.*

[31] Z. J. Jakubowski and K. Zyskowska, On an estimate of a functional in the class of holomorphic univalent functions, Math. Bohem. **118** *(1993), no. 3, 281–296.*

[32] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math. **7** *(2006), no. 2, Art. 50, 5 pp.*

[33] Z. J. Jakubowski and K. Zyskowska, On an estimate of a functional in the class of holomorphic univalent functions, Math. Bohem. **118** *(1993), no. 3, 281–296.*

[34] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math. **7** *(2006), no. 2, Art. 50, 5 pp.*

[35] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, J. Comput. Appl. Math. **105** *(1999), no. 1-2, 327–336.*

[36] K. Khatter S. K. Lee and S. Sivaprasad Kumar, Bounds for the second Hankel determinant and the Fekete-Szegö coefficient functional of certain analytic functions, Bull. Malays. Math. Soc., online.

[37] *F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20(1969), 8-12.*

[38] *K. Kiepiela, M. Pietrzyk and J. Szynal, The sharp bound for some coefficient functional within the class of holomorphic bounded functions and its applications, Rocky Mountain J. Math.* **31** *(2001), no. 1, 313–326.*

[39] *Y. C. Kim, J. H. Choi and T. Sugawa, Coefficient bounds and convolution properties for certain classes of close-to-convex functions, Proc. Japan Acad. Ser. A Math. Sci.* **76** *(2000), no. 6, 95–98.*

[40] *P. Koebe, Über die uniformisierung beliebiger analytischer Kurven,* Nachr. Akad. Wiss. Gottingen *(1907), 191-210.*

[41] *W. Koepf, On the Fekete Szegő problem for close-to-convex functions, Proc. Amer. Math. Soc.* **101** *(1987), no. 1, 89–95.*

[42] *B. Kowalczyk, A. Lecko, Fekete-Szegö problem for close-to-convex functions with respect to the Koebe function. Acta Math Sci Ser B Engl Ed, 2014, 34(5): 1571–1583.*

[43] *S. S. Kumar and V. Kumar, Fekete-Szegö problem for a class of analytic functions, Stud. Univ. Babeş-Bolyai Math.* **58** *(2013), no. 2, 181–188.*

[44] *S. Kumar, S. Nagpal and V. Ravichandran, Coefficient inequalities for Janowski starlikeness, Pro. Jangjeon Math. Soc.,* **19** *(2016), no. 1, 83–100.*

[45] *S. K. Lee, V. Ravichandran and S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, J. Inequal. Appl.* **2013***, 2013:281, 17 pp.*

[46] *Lindelof, E. Memoire sur certaines inegalités dans la théorie des fonctions monogénes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel,Acta Soc. Sci. Fenn.* **35***(1908), No. 7.*

[47] *J.E. Littlewood, On inequalities in the theory of functions, Proc. London. Math. Soc.,* **23** *(1925), 481–519.*

[48] *J.E. Littlewood, Lectures on the Theory of Functions., Oxford University Press: London, 1944.*

[49] *M.-S. Liu, J.-F. Xu and M. Yang, Upper bound of second Hankel determinant for certain subclasses of analytic functions, Abstr. Appl. Anal.* **2014***, Art. ID 603180, 10 pp.*

[50] *M.-S. Liu, Y.-C. Zhu and H. M. Srivastava, Properties and characteristics of certain subclasses of starlike functions of order β, Math. Comput. Modelling* **48** *(2008), no. 3-4, 402–419.*

[51] *W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge.*

[52] *W. C. Ma and D. Minda, Uniformly convex functions. II, Ann. Polon. Math.* **58** *(1993), no. 3, 275–285.*

[53] *R. Mendiratta, S. Nagpal and V. Ravichandran, A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli, Internat. J. Math.* **25** *(2014), no. 9, 1450090.*

[54] *R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, Bull. Malays. Math. Sci. Soc.* **38** *(2015), no. 1, 365–386.*

[55] *R. Mendiratta, S. Nagpal, and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, Bull. Malays. Math. Sci. Soc.* **38** *(2015), no. 1, 365–386.*

[56] E. P. Merkes, M. S. Robertson and W. T. Scott, On products of starlike functions, Proc. Amer. Math. Soc. **13** (1962), 960–964.

[57] A. K. Mishra, J. K. Prajapat and S. Maharana, Bounds on Hankel determinant for starlike and convex functions with respect to symmetric points, Cogent Mathematics (2016), 3: 1160557

[58] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20(1969), 8-12.

[59] J. W. Noonan and D. K. Thomas, On the Hankel determinants of areally mean p-valent functions, Proc. London Math. Soc. (3) **25** (1972), 503–524.

[60] J. W. Noonan, Coefficient differences and Hankel determinants of areally mean p-valent functions, Proc. Amer. Math. Soc. **46** (1974), 29–37.

[61] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of areally mean p-valent functions, Trans. Amer. Math. Soc. **223** (1976), 337–346.

[62] K. I. Noor, Higher order close-to-convex functions, Math. Japon. **37** (1992), no. 1, 1–8.

[63] K. I. Noor, On the Hankel determinant problem for strongly close-to-convex functions, J. Natur. Geom. **11** (1997), no. 1, 29–34.

