Chapter 1

INTRODUCTION

1.1 Introduction and fundamental concepts

Dynamic stability or instability of elastic structures has drawn considerable attention in the past years. The beginning of the subject can be traced to the investigation of Koning and Taub (1933), who considered the response of an imperfect (half-sine wave), simply supported column subjected to a sudden axial load of specified duration. Since then, many studies have been conducted by various investigators on structural systems that are either suddenly loaded or subjected to time-dependent loads (periodic or nonperiodic), and several attempts have been made to find common response features and to define critical conditions for these systems. As a result of this, the term *dynamic stability* encompasses many classes of problems and many different physical phenomena. In general, problems that deal with the stability of motion have concerned researchers for many years in many fields of engineering. Definitions for stability and for the related criteria and estimates of critical conditions as developed through the years are given by Stoker (1955). In particular, the contributions of Routh (1975) Thompson and Tait (1923) deserve particular attention. Some of these criteria find wide uses in problems of control theory (Lefschetz, 1965), of stability and control of aircraft (Seckel, 1964), and in other areas (Crocco and Cheng, 1956). The class of problems falling in the category of parametric excitation, or parametric resonance, includes the best defined, conceived, and understood problems of dynamic stability. An excellent treatment and bibliography can be found in the book by Bolotin (1964). Another reference on the subject is Stoker's book (1950). The problem of parametric excitation is best defined in terms of an example. Consider an Euler column, which is loaded at one end by a periodic axial force. The other end is immovable. It can be shown that, for certain relationships between the exciting frequency and the column natural frequency of transverse vibration, transverse vibrations occur with rapidly increasing amplitudes. This is called parametric resonance and the system is said to be dynamically unstable. Moreover, the loading is called parametric excitation.

Other examples of parametric resonance include (1) a thin flat plate parametrically loaded by inplane forces, which may cause transverse plate vibrations; (2) parametrically loaded shallow arches (symmetric loading), which under certain conditions vibrate asymmetrically with increasing amplitude; and (3) long cylindrical, thin shells (or thin rings) under uniform but periodically applied pressure, which can excite vibrations in an asymmetric mode. Thus, it is seen that, in parametric excitation, the loading is parametric with respect to certain deformation forms. This makes parametric resonance different from the usual forced vibration resonance. Characteristics of parametric excitation is discussed in section 1.3 in detail.

Moreover, there exists a large class of problems for which the load is applied statically but the system is nonconservative. An elastic system is conservative when subjected to conservative loads; the reader is also referred to Ziegler's book (1968) for a classification of loads and reactions. An excellent review on the subject of stability of elastic systems under nonconservative forces is given by Herrmann (1967). He classifies all problems of nonconservative systems into three groups. The first group deals with follower-force problems, the second with problems of rotating shafts (whirling), and the third with aeroelasticity (fluid-solid interaction; flutter). All of these groups, justifiably or not, are called problems of dynamic stability. In the opinion of the author, justification is needed only for the first group. Ziegler (1956) has shown that critical conditions for this group of nonconservative systems can be obtained only through the use of the dynamic or kinetic approach to stability problems. The question of applicability of the particular approach was clearly presented by Herrmann and Bungay (1964) through a two-degree-of-freedom model. They showed that in some nonconservative systems, there exist two instability mechanisms, one of divergence (large deflections may occur) and one of flutter (oscillations of increasing amplitude). They further showed that the critical load for which the flutter type of instability occurs can be determined only through the kinetic approach, while the divergence type of critical load can be determined by employing any one of the three approaches (classical, potential energy, or kinetic). It is understandable, then, why many authors refer to problems of follower-forced systems as dynamic stability problems. Furthermore, the problem of flow-induced vibrations in elastic pipes is another fluid-solid interaction problem that falls under the general heading of dynamic stability. The establishment of stability concepts, as well as of estimates for critical conditions, is an area of great practical importance. A few references (Au-Yang and Brown, 1977; Benjamin, 1961; Blevins, 1977; Chen, 1975a, b; Chen, 1978; Chen, 1981; Gregory, 1966; Hill and Swanson, 1970; Junger and Feit, 1972; King, 1977; Paidoussis, 1970; Paidoussis and Deksnis, 1970; Reusselet and Hermann, 1977; Scanlan and Simin, 1978) are provided for the interested reader. In addition, a few studies have been reported that deal with the phenomenon of parametric resonance in a fluid-structure interaction problem (Bohn and Hermann, 1974; Ginsberg, 1973; Paidoussis and Issid, 1976; Paidoussis and Sundararajan, 1975). For completeness, one should refer to a few studies of aeroelastic flutter (Dowell, 1969, 1970; Dugundji, 1972; Kornecki, 1970; Kuo et al., 1972).

1.2 Parametric Excitation

Nayfeh and Mook (1979) [1] and Butikov (2005) [2] gave detailed explanation about parametric excitation. Parametric excitation is quite different from normal direct force. All systems we are familiar with are modeled using equations in which the homogeneous part did not contain functions of time. Even if an excitation is introduced into the model, an external excitation is added to a system in a separate term. However, there are many systems for which this type of equation is not applicable. Let us assume a simple differential equation which contains time variable coefficients.

$$\ddot{x} + p_1(t)\dot{x} + p_2(t)x = f(t) \tag{1.2.1}$$

In Eq. (1.2.1), although the external excitation is set to zero, i.e., $f(t) = 0$, the time dependant terms in the equation can act as an excitation. Because this type of excitation acts from within the parameters of the system, it is usually referred to as *parametric excitation*. As shown in Eq. (1.2.1), parametric excitation can coexist with external excitation.

Equation (1.2.1) is linear, even though its coefficients are not constant, and its general solution can be obtained by adding a particular solution of the complete equation to the general solution of the homogeneous equation. If $x_1(t)$ and $x_2(t)$ are two independent solutions of the homogeneous equation, its general solution can be obtained by the linear combination $x(t) = C_1x_1(t) + C_2x_2(t)$.

Consider a system modeled with a linear second-order differential equation of the type of Eq. (1.2.1) but without external excitation and with functions $p_1(t)$and $p_2(t)$, which are periodic in time with period *T*. The study of equation of this type was published by Floquet in 1883, and, hence, is usually referred to as *Floquet theory*.

$$\ddot{x} + p_1(t)\dot{x} + p_2(t)x = 0 \tag{1.2.2}$$

By introducing the transformation,

$$x = \tilde{x} exp\left[-\frac{1}{2}\int p_1(t)dt\right]$$

Equation (1.2.2) can be written as

$$\ddot{\tilde{x}} + p(t)\tilde{x} = 0 \tag{1.2.3}$$

where,

$$p(t) = p_2 - \frac{1}{4}p_1^2 - \dot{p}_1$$

For this transformation to be valid $p_1(t)$ is differentiable with respect to time.

This means that the free behavior of the damped system can be obtained from that of an undamped system by multiplying the time history of the latter by an appropriate decaying factor and slightly modifying the frequency by a change of the stiffness. It can be also applied for linear systems with constant parameter. Equation (1.2.3) is usually referred to as *Hill's equation*, because it was first studied by Hill in 1886 in the determination of the perigee of lunar orbit. Vibrations in a system described by Hill's equation are called *parametrically excited* or simply *parametric*.

As we mentioned before, the responses from the parametrically excited systems are different from both *free* vibrations, which occur when the coefficients in the homogeneous differential equation of motion are constant, and *forced* vibrations, which occur when an additional time dependent forcing term is added to the right side of the equation of motion with constant coefficients. More detailed explanation about the characteristics of the parametric excitation will be addressed in Section 1.3.

The most common resonances in parametrically excited systems are the principal parametric resonance, which occurs when the excitation frequency is nearly equal to twice the natural frequency.

1.3 Characteristics of the Parametric Excitation

There are several important differences that distinguish parametric resonance from the ordinary resonance caused by external force acting directly on the system. Butikov[2] described the special characteristics of the parametric excitation in his paper. The growth of the amplitude of the vibrations during parametric excitation is provided by the force that periodically changes the parameter. Parametric resonance is possible when one of the following conditions for the frequency ω of modulation is fulfilled;

$$\omega = \frac{2\omega_0}{n} \quad (n=1, 2, 3, \ldots)$$

In other words, parametric resonance occurs when the parameter changes twice during one period, once during one period, twice during three periods, and so on. However, the maximum energy transfer to the vibrating system occurs when the parameter is changed twice during one period of the natural frequency.

Another important distinction between parametric excitation and forced vibration is the dependence of the growth of energy on the energy already stored in the system. While for forced excitation the increment of energy during one period is proportional to the amplitude of vibrations, i.e., to the square root of the energy, at parametric resonance the increment of energy is proportional to the energy stored in the system. Also, energy losses caused by damping are also proportional to the energy already stored.

