SOLUTION OF LARGE STRAIN HYPERELASTIC PROBLEMS USING FINITE ELEMENT METHOD

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I, **Sushant Verma,** hereby certify that the work which is being presented in this thesis entitled **"Solution of Large Strain Hyperelastic Problem using Finite Element Method"** is submitted in the partial fulfillment of the requirement for degree of **Master of Technology (Computational Design)** in Department of Mechanical Engineering at **Delhi Technological University** is an authentic record of my own work carried out under the supervision of **Associate Prof. Dr.Atul Kumar Agrawal.** The matter presented in this thesis has not been submitted in any other University/Institute for the award of Master of Technology Degree. Also, it has not been directly copied from any source without giving its proper reference.

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CERTIFICATE

This is to certify that this thesis report entitled, **"Solution of Large Strain Hyperelastic Problem Using Finite Element Method"** being submitted by **Sushant Verma (Roll No. 2K14/CDN/22)** at Delhi Technological University, Delhi for the award of the Degree of Master of Technology as per academic curriculum. It is a record of bonafide research work carried out by the student under my supervision and guidance, towards partial fulfillment of the requirement for the award of Master of Technology degree in Computational Design. The work is original as it has not been submitted earlier in part or full for any purpose before.

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ABSTRACT

The hyperelastic material (elastomer) is a cross linked polymer consisting of molecules chain-linked together by bonds. The properties of this material (Excellent damping, energy absorption characterisitcs , flexibility, resilience, long service life ,moldability etc) increase its applications in various fields. Many machine elements incorporate the elastomer components such as tyres ,center bearing supports, pressurised cylinder, cables etc. For all the components to perform their functions properly it become necessary to know the behaviour of these material under various loading conditions.

Present work describes the behaviour of the hyperleastic material subjected to large strain by using the code Flagshyp(Finite Element Large Strain Hyperleastic Program) written by Javier Bonet and Richard D.Wood . In this Work three hyperelastic constitutive models i.e. Odgen model, Incompressible neo-hookean model and Mooney-Rivlin model have been used to solve the hyperelastic problems (truss , plate with hole and pressurised Cylinder) subjected to large strain.

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Introduction

1.1 Background

A material is said to be hyperelastic if its stress-strain relationships determined from a strain energy density function. An example of hyperelastic material/Cauchy elastic material (elastomer) is a cross-link polymer made up of molecular chain linked together by bonds as shown in fig 1.1; these bonds can pull back to its original shape after unloading.

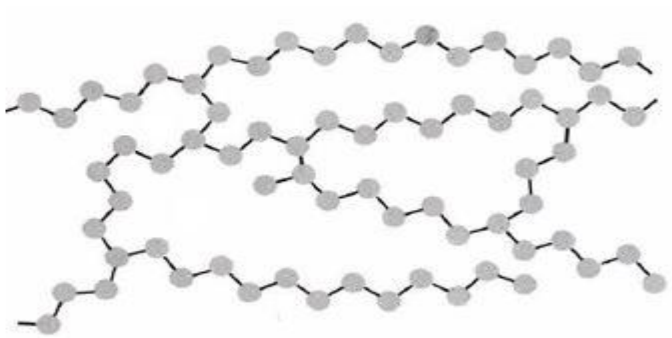

Fig 1.1 Cross linked polymer

Now a days, this material is used extensively in many industries because of its wide availability, low cost, excellent damping, energy absorption characteristics, flexibility, resilience, long service life, ability to seal against moisture heat and pressure, moldability and variable stiffness. According to the Rubber Industry Report (2014) published by the secretariat of the international rubber study group[1], the global consumption of elastomer(rubber) will reach to 30.4 million ton in 2019 compared to 23.9 million ton in 2010.

Fig 1.2 Mechanical Components of Elastomers

Many machine elements incorporate the elastomer component, such as automobile industry tyres, engine and transmission mounts, center-bearing-supports etc. For the parts to perform their function satisfactorily, it become necessary that all its components perform their functions properly, which make it important the analysis of this material with good accuracy under different loading conditions.

1.2 Motivation

The motivation of this thesis came from the wide application of elastomer components in various fields. For efficiently use of hyperelastic material it become necessary to know its behaviour under different conditions like large displacement- large strains, large rotation –small strains etc. But the linear elastic model do not accurately describe the material behaviour as shown in fig 1.3. Hyperelastic constitutive law provide a mean of modeling the stress-strain behaviour of such material. Concerning the application of finite element methods to the solution of problem involving large displacement and rotation we note that there are basically two different approaches used in FEM first is the total lagrangean formulation and the second is updated lagrangean formulation.