[64] K. I. Noor, On certain analytic functions related with strongly close-to-convex functions, Appl. Math. Comput. **197** (2008), no. 1, 149–157.

[65] K. Noshiro, On the theory of schlicht functions, Journal of the Faculty of Science, Hokkaido Imperial University. Ser. 1, **2** (1934), no. 3, 129–155.

[66] H. Orhan, N. Yagmur and M. Caglar, Fekete-Szegö problem for a generalized subclass of analytic functions, Kyungpook Math. J. **53** (2013), no. 1, 13–23.

[67] K. S. Padmanabhan and R. Parvatham, Some applications of differential subordination, Bull. Austral. Math. Soc. **32** (1985), no. 3, 321–330.

[68] C. Pommerenke, Univalent functions, Vandenhoeck & Ruprecht, Göttingen, 1975. MR0507768

[69] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. London Math. Soc. **41** (1966), 111–122.

[70] Ch. Pommerenke, On the Hankel determinants of univalent functions, Mathematika **14** (1967), 108–112.

[71] J. K. Prajapat, D. Bansal, A. Singh and A. K. Mishra, Bounds on third Hankel determinant for close-to-convex functions, Acta Univ. Sapientiae Math. **7** (2015), no. 2, 210–219.

[72] D. V. Prokhorov and J. Szynal, Inverse coefficients for (α,β)-convex functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A **35** (1981), 125–143 (1984).

[73] M. Raza and S. N. Malik, Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli,J. of Ineq. and App. (2013), 2013:412.

[74] V. Ravichandran, Starlike and convex functions with respect to conjugate points, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **20** (2004), no. 1, 31–37.

[75] V. Ravichandran and S. Verma, Bound for the fifth coefficient of starlike functions, C. R. Acad. Sci. Paris. Ser. I **353** (2015), 505–510.

[76] W. Rogosinski, On subordinate functions, Proc. Cambridge Philos. Soc. **35** (1939), 1–26.

[77] W. Rogosinski, On the coefficients of subordinate functions, Proc. London. Math. Soc., **48** (1943), 48–82.

[78] M. S. Robertson, Certain classes of starlike functions, Michigan Math. J. **32** (1985), no. 2, 135–140.

[79] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan **11** (1959), 72–75.

[80] T. N. Shanmugam, C. Ramachandran and V. Ravichandran, Fekete-Szegő problem for subclasses of starlike functions with respect to symmetric points, Bull. Korean Math. Soc. **43** (2006), no. 3, 589–598.

[81] G. Shanmugam, B. Adolf Stephen and K. O. Babalola, Third Hankel determinant for α-starlike functions, Gulf J. of Math. **2**, (2014) 107–113.

[82] T. N. Shanmugam, Convolution and differential subordination, Internat. J. Math. Math. Sci. **12** (1989), no. 2, 333–340.

[83] H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, Canad. J. Math. **37** (1985), no. 1, 48–61.

[84] Y. J. Sim and O. S. Kwon, On certain classes of convex functions, Int. J. Math. Math. Sci. **2013**, Art. ID 294378, 6 pp.

[85] S. Sivaprasad Kumar and V. Kumar, Fekete-Szegö problem for a class of analytic functions defined by convolution, Tamkang J. Math. **44** (2013), no. 2, 187–195.

[86] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, Zeszyty Nauk. Politech. Rzeszowskiej Mat. No. 19 (1996), 101–105.

[87] J. Sokól, Radius problems in the class $\mathcal{S}_L^*$, Appl. Math. Comput. **214** (2009), no. 2, 569573.

[88] Y. Sun, Y.-P. Jiang and A. Rasila, Coefficient estimates for certain subclasses of analytic and bi-univalent functions, Filomat **29** (2015), no. 2, 351–360.

[89] D. Vamshee Krishna and T. Ram Reddy, Coefficient inequality for certain subclasses of analytic functions associated with Hankel determinant, Indian J. Pure Appl. Math. **46** (2015), no. 1, 91–106.

[90] D. Vamshee Krishna, B. Venkateswarlu and T. RamReddy, Third Hankel determinant for certain subclass of p-valent functions, Complex Var. Elliptic Equ. **60** (2015), no. 9, 1301–1307.

[91] B. A. Uralegaddi, M. D. Ganigi and S. M. Sarangi, Univalent functions with positive coefficients, Tamkang J. Math.**25** (1994), no. 3, 225230.

[92] D. Vamshee Krishna, B. Venkateswarlu and T. RamReddy, Third Hankel determinant for the inverse of a function whose derivative has a positive real part, Matematychni Studii. **42**, no. 1, 54–60

[93] S. E. Warschawski, On the higher derivatives at the boundary in conformal mapping, Trans. Amer. Math. Soc. **38** (1935), no. 2, 310–340.

[94] N. Yagmur and H. Orhan, Fekete-Szegő problem for subclasses of generalized uniformly starlike functions with respect to symmetric points, Math. Bohem. **139** (2014), no. 3, 485–509.

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[2] Kanika Khatter, See Keong Lee, S. Sivaprasad Kumar, Bulletin of the Malaysian Mathematical Sciences Society (2016). doi:10.1007/s40840-016-0414-3

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