In the case of direct forced excitation energy losses restrict the growth of the amplitude because these losses grow with the energy faster than does the investment in energy arising from the work done by the external force. In the case of parametric resonance, both the investment in energy caused by the modulation of a parameter and the losses by damping are proportional to the energy stored, and so their ratio does not depend on the amplitude. Therefore, parametric resonance is possible only when a *threshold* is exceeded, that is, when the increment in energy during a period (caused by the parameter variation) is larger than the amount of energy dissipated during the same time. The critical (threshold) value of the modulation depth depends on damping. However, if the threshold is exceeded, the losses of energy by damping in a linear system cannot restrict the growth of the amplitude.

1.4 Nonlinearities

As mentioned earlier, parametrically excited linear, undamped systems have solutions that grow indefinitely with time. In other words, if the system is truly linear, the amplitude grows until the system is destroyed. However, most systems have some degree of nonlinearity which comes into play as soon as the amplitude of the motion becomes appreciable, and it modifies the response. For instance, as the amplitude grows, the nonlinearity limits the growth, resulting in a limit cycle. Thus although the linear theory is useful in determining the initial growth or decay, it may be inadequate if the system possesses any nonlinearity. Hence, nonlinearity identification of the system is a very important problem.

In theory, nonlinearity exists in a system whenever there are products of dependent variables and their derivatives in the equations of motion, boundary conditions, and/or constitutive laws, and whenever there are any sort of discontinuities or jumps in the system. Evan-Iwanowski (1976) [3], Nayfeh and Mook (1979) [1], and Moon (1987) [4] explain the various types of nonlinearities in detail along with examples. Also, Pramod Malatkar (2003) [5] has briefly explained a variety of nonlinearities in his dissertation. Here are some nonlinearities that need to be considered when we design mechanical systems.

Damping dissipation is essentially a nonlinear phenomenon. Linear viscous damping is an idealization. Coulomb friction, aerodynamic drag, hysteretic damping, etc. are examples of nonlinear damping.

Geometric nonlinearity exists in systems undergoing large deformations or deflections. This nonlinearity arises from the potential energy of the system. In structures, large deformations usually result in nonlinear strain- and curvature-displacement relations. This type of nonlinearity is present, for example, in the equation governing the large-angle motion of a simple pendulum, in the nonlinear strain-displacement relations due to mid-plane stretching in strings, and due to nonlinear curvature in cantilever beams.

Inertia nonlinearity derives from nonlinear terms containing velocities and/or accelerations in the equations of motion. It should be noted that nonlinear damping, which has similar terms, is different from nonlinear inertia. The kinetic energy of the system is the source of inertia nonlinearities. Examples include centripetal and Coriolis acceleration terms. It is also present in the equations describing the motion of an elastic pendulum (a mass attached to a spring) and those describing the transverse motion of an inextensional cantilever beam.

Nonlinearities can also appear in the boundary conditions. For example, a nonlinear boundary condition exists in the case of a pinned-free rod attached to a nonlinear torsional spring at the pinned end.

Many other types of nonlinearities exist: like the ones in systems with impacts, with backlash or play in their joints, etc. It is interesting to note that the majority of physical systems belong to the class of weakly nonlinear (or quasi-linear) system. For certain phenomena, these systems exhibit a behavior only slightly different from that of their linear counterpart. In addition, they also exhibit phenomena which do not exist in the linear domain. Therefore, for weakly nonlinear structures, the usual starting point is still the identification of the linear natural frequencies and mode shapes. Then, in the analysis, the dynamic response is usually described in terms of its linear natural frequencies and mode shapes. The effect of the small nonlinearities is seen in the equations governing the amplitude and phase of the structure response.

1.5 FUNCTIONALLY GRADED MATERIALS [6]

The idea of functionally graded materials (FGMs) was advanced substantially in the early 1980's in Japan, where this new material concept has been proposed to increase adhesion and to minimize the thermal stresses in metallic–ceramic composites developed for reusable rocket engines. Meanwhile, FGMs concepts have triggered world-wide research activity and have been applied to metals, ceramics, and organic composites to generate improved components with superior physical properties. From the most informative and widely acceptable opinion, FGMs have been characterized by gradual space changes in their composition, structure, and consequently in their properties. Usually, they are composites in the common sense, but graded structures can also be obtained in traditional, monolithic materials on the basis of a variety of microstructures formed during some kind of material processing. . They are materials in which the volume fraction of two or more materials is varied as a power-law, sigmoid or exponential distribution continuously as a function of position along certain dimension(s) of the structure. These materials do not contain well distinguished boundaries or interfaces between their different regions as in the case of conventional composite materials. Because of this, such materials possess good chances of reducing mechanical and thermal stress concentration in many structural elements, which can be developed for specific applications. The structures are not simply inhomogeneous, but their heterogeneity is usually typical in one direction for the entire volume of a material.

FGMs have played important role in many industrial and defense applications. Normally, a component can be fabricated using any metal. For some specific applications such as in aerospace engineering where the component's weight and durability in high temperature environment are so crucial, the components need to be fabricated using special material; usually made of a mixture of ceramic and metals. The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture caused by stress due to high temperature gradient in a very short period of time.

1.6 RESEARCH MOTIVATION

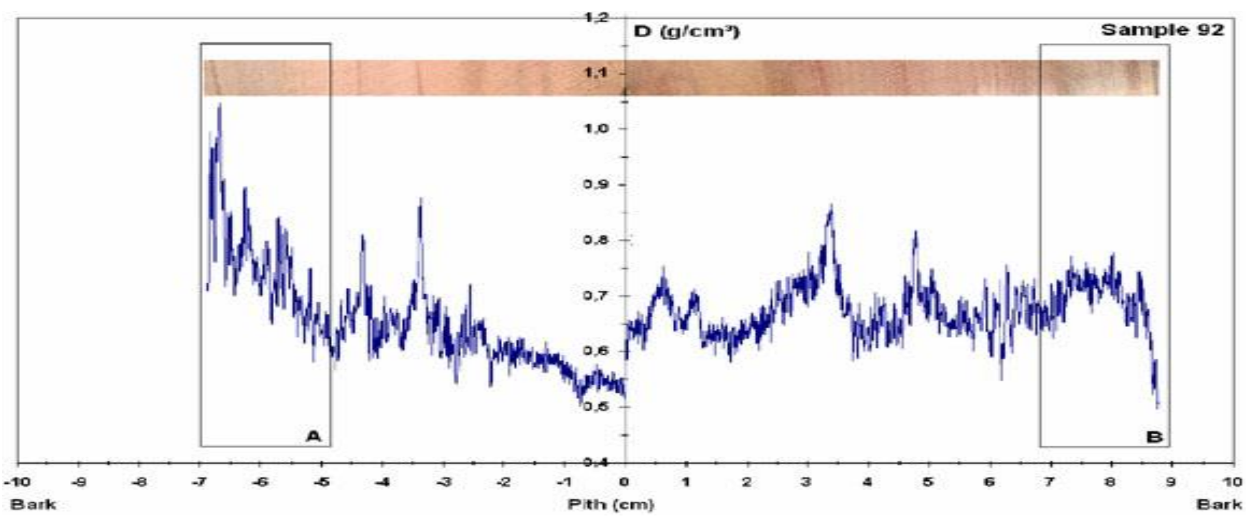

Fig.1 Wood density profile of Eucalyptus. A-healthy and B- affected by area of white rot fungi. [7]

By observing the structural configuration and stability of the trunk and branches of different trees in the present investigation the author has considered radially functionally graded circular columns. Author has classified commonly seen trees into three types according to the volume fraction of fibers from centre to the periphery producing a radial modulus gradient. Type-I: Mango tree, Jamun tree, Banyan tree etc. where volume fraction of fibers radially increases from outer periphery towards centre. In this type the trunk and branches are stable with buckled configuration but vulnerable to self weight and external forces such as wind, rain and snow. Type-II: Palm tree, coconut tree, bamboo etc. where gradient of volume fraction of fiber is opposite to that of Type I. Here, these types of trees are slender and stable under external loads. Type- III: Eucalyptus tree, Pine tree, Deodar tree etc. where volume fraction of fiber variation similar like type- II with lesser gradient resulting a better stable structure like Type-II._In this present study author has studied the dynamic instability of the circular columns analogous to above each type i.e. functionally graded columns with different boundary conditions.

1.7 LITERATURE REVIEW

In elastic systems, structural members such as columns, plates and shell panels under time dependent compressive axial load leads to the problem of dynamic instability. When the loading is time dependent, the systems are called parametrically excited systems and the instability is referred as parametric instability or dynamic instability. Dynamic instability occurs in structural components (beams, plates and shells) for certain combinations of load amplitude, disturbing frequency and frequency of transverse vibration of plates. Most of the early development in this field is due to Russian researchers. Bolotin[8] (1964) has reviewed literature on the dynamic instability of isotropic beams, plates and shells in his book. The first observation of parametric resonance is attributed to Faraday in 1831 and first mathematical explanation of phenomenon is given by Rayleigh in 1883. The mathematical formulation of the problem leads to Mathieu-Hill type of linear differential equation with periodic coefficients. Boundedness of the solution corresponding to stability or instability is Hill’s infinite determinant method originally developed by Hill in 1886 for a single degree of freedom system. This method has been further extended for a multiple degree of freedom system by Bolotin (1964), Valeev (1961). Lot of work is available in dynamic stability of columns, plates and shells.