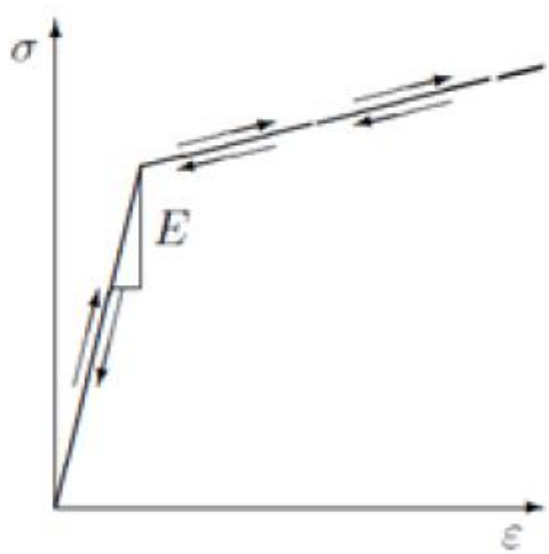

Fig 1.3 stress-strain curve of elastomers

1.3 Hyperelastic Constitutive Law

There are five models that are used to describe the stress-strain behaviour of hyperelastic materials[2].

1.3.1 Ogden Model

This model describe as an incompressible hyperelastic material in principle direction. The strain energy for this model is define as,

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{i+j=1}^{N} \frac{2\mu_i}{a_i^{\,2}} [\lambda_1^{\,a_i} + \lambda_2^{\,a_i} + \lambda_3^{\,a_i}] + \frac{K}{2}(J-1)^2 \tag{1.1}$$

Where μ_i, a_i, K are material properties and $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches. For rubber elastic model with limited compressibility, $K \gg \mu_i$.

1.3.2 Mooney-Rivlin Model

The strain energy function for this model is given as

$$\Psi = \frac{\mu_1}{2}(I_C - 3) + \frac{\mu_2}{2}(II_C - 3) + \frac{K}{2}(J-1)^2 \tag{1.2}$$

where μ_1, μ_2, K are the material properties and I_C, II_C are the invariants of right Cauchy green deformation tensor **C** . For rubber elastic model with limited compressibility, $K \gg \mu_i$.

1.3.3 Neo-Hookean Model

The strain energy function is given as,

$$\Psi = \frac{\mu_1}{2}(I_C - 3) + \frac{K}{2}(J-1)^2 \tag{1.3}$$

Where J is the jacobian. For fully incompressible case, $K(J-1) = \frac{p}{3}$.

1.3.4 Arruda Boyce Model

This model is defined as,

$$\Psi = \mu\frac{1}{2}(I_C - 3) + \frac{1}{20\gamma^2}\left(I_C{}^2 - 9\right) + \frac{11}{1050\gamma^4}(I_C^3 - 27) + \cdots\frac{K}{2}(J-1)^2 \qquad (1.4)$$

where μ, γ are material constant

1.3.5 Yeoh Model

The strain energy density function for yeoh model is defined as

$$\Psi = \sum_{i+j=1}^{3} C_{ij}(I_C - 3)^i(II_C - 3)^j + \sum_{i=1}^{3}\frac{K_i}{2}(J-1)^{2i} \qquad (1.5)$$

where C_{ij}, K_i are the material constant.

1.4 Literature Review

Now a days, analysis of non-linear behavior of components is most likely to involve a simulation, because of the easy availability of finite element softwares to provide good results with accuracy it is necessary to understand the fundamental of linear and non-linear continuum mechanics, non-linear finite element formulation and solution procedure. A number of books are available to provide background on this subject, for example the book of JN Reddy [3], Klaus Hackl and Mehndi goodarazi [4] explain the basic concepts of linear and non-linear continuum mechanics(stress and tensor, elasticity, Hooke's law, displacement functions. The book of Javier Bonet and Richard D.Wood[5], Morton E.Gurtin[6], J.Tensley Oden [7], explain the concepts dealing with different types of non-linearities viz geometrical non-linearity, material non-linearity and contact non-linearity; kinematics concepts viz deformation tensors, strains; kinetics concepts viz Cauchy stress, first and second piola kirchoff stress and different governing equations involved in it.