1.7.1 Current knowledge of isotropic and composite plates

There exists a large body of literature in the dynamic instability of plates. Prabhakara and Datta[9] studied the free vibration and static stability of rectangular plate using finite element method. Deolasi and Datta[10] investigated the parametric stability behavior of a plate subjected to in-plane compressive or tensile periodic edge loading. While on the other hand Ganapathi et al.[11] determine the non linear instability behavior of plates subjected to periodic in-plane load.. Authors have also published literature in the field of composite plate. Patel et al.[12] studied the dynamic stability of laminate composite plate supported on the elastic foundations subjected to periodic in-plane loads. Balamurugan et al.[13] carried out an investigation on the dynamic instability of anisotropic laminated composite plates considering geometric non linearity. Chattopadhyay and Radu[14] used higher order shear deformation to investigate the instability associated with composite plates. Recently, Ramachandra and Panda [15] investigated the dynamic instability of shear deformable composite plates subjected to non-uniform in-plane loading for four different sets of boundary conditions.

1.7.2 Current knowledge of isotropic and composite shells

Dynamic stability of shells has been extensively studied by various authors. Lam and Ng[16] studied dynamic stability of thin isotropic cylindrical shell under combined static and periodic axial forces using four common thin shell theories. They [17] also determine the effects of length to radius and thickness to radius ratio of the cylinder on the instability region. J.N. Reddy [18] et al. examined the dynamic stability of anti symmetric cross-ply circular cylindrical shells of different lamination schemes. Sahu and Datta [19] investigated the instability behavior of curved panels with cutout subjected to in-plane static and periodic compressive edge loadings using finite element analysis taking into the account first order shear deformation theory. They [20] also determine the instability characteristics of doubly curved panels.

1.7.3 Current knowledge of columns

Parametric instability of columns under periodic axial loading has been investigated by many authors [8, 21] by classical method. Existence of combination resonance in addition to a simple parametric resonance is focussed by researcher [22, 23]. Dynamic instability of columns were studied by Brown et al. [24] using finite element method. Dynamic instability of cantilever column subjected to distributed loading [25] and intermediate periodic axial concentrated loading [26] along its length are investigated by using finite element formulation. So far to the best of the author's knowledge, there is no work available in the published literature on the dynamic instability analysis of functionally radially graded columns subjected to dynamic axial loading. In the published literature most of the investigator studied the principal zone of instability i.e. first zone of instability because of its practical importance. For the sake of completeness and academic interest, the authors have studied second, third and fourth zone of instability. In the present study columns with different boundary condition have been considered.

1.8 OBJECTIVES

The objectives of the present investigation are

a) *To plot the different zones of instability of radially functionally graded circular columns with different end conditions subjected to periodic axial load.*
b) *To determine the behaviour of the geometrically non linear circular column and studying the instability of different points on amplitude frequency curve through phase portraits.*

Chapter 2

INSTABILITY REGIONS
OF
RADIALLY FG CIRCULAR COLUMN

2.1 MATHEMATICAL FORMUATION

Consider a simply supported column of length L and uniform circular cross-section subjected to the dynamic axial load $P(t)$ as shown in the figure 2.

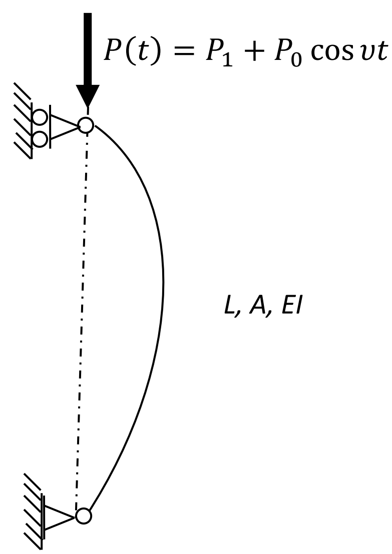

Fig.2 Column under axial load

The governing partial differential equation for the transverse vibration of column under periodic axial loading can be expressed as

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + P(t)\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.1.1)$$

The applied axial loading has a static and a dynamic component viz, $P(t) = P_1 + P_0 \cos \upsilon t$. Adopting Galerkin's approximation with suitable beam function [15] for different boundary conditions, the governing partial differential Eq. (2.1.1) are reduced to ordinary differential equation (Mathieu type of equation) describing the column dynamic instability behaviour as,

$$[M]\{\ddot{x}\} + \big[[K] - P_1[K_G] - P_0[K_G]\cos \upsilon t\big]\{x\} = \{0\} \qquad (2.1.2)$$

where $M, K \ and \ K_G$ are respectively the mass, stiffness and geometric stiffness matrices.

Beam functions used for different boundary conditions are as follows,

For simply supported column (SS),

$$w(x) = \sin\frac{m\pi x}{l} \qquad (m = 1,2,3 \dots)$$

For fixed column (CC),

$$w(x) = \cos\xi_m\left(\frac{x}{l} - \frac{1}{2}\right) + \frac{\sin\left(\frac{\xi_m}{2}\right)}{\sinh\left(\frac{\xi_m}{2}\right)}\cosh\xi_m\left(\frac{x}{l} - \frac{1}{2}\right) \qquad (m = 2,4,6 \dots)$$

where ξ_m are the roots of equation $\tan\left(\frac{\xi_m}{2}\right) + \tanh\left(\frac{\xi_m}{2}\right) = 0$

$$w(x) = \sin\xi_m\left(\frac{x}{l} - \frac{1}{2}\right) - \frac{\sin\left(\frac{\xi_m}{2}\right)}{\sinh\left(\frac{\xi_m}{2}\right)}\cosh\xi_m\left(\frac{x}{l} - \frac{1}{2}\right) \qquad (m = 3,5,7 \dots)$$

where ξ_m are the roots of equation $\tan\left(\frac{\xi_m}{2}\right) - \tanh\left(\frac{\xi_m}{2}\right) = 0$

For clamped-Hinged column (CS),

$$w(x) = \sin\xi_m\left(\frac{x}{2l} - \frac{1}{2}\right) - \frac{\sin\left(\frac{\xi_m}{2}\right)}{\sinh\left(\frac{\xi_m}{2}\right)}\sinh\xi_m\left(\frac{x}{2l} - \frac{1}{2}\right) \qquad (m = 2,3,4 \dots)$$

where ξ_m are the roots of equation $\tan\left(\frac{\xi_m}{2}\right) - \tanh\left(\frac{\xi_m}{2}\right) = 0$

For cantilever column (CF),

$$w(x) = \cosh\left(\lambda_j \frac{x}{l}\right) - \cos\left(\lambda_j \frac{x}{l}\right) - \frac{\cosh\left(\lambda_j \frac{x}{l}\right) + \cos\left(\lambda_j \frac{x}{l}\right)}{\sinh\left(\lambda_j \frac{x}{l}\right) + \sin\left(\lambda_j \frac{x}{l}\right)}\left(\sinh\left(\lambda_j \frac{x}{l}\right) - \sin\left(\lambda_j \frac{x}{l}\right)\right)$$

$$\lambda_j = 1.87511$$

In the present investigation, only the linear dynamic instability of columns with different boundary conditions is studied and hence the nonlinear stiffness matrix is not considered. Hence in the above equation $[K]$ is the linear stiffness matrix. In Eq. (2.1.2), $P_1\ and\ P_0$ are varied as $P_1 = \alpha P_{cr}$ and $P_0 = \beta P_{cr}$ where α and β are static and dynamic load factors respectively and P_{cr} is the static buckling load. The effect of axial load is reflected in the computation of K_G matrix. The Eq. (2.1.2) is a second order differential equation with periodic coefficients. The critical buckling load is evaluated from the solution of linear eigenvalue problem by neglecting the mass and time dependant load terms. Similarly the solutions of the eigenvalue problem associated with the Eq. (2.1.2) neglecting terms containing P_1 and P_0 gives the natural frequencies. The regions of instability are located by boundaries of instability having periodic solutions with period T and 2T to Eq. (2.1.2). These are determined by the method suggested by Bolotin [8] in this investigation. These solutions are sought as,

$$f(t) = \sum_{k=1,3,5\ldots} a_k \sin\frac{kvt}{2} + b_k \cos\frac{kvt}{2} \tag{2.1.3}$$

$$f(t) = b_0 + \sum_{k=2,4,6\ldots} a_k \sin\frac{kvt}{2} + b_k \cos\frac{kvt}{2} \tag{2.1.4}$$

,where $a_k$and b_k are arbitrary constants. Here, it may be noted, two solutions of identical periods bound the region of instability, two solutions of different periods bound the region of instability. Substituting Eqs. (2.1.3) or (2.1.4) in Eq. (2.1.2) and equating the coefficients of identical $\sin\frac{kvt}{2}$ and $\cos\frac{kvt}{2}$ leads to a system of homogeneous algebraic equations in $a_k$and b_k. For a nontrivial solution the determinant of the coefficient matrix of $a_k$and $b_k$must vanish. The size of the above determinant is infinite as we have assumed the solution in the form of infinite series. The determinants are shown to be belonging to a class of converging determinant known as normal determinant [8]. The first order and second order approximation to boundaries of principal regions of instability corresponding to period 2T of undamped system is obtained by solving following two eigenvalue problems respectively.

$$|K^* \pm 0.5\beta P_{cr}K_G - 0.25Mv_1^2| = 0 \quad (2.1.5a)$$

$$\begin{vmatrix} K^* \pm 0.5\beta P_{cr}K_G & 0.5\beta P_{cr}K_G \\ \beta P_{cr}K_G & K^* - 2.25Mv_1^2 \end{vmatrix} - v_2^2 \begin{vmatrix} 0.25M & 0 \\ 0 & 0 \end{vmatrix} = 0 \quad (2.1.5b)$$

Secondary zone of instability with first order approximation (Eq. 2.1.6a and 2.1.6b) and second order approximation (Eq. 2.1.6c and Eq. 2.1.6d) corresponding to period T are determined from,

$$|K^* - Mv_1^2| = 0 \quad (2.1.6a)$$

$$\begin{vmatrix} K^* & -0.5\beta P_{cr}K_G \\ -\beta P_{cr}K_G & K^* \end{vmatrix} - v_1^2 \begin{vmatrix} 0 & 0 \\ 0 & M \end{vmatrix} = 0 \quad (2.1.6b)$$

$$\begin{vmatrix} K^* & -0.5\beta P_{cr}K_G \\ -\beta P_{cr}K_G & K^* - 4Mv_1^2 \end{vmatrix} - v_2^2 \begin{vmatrix} M & 0 \\ 0 & 0 \end{vmatrix} = 0 \quad (2.1.6c)$$

$$\begin{vmatrix} K^* & -0.5\beta P_{cr}K_G & 0 \\ -\beta P_{cr}K_G & K^* & -0.5\beta P_{cr}K_G \\ 0 & -0.5\beta P_{cr}K_G & K^* - 4Mv_1^2 \end{vmatrix} - v_2^2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad (2.1.6d)$$

Where, $K^* = K - P_1K_G$

Similarly third and fourth instability region can be computed by following the same procedure. However they are not given here for the sake of brevity.

2.2 RESULTS AND DISCUSSIONS

2.2.1 Effect of Damping

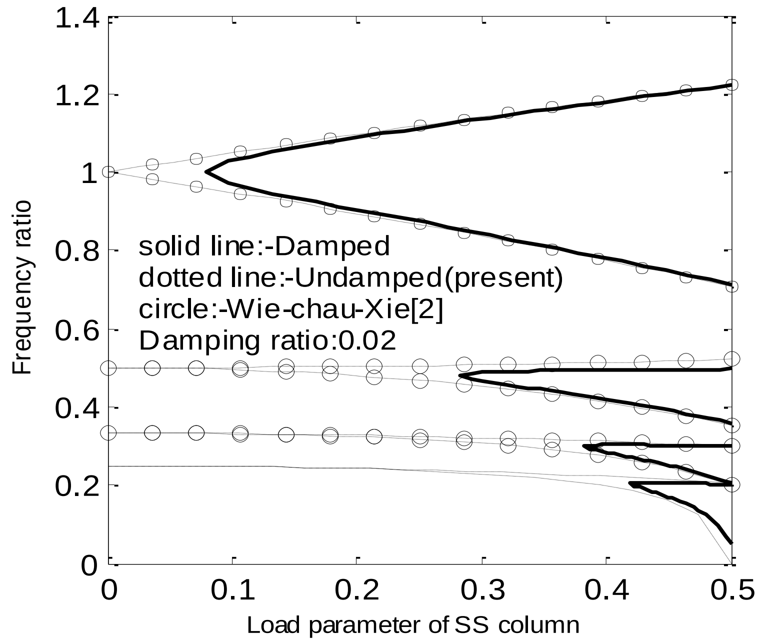

Fig.3 Instability zones of SS circular column under damped and undamped condition. Frequency ratio$(\lambda_i) = \frac{\upsilon}{2\omega_{SS}}$, Load parameter (μ)=$\frac{P_0}{2P_S}$

Figure 3. illustrates the stability boundaries for simply supported circular column under damped and undamped condition. The width of instability for all four zones increases with the increase of load parameter for both damped and undamped condition. The width of principal instability zone is maximum compared to all other instability zones and hence of greater importance. At higher load parameter the instability regions come closer. Hence, causing the column unstable for most of the frequency ratio$\left(\frac{\upsilon}{2\omega_{SS}}\right)$,where υ is the excitation frequency and ω_{SS} is the natural frequency of SS column. It is also observed that for damped column some minimum value of load parameter$\left(\mu = \frac{P_0}{2P_S}\right)$ is required ,where P_S is critical static buckling load of SS column.For damping ratio(ξ) of 0.02 the minimum load parameter required for instability are 0.08, 0.28, 0.38 and 0.41 under first, second ,third and fourth zone of instability region respectively. The boundary of instability of undamped column is validated with Wie-Chau Xie[21] for first, second and third zone of instability region.

2.2.2 Effect of end restraining

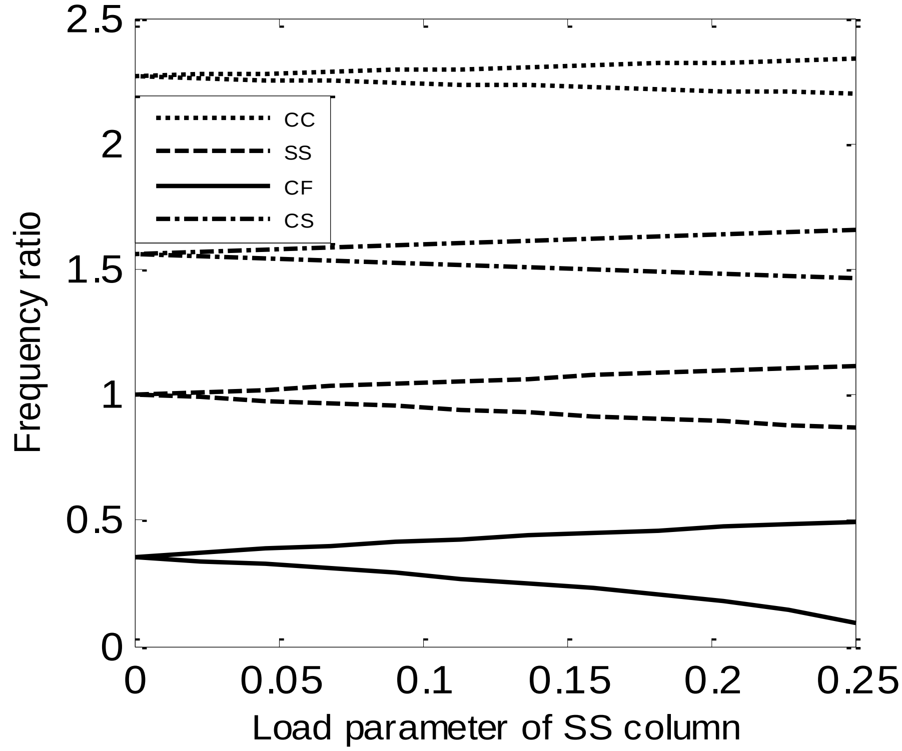

Fig.4 Effect of boundary conditions on principal instability regions of isotropic circular column. Frequency ratio(λ_i).

Figure 4. shows the effect of boundary condition on the principal instability region of column by taking four different boundary condition: simply supported(SS),clamped at one end and other end hinged(CS),clamped at both ends(CC)and clamped at one end and loading end free(CF). The zone width for instability region decreases with the increase in the end restraining. The zone widths for load parameter of 0.25 are 0.3987λ_i, 0.2520λ_i, 0.1869λ_i and 0.1375λ_i for CF, SS, CS and CC respectively.

2.2.3 Effect of approximation on width of instability region

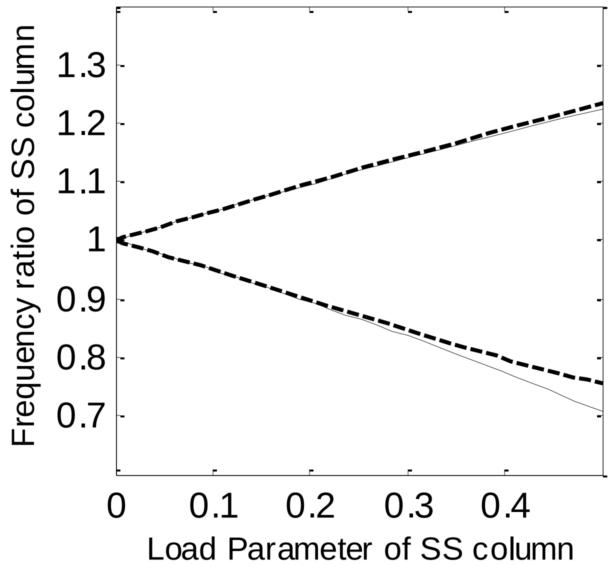

Fig.5 Effect of approximation on instability region.