1.4.1 Literature review on Finite element analysis for a large strain hyperelastic material

Meral Bayraktar [8] studied the hyperelastic constitutive models for isotropic non linear material based on strain energy potential viz polynomial model, neo-hookean model, yeoh model, arruda and boyce model, vander wall model, odgen model and mooney –rivlin model. The ability of model was compared in predicting uniaxial deformation states based on the experimental data from dumb-bell test specimen under uniaxial loading. The result showed that the mooney-rivlin model, neo-hookean model and odgen model are more suitable for small, moderate and large deformation if the material is incompressible. E.Boudaia et al. [9] analysed the large deformation problem of ogden hyperelastic model based on the finite element method follow updated lagrangean formulation.

Dr Mushin N. Hamza et al. [10] determined and compared the hyperelastic constitutive model for incompressible elastomers, viz Odgen model and mooney-rivlin model; the material parameter is obtained from using least square method and non-linear least square method respectively. The results have been compared with the behaviour of material obtained from treloar experiments. The results showed that the mooney rivlin is most suitable when deformation is 100% while odgen is more appropriate when deformation exceed 100%.

Klaus-Jurgen Bathe et al. [12] derived and compared the two finite element incremental formulations viz Updated lagrangean formulation and total lograngean formulation for non linear static and dynamic analyses. The formulation include large displacement, large strain and material non linearities. The formulations are listed below

Total Lagrangean Formulation:

$$\int_{0_v} {}_0C_{ijrs}\, {}_0\varepsilon_{rs}\, \delta\, {}_0\varepsilon_{ij}\, {}^0dv + \int_{0_v} {}^t_0S_{ij}\, \delta\, {}_0\eta_{ij}\, {}^0dv = {}^{t+\Delta t}R - \int_{0_v} {}^t_0S_{ij}\, \delta\, {}_0e_{ij}\, {}^0dv \qquad (1.6)$$

Updated Lagrangean Formulation:

$$\int_{t_v} {}_tC_{ijrs}\, {}_t\varepsilon_{rs}\, \delta\, {}_t\varepsilon_{ij}\, {}^tdv + \int_{t} {}^t\zeta_{ij}\, \delta\, {}_t\eta_{ij}\, {}^tdv = {}^{t+\Delta t}R - \int_{t_v} {}^t\zeta_{ij}\, \delta\, {}_te_{ij}\, {}^tdv \qquad (1.7)$$

Where $\mathbf{0}, \boldsymbol{t}\ \boldsymbol{and}\ \boldsymbol{t}+\Delta \boldsymbol{t}$ are time step/load level showing the reference configuration, new reference configuration and current configuration respectively; ${}_0\boldsymbol{C}_{ijrs}$, ${}_t\boldsymbol{C}_{ijrs}$ are the components of constitutive tensor at time $\boldsymbol{t}$ referred to the configuration at time $\mathbf{0}$ and $\boldsymbol{t}$ respectively; ${}_0\varepsilon_{rs}$, ${}_t\varepsilon_{rs}$components of strain increment tensor referred to the configuration at time $\mathbf{0}$ and $\boldsymbol{t}$ respectively; ${}_0\boldsymbol{e}_{ij}$, ${}_t\boldsymbol{e}_{ij}$ are linear part of strain increment at time $\mathbf{0}$ and $\boldsymbol{t}$ respectively; ${}_0\boldsymbol{\eta}_{ij}$, ${}_t\boldsymbol{\eta}_{ij}$ are non-linear part of strain increment at time $\mathbf{0}$ and $\boldsymbol{t}$ respectively; ${}^t_0\boldsymbol{S}_{ij}$ are the components of second piola kirchoff stress tensor at time $\boldsymbol{t}$ referred to configuration at time $\mathbf{0}$; ${}^t\boldsymbol{\zeta}_{ij}$ are the components of Cauchy stress tensor at time $\boldsymbol{t}$ and ${}^{t+\Delta t}\boldsymbol{R}$ is the external virtual work corresponding to configuration at time $\boldsymbol{t}+\Delta\boldsymbol{t}$**.** Plate with hole and cantilever beam problems were considered. Result show that both give same results but UL formulation requires less time to calculate the element matrices whereas the advantage of TL is that the derivative of interpolation functions are with respect to the initial configuration; therefore need to be performed only once and the effort of implementing a new non-linear constitutive relation is less than that of UL method.