Figure 5 shows the importance of approximation in determining the width of instability region. Higher order approximation in-cooperates more number of terms in the Fourier series and also the instability region does not remain symmetric. With the increase in load parameter the correction increases.

2.2.4 Instability region of SS column in different modes

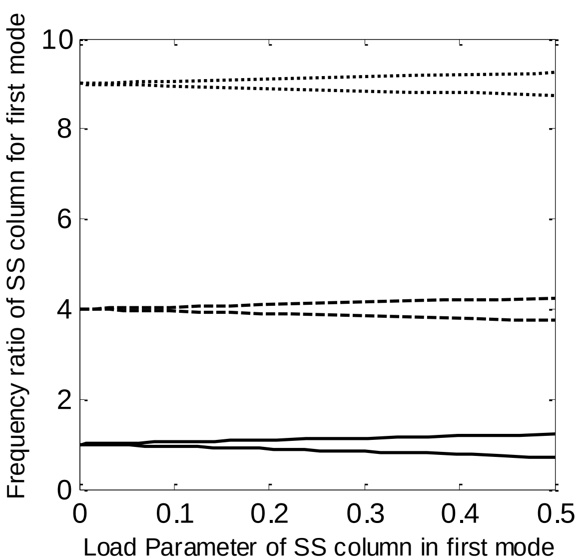

Fig.6 Instability region of SS column for different modes.

Figure 6 reflects that the width of principal instability region remain unaffected in different modes.

2.2.5 Effect of radially FGM

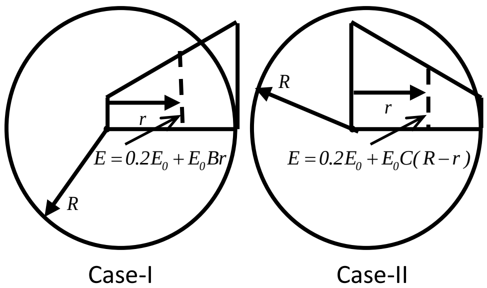

Fig 7 Modulus of elasticity increases linearly (Case-I) and decreases linearly (Case-II) in radial direction

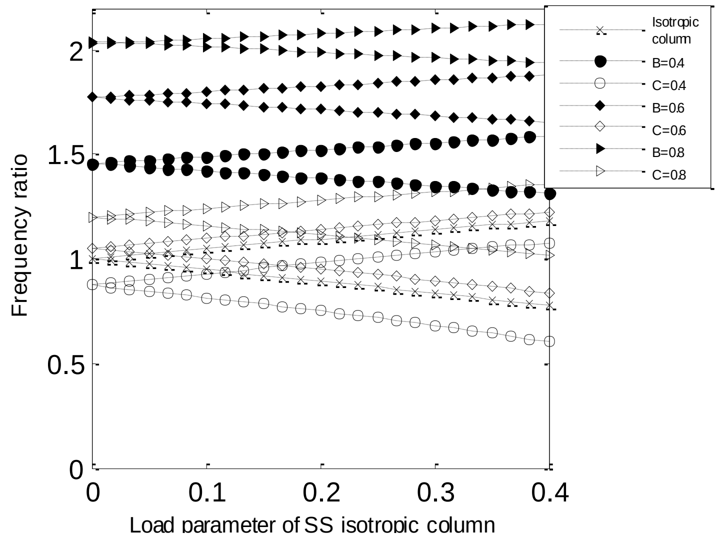

Fig.8 Principal instability regions of simply supported circular column with linearly varying modulus of elasticity along radial direction with different gradient. Frequency ratio (λ_i)

Figure 7. shows the variation of modulus of elasticity from centre of the column to the periphery. In case I the modulus of elasticity increases linearly from centre to the periphery and the opposite variation is case II. Three different linear variations are considered for three sets of B and C values i.e. 0.4, 0.6, and 0.8. In a particular case of B and C value of 0.4 the zone width is $0.1374\lambda_i$ and $0.2308\lambda_i$respectively.

It is observed from Fig. 8. that for same gradient modulus of elasticity for both cases, the zone width of the column for case I is narrower than case II.

2.2.6 Effect of bending stiffness

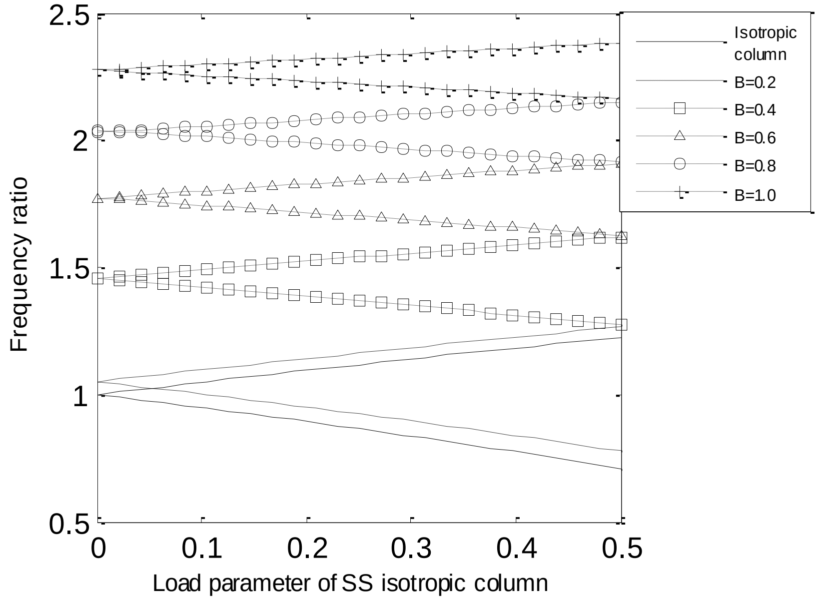

Fig 9. Effect of bending stiffness on principal instability region of simply supported circular column with linearly varying modulus of elasticity. Frequency ratio(λ_i)

Figure 9. demonstrate the effect of bending stiffness on the zone width of simply supported circular column for case I mentioned in Fig. 3. As the bending stiffness increases radially the width of instability region decreases. Zone widths for $B = 0.2,0.4,0.6,0.8,1.0$ are $0.1912\lambda_i$, $0.1374\lambda_i$, $0.1129\lambda_i$, $0.0889\lambda_i$, $0.0879\lambda_i$ respectively and zone width for isotropic column for same load parameter$(\mu = 0.2)$ is $0.2010\lambda_i$.

Chapter 3

AMPLITUDE-FREQUENCY RELATIONSHIP
OF
GEOMETRICALLY NON-LINEAR COLUMN

3.1 MATHEMATICAL FORMULATION

This chapter deals with the behavior of the structure when non-linearity has been introduced. Here SS, CS and CC column is considered. Using the Hamilton's principle the generalized governing differential equation taking into the account damping and the geometric non-linearity is as follows,

$$\mathrm{EIv_{xxxx}} + \mathrm{P(t)v_{xx}} - \frac{\mathrm{EA}}{2\mathrm{L}}\int_0^{\mathrm{L}} \mathrm{v_x^2 dx.\, v_{xx}} + \beta \mathrm{v_t} + \rho \mathrm{Av_{tt}} = 0 \qquad (3.1.1)$$

EI is the flexural rigidity of the column, β is the damping coefficient, ρ is the mass density per unit volume and A is the cross-sectional area.

Using the appropriate shape functions given in chapter 2 for each column and applying Galerkin's approximations the above governing differential equation is reduced into equation of motion.

The generalized equation of motion is,

$$q'' + 2\epsilon\xi\omega q' + \omega q[1 - M\epsilon\mu\cos\upsilon t + B\epsilon\gamma q^2] = 0 \qquad (3.1.2)$$

where, ξ is the damping ratio, ω is the natural frequency of the column, υ is the excitation frequency of the load $P(t) = P_0 \cos\upsilon t$, ϵ is the small parameter such that $0 < \epsilon \lll 1$, M and B are the numerical parameter introduced in the present study in order to generalize the equation of motion for columns with different end conditions.

Table 1. Values of numerical parameters for different columns

Column	M	B
SS	2	1
CS	1.91083	1.1133
CC	1.9405	1.5813

For the steady state solution apply the time scaling $\tau = \upsilon t$ and let $\upsilon = \omega_0(1-\epsilon\Delta)$ where ω_0 is the reference frequency and Δ is the detuning parameter. With the notation $\kappa = \frac{\omega}{\omega_0}$ the equation (3.1.2) becomes,

$$q'' + \kappa^2 q = \epsilon\left(-2\frac{\omega}{\upsilon}\xi q' - 2\Delta\kappa^2 q + M\frac{\omega^2}{\upsilon^2}\mu q\cos\tau - B\gamma\frac{\omega^2}{\upsilon^2}q^3\right) \quad (3.1.3)$$

Applying the transformation,

$$q_C(\tau) = a(\tau)\, cos\Phi(\tau), \qquad q'_C(\tau) = -a(\tau)k\, sin\Phi(\tau), \qquad \Phi(\tau) = k\tau + \varphi(\tau)$$

results in,

$$a' = \epsilon a\left\{\frac{\xi\omega}{\upsilon}(1-\mathbf{cos\,2\Phi}) + \kappa\,\Delta\,\mathbf{sin2\Phi} + \frac{B\gamma\omega^2}{\kappa\upsilon^2}a^2\left(\frac{1}{8}\mathbf{sin\,4\Phi} + \frac{1}{4}\mathbf{sin\,2\Phi}\right) - \frac{M\mu\omega^2}{4\kappa\upsilon^2}(\sin(2\Phi - \tau) + \mathbf{sin(2\Phi+\tau)})\right\} \quad (3.1.4)$$

$$\varphi' = \epsilon\left\{\frac{\xi\omega}{\upsilon}\mathbf{sin2\Phi} + (1+\mathbf{cos\,2\Phi}) + \frac{B\gamma\omega^2}{\kappa\upsilon^2}a^2\left(\frac{1}{8}\mathbf{cos\,4\Phi} + \frac{1}{4}\mathbf{cos\,2\Phi} + \frac{3}{8}\right) - \frac{M\mu\omega^2}{4\kappa\upsilon^2}\left(\boldsymbol{cos\tau} + \frac{1}{2}\cos(2\Phi-\tau) + \frac{1}{2}\mathbf{cos(2\Phi+\tau)}\right)\right\} \quad (3.1.5)$$

For Subharmonic Resonance $\kappa = 1/2$

For $\kappa = 1/2$, i.e. $\omega_0 = 2\omega$ or υ is in the vicinity of 2ω, all sinusoidal terms in bold in above equations (3.1.4 and 3.1.5) vanish when averaged. Noting that

$$\frac{\omega^2}{\kappa\upsilon^2} = \frac{\omega^2}{\kappa{\omega_0}^2(1-\epsilon\Delta)^2} \approx \frac{\omega^2}{\kappa{\omega_0}^2} = \frac{1}{2}$$

One obtains the averaged equations as

$$a' = -\epsilon\left(\frac{\xi\omega}{\upsilon} + \frac{M\mu}{8} sin2\varphi\right)\bar{a} \qquad (3.1.6)$$

$$\varphi' = \epsilon\frac{1}{2}\left(\Delta + \frac{3B\gamma}{8}\bar{a}^2 - \frac{M\mu}{4} cos2\bar{\varphi}\right) \qquad (3.1.7)$$

The steady state solutions $\bar{a}_0$ and $\bar{\varphi}_0$ are given by $\bar{a}, = \bar{\varphi}' = 0$

If $\bar{a} = 0$ then $q(\tau) = 0$.

If $\bar{a} \neq 0$, then from equation (3.1.6), one obtains

$$\sin2\overline{\varphi}_0 = -\frac{8\xi\omega}{M\upsilon\mu} \qquad (3.1.8)$$

Which gives

$$\cos2\overline{\varphi}_0 = \pm\left[1 - \left(\frac{8\xi\omega}{M\upsilon\mu}\right)^2\right]^{1/2} \qquad (3.1.9)$$

Equation (3.1.7) yields

$$\Delta = -\frac{3}{8}B\gamma\overline{a}_0^2 + \frac{M\mu}{4}\cos 2\,\varphi_0 \qquad (3.1.10)$$

Since $\epsilon\Delta = 1 - \upsilon/\omega_0$, equation (3.1.10) becomes

$$\frac{\upsilon}{2\omega} - 1 = \epsilon\left\{\frac{3B\gamma}{8}\overline{a}_0^2 \pm \left[\left(\frac{M\mu}{4}\right)^2 - \left(\frac{2\xi\omega}{\upsilon}\right)^2\right]^{1/2}\right\} \qquad (3.1.11)$$

Which gives the amplitude-frequency relation.

For undamped case $= 0$,

$$\frac{\upsilon}{2\omega} - 1 = \epsilon\left\{\frac{3B\gamma}{8}\overline{a}_0^2 \pm \frac{M\mu}{4}\right\} \qquad (3.1.12)$$

3.2 STABILITY OF STEADY STATE SOLUTION

3.2.1 Non-Trivial solution

In this case, equation (3.1.6) and (3.1.7) can be written as, for $\xi = 0$,

$$\frac{d\bar{a}}{d\bar{\varphi}} = -\frac{M\mu\bar{a}\sin 2\bar{\varphi}}{4\Delta + \frac{3\beta\gamma}{8}\bar{a}^2 - \mu M\cos 2\bar{\varphi}} \qquad (3.2.1)$$

$$\left(4\Delta + \frac{3}{2}B\gamma\bar{a}^2\right)d\bar{a} - M\mu(\cos 2\bar{\varphi}\,d\bar{a} - \bar{a}\sin 2\bar{\varphi}\,d\bar{\varphi}) = 0 \qquad (3.2.2)$$

Multiplying this equation by $\bar{a}$ and integrating both sides of the equation yield

$$\Delta\bar{a}^2 + \frac{3}{16}B\gamma\bar{a}^4 - \frac{\mu M}{2}\bar{a}^2\cos 2\bar{\varphi} = C \qquad (3.2.3)$$

If one puts $x = \bar{a}\cos\bar{\varphi}$ and y= $\bar{a}\sin\bar{\varphi}$, equation (3.142) becomes

$$\left(\Delta - \frac{\mu M}{4}\right)x^2 + \left(\Delta + \frac{\mu M}{4}\right)y^2 + \frac{3B\gamma}{16}(x^2 + y^2)^2 = C \qquad (3.2.4)$$

By plotting equation (3.2.4) for various values of μ and γ, the stability of the steady-state solution can be determined.

1. When $\Delta = \frac{1}{4}M\mu$
2. When $\Delta = 0$
3. When $\Delta = -\frac{1}{4}\mathrm{M}\mu$

3.2.2 Trivial solution

For the trivial solution, $\bar{a}_0 = 0$. The damped and undamped cases are studied in the following. For the damped case when $\xi \neq 0$, equations (3.1.6) and (3.1.7) can be converted to rectangular coordinates by putting $x = \bar{a}\cos\bar{\varphi}$ and y= $\bar{a}\sin\bar{\varphi}$,

$$x' = \bar{a}'\cos\bar{\varphi} - \bar{a}\sin\bar{\varphi}\,.\,\bar{\varphi}'$$

$$= -\epsilon\left(\frac{\xi\omega}{\upsilon} + \frac{M\mu}{8}\sin 2\bar{\varphi}\right)\bar{a}\cos\bar{\varphi} - \epsilon\frac{1}{2}\left(\Delta + \frac{3B\gamma}{8}\bar{a}^2 - \frac{M\mu}{4}cos2\bar{\varphi}\right)\bar{a}\sin\bar{\varphi} \qquad (3.2.8)$$

Which yields

$$x' = -\epsilon\left[\frac{\xi\omega}{\upsilon}x + \left(\frac{\Delta}{2} + \frac{M\mu}{8}\right)y + \frac{3B\gamma}{16}(x^2 + y^2)y\right] \qquad (3.2.9)$$

And similarly

$$y' = -\epsilon\left[\frac{\xi\omega}{\upsilon}y - \left(\frac{\Delta}{2} - \frac{M\mu}{8}\right)x - \frac{3B\gamma}{16}(x^2 + y^2)x\right] \qquad (3.2.10)$$

To investigate the stability of the trivial solution $x_0 = y_0 = 0$, consider the linearized equations

$$x' = -\epsilon\left[\frac{\xi\omega}{\upsilon}x + \left(\frac{\Delta}{2} + \frac{M\mu}{8}\right)y\right],\ y' = -\epsilon\left[\frac{\xi\omega}{\upsilon}y - \left(\frac{\Delta}{2} - \frac{M\mu}{8}\right)x\right] \qquad (3.2.11)$$

Which admit solutions of the form $x, y \propto e^{\rho\tau}$, where the characteristic numbers are given by

$$\begin{vmatrix} \rho + \epsilon\frac{\xi\omega}{\upsilon} & \epsilon\left(\frac{\Delta}{2} + \frac{M\mu}{8}\right) \\ -\epsilon\left(\frac{\Delta}{2} - \frac{M\mu}{8}\right) & \rho + \epsilon\frac{\xi\omega}{\upsilon} \end{vmatrix} = 0 \qquad (3.2.12)$$

i.e.

$$\left(\rho + \epsilon\frac{\xi\omega}{\upsilon}\right)^2 + \epsilon^2\left(\frac{\Delta^2}{4} - \frac{M^2\mu^2}{64}\right) = 0 \qquad (3.2.13)$$

or

$$\rho = \epsilon\left[-\frac{\xi\omega}{\upsilon} \pm \left(\frac{M^2\mu^2}{64} - \frac{\Delta^2}{4}\right)^{1/2}\right] \qquad (3.2.14)$$

The trivial solution is unstable when ρ is real and $\rho_{max} > 0$, implying that

$$-\frac{\xi\omega}{\upsilon} + \left(\frac{M^2\mu^2}{64} - \frac{\Delta^2}{4}\right)^{1/2} > 0 \qquad (3.2.15)$$

Or, since$\upsilon \approx 2\omega$,

$$|\Delta| < \left(\frac{M^2\mu^2}{16} - \xi^2\right)^{1/2} \qquad (3.2.16)$$

Otherwise the solution is Stable.