David S.Malkus, et al. [13] studied the application of penalty method for incompressible non-linear problem. They examine the tangent stiffnes matrix $\boldsymbol{K}$ should be positive definite throughout the whole sequence of iteration for maintaining the stability of solution process for that the load increment and refinement strategies explained by oden [7].

1.5 Objective

The main aim of the research is to analyze the behaviour of hyperelatic material (incompressible elastomers) subjected to large strain through the code Flagshyp written by Javier Bonet and Richard D. Wood.

1.6 Organization of Thesis

The chapters of the thesis are arranged in the following manner.

1.Chapter one presents the background of the thesis. The main aim of the chapter is to provide the basic idea of the work described in the thesis.

2. Chapter two presents the literature review along with the objectives of the research.

3. Chapter three describes the different types of non-linearities, kinetics and kinematics involved in it and the constitutive behaviour of hyperelastic material

4. Chapter four describe the Finite element formulations (Total lagrangean formulation and updated lagrangean formulation) and solution procedure for non-linear analysis.

5 Chapter five describes the updated lagrangean formulation and newton raphson iterative method used in the Flagshyp program.

6. Chapter six describes various subroutines of the flagshyp code.

7. Chapter seven provides the results obtained through the code for various problems.

8. Chapter eight concludes and presents suggestions for future work.

Chapter 2

Non Linear Continuum Mechanics

2.1 Introduction

This chapter will introduce the linear and non-linear concepts dealing with different types of mechanical non linearities(geometric non linearity and material non linearity), kinematic concepts involved in it, deformation tensors, strains, polar decomposition theorem, volume change and area change. Thereafter the stress concepts i.e Cauchy stress tensor, first piola kirchoff stress tensor and second piola kirchoff stress tensor are described followed by the derivation of differential equations enforcing translational and rotational equilibrium and principle of virtual work in reference and current configurations. At the end of this chapter the constitutive behaviour of hyperelastic material is discussed. The material discussed in this chapter is available in text book "Introduction to Continuum Mechanics" by Morton Gurtin.

2.2 Linear and Non-Linear analysis

2.2.1 Linear Analysis

When forces are applied to a body, the body deforms which induce the stresses to make the state of equilibrium. The displacement, stress, strain and reaction forces are calculated by solving the equation

$$\boldsymbol{Kd} = \boldsymbol{F} \tag{2.1}$$

Where $\boldsymbol{K}$is the stiffness matrix, $\boldsymbol{d}$ is the displacement vector and $\boldsymbol{F}$ is the external force

The requirement for these equation to be applicable are that the deformation and rotation should be small(geometrically linear), the material behaviour sould be linearly elastic(obey hooke's law) and boundary constraint should be constant.

2.2.2 Non-Linear Analysis

In the linear analysis two assumption are made the material behaviour is linear elastic (materially linear) and that deformations are small (geometrically linear). The former assumption prescribes the validity of the Hooke's law and latter excludes the modelling of irreversible material behaviour such as plastification or damage but these assumptions make it inadequate for design analysis of real structures and machines due to which the necessity of non linear analysis become important.

There are mainly two types of Non linearity

1.Material Non Linearity

2. Geometrical Non Linearity

2.2.2.1 Material Non-Linearity

The stress-strain relationship is shown as,

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} \tag{2.2}$$

In actual condition stress tensor cannot be obtained by linear mapping of the strain tensor with the help of material tensor $\boldsymbol{C}$. The stress tensor $\boldsymbol{\sigma}$ is an arbitrary function of the strain tensor $\boldsymbol{\varepsilon}$, and other values $\boldsymbol{\alpha}$, described as internal variables or as time history variables that characterize non-elastic deformations or damage

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) \tag{2.3}$$

where the partial derivative of the stress tensor w.r.t the strain tensor defines the **tangential material tensor**

$$\mathbf{C}_{\tan}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) = \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\alpha})}{\partial \boldsymbol{\varepsilon}} \tag{2.4}$$

In order to make the solution of the initial value problems possible, the stress function (2.3) has to be supplemented with the so called equations of internal variables in the form

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\alpha}}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) \tag{2.5}$$

In the special case of non-linear elastic material laws, the stress state is only the function of the strain state

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \qquad \mathbf{C}_{\tan}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) = \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \tag{2.6}$$

and the equations are dropped. For the simulation of the non-linear material law

- non-linear elastic,
- elasto-plastic
- and elasto-damaged

material models and ground type combinations are in use.