3.3 RESULTS AND DISCUSSIONS

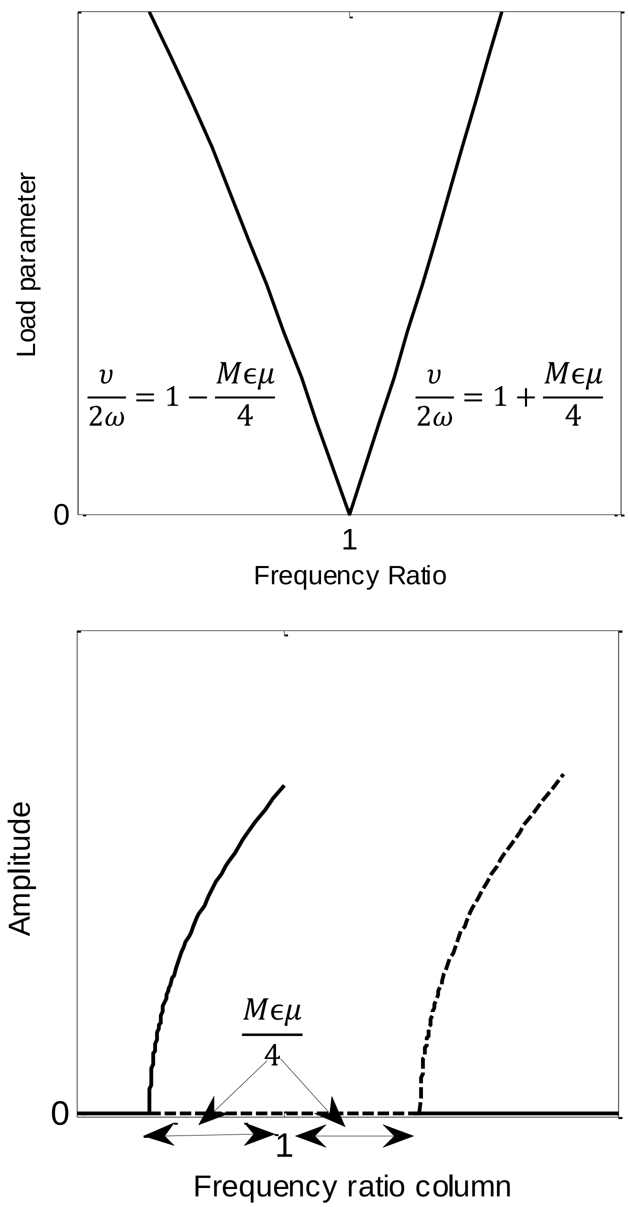

Fig. 10 Amplitude-Frequency

The steady state solution represented by solid line is stable while the dash line is unstable in Amplitude-Frequency curve.

1. When $\Delta = \frac{1}{4} M\mu$, equation (3.2.4) becomes

$$\frac{\mu M}{2} y^2 + \frac{3B\gamma}{16}(x^2 + y^2) = C \tag{3.3.1}$$

The phase portrait is drawn. It is seen that the steady-state solution corresponding to the singular point $x = y = 0$ is a centre and is hence stable.

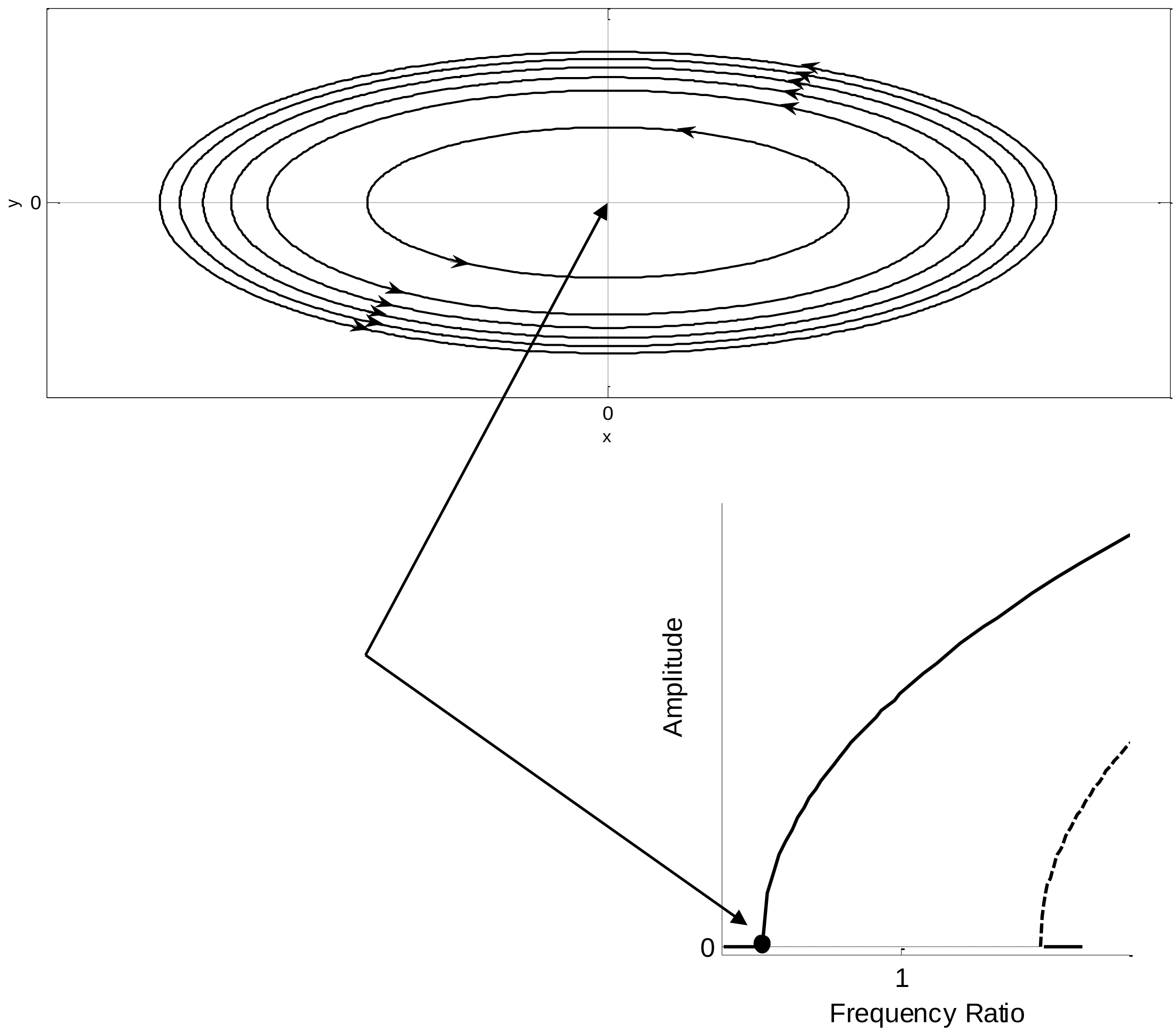

Fig. 11a Phase Portraits

2. When $\Delta = 0$, equation (3.2.4) becomes

$$\frac{1}{4}\mu M(y^2 - x^2) + \frac{3B\gamma}{16}(x^2 + y^2)^2 = C \tag{3.3.2}$$

With the phase portrait shown. The steady-state solution corresponding to $x = y = 0$ is a saddle point and is therefore unstable. The steady-state solutions with $\bar{a}_0 \neq 0$ are related to two centres on the x-axis and are hence stable.

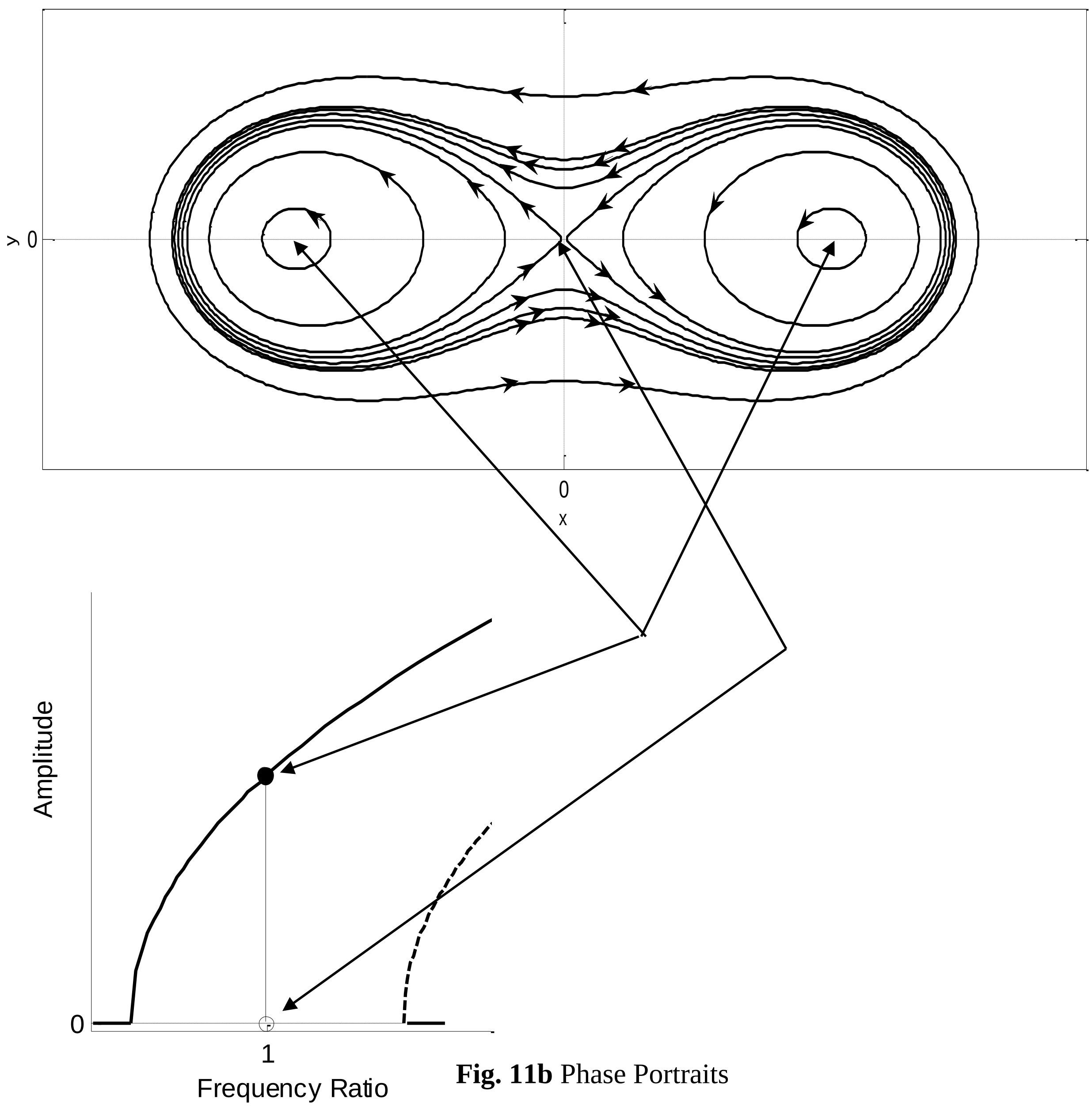

Fig. 11b Phase Portraits

3. When $\Delta = -\frac{1}{4}\text{M}\mu$, equation (3.2.4) is reduced to

$$-\frac{1}{2}\mu M x^2 + \frac{3B\gamma}{16}(x^2 + y^2) = C \qquad (3.3.3)$$

And the phase portrait is shown. Similar to the case when $\Delta = 0$, the steady-state solution corresponding to $x = y = 0$ is a saddle point and is unstable. The steady-state solutions with $\bar{a}_0 = 0$ correspond to two singular points that are centres and are stable.

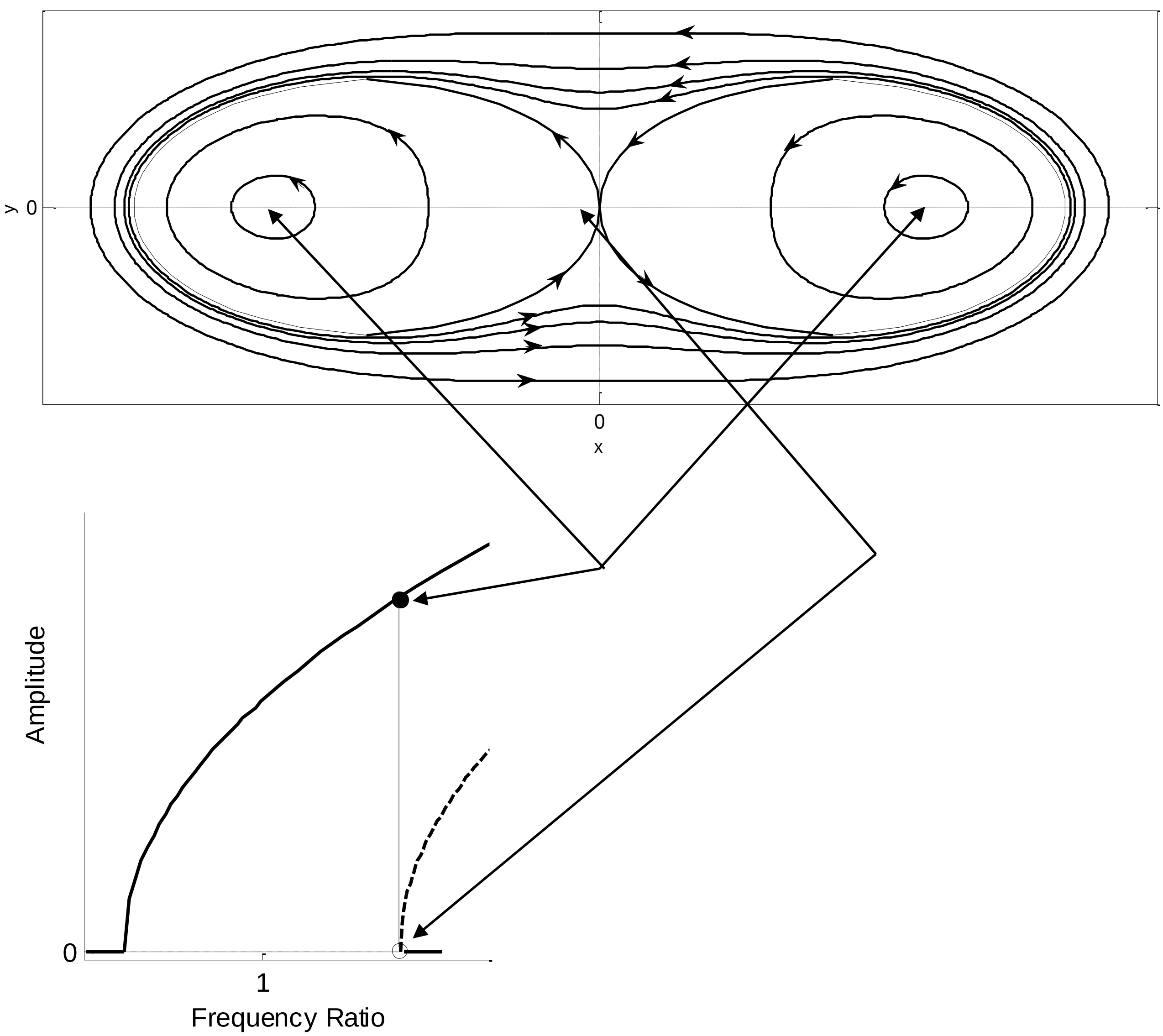

Fig. 11c Phase Portraits

Chapter 4

CONCLUSIONS

In the present study, dynamic instability of functionally graded cylindrical column with different end conditions subjected to dynamic axial loading is studied. The width of instability region for period 2T is more compared to the width of period T and hence, has greater practical importance. At higher load parameter the instability regions come closer. Hence, causing the column unstable for most of the frequency ratios. For damped column to be unstable, the minimum value of load parameter value increases with the increase of zone number i.e. from first zone to fourth zone. The boundary conditions of the column have a significant influence on the dynamic instability region. The width of instability region decreases with the increase in the end restraint. When the gradient of modulus of elasticity increases from centre to periphery the width of the zone of instability is narrower than the column with opposite variation of modulus of elasticity with same extreme values.

The instability points on amplitude-frequency relation have been studied by plotting the phase portraits.

FUTURE SCOPE

- In civil engineering, an economic design often requires a variable cross-section to be utilized. Non-prismatic or non-uniform columns or beams have wide applications in practice, e.g. in long-span bridges and industrial structures. The dynamic stability of these non-prismatic columns and beams could be determined once their shape function is known to us.
- Apart from columns and beams lot of work could be done in functionally graded plates and shells. Shells could be cylindrical or spherical.

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