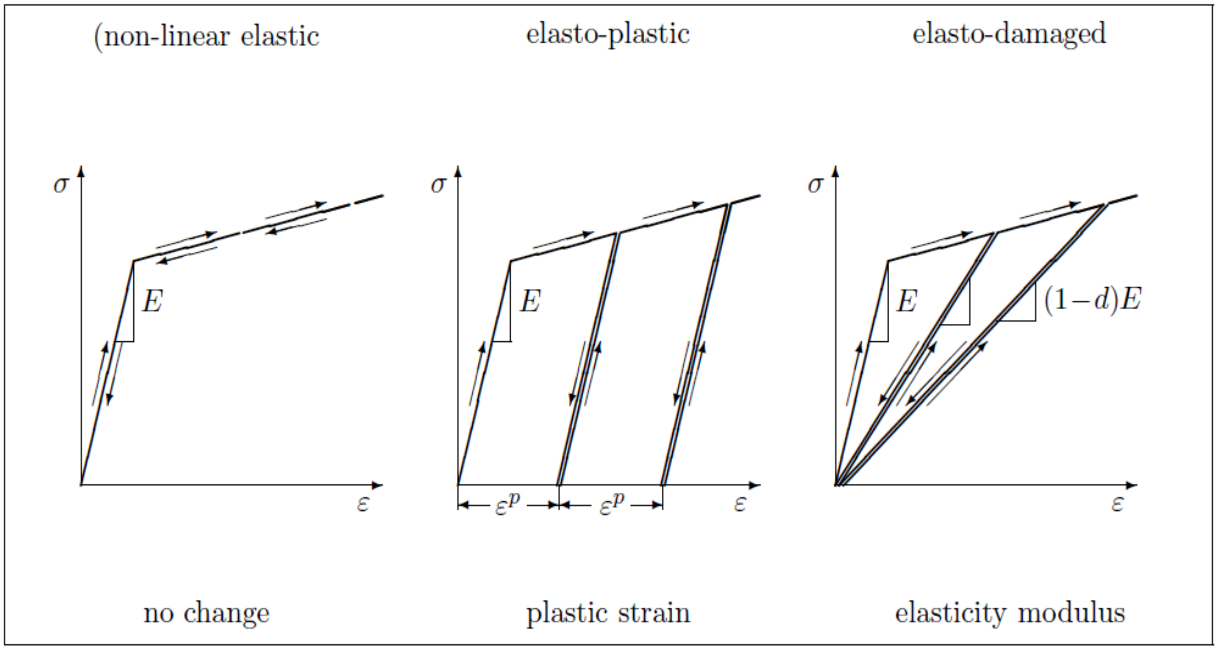

Fig 2.1 Cyclic loading and unloading of non linear elastic, elasto-plastic, and elasto-damaged bilinear material models

As shown in figure 2.1. In the non-linear elastic case,the loading and unloading path are same. In elasto-plastic case loading and unloading path are not same. Here after full loading plastic strains remain. In elasto-damaged case, no permanent strain is left but loading and unloading path are not same. A load introduction is effected by degradation of the stiffness with damage parameter ***d***.

2.2.2.2 Geometrical Non-Linearity

This non-linearity involve nonlinearities in kinematics quantities such as the green lagrange strain which has been described as function of the displacement gradient $\boldsymbol{\nabla u}$ and its transpose$\boldsymbol{\nabla^T u}$. the motion of the material point from reference to current configuration is describe with help of the displacement vector $\boldsymbol{u} = \boldsymbol{X} - \boldsymbol{x}$, the Green Lagrange strain tensor is given by,

$$\boldsymbol{E} = \frac{1}{2}[\boldsymbol{F}^T.\boldsymbol{F} - 1] = \nabla^{sym}\boldsymbol{u} + \frac{1}{2}\nabla^T\boldsymbol{u}.\nabla\boldsymbol{u} = \frac{1}{2}[\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u} + \nabla^T\boldsymbol{u}.\nabla\boldsymbol{u}] \tag{2.7}$$

It can be represented as a function of the material deformation gradient(discuss in section 2.1.1)

$$\boldsymbol{F} = \frac{\partial}{\partial \boldsymbol{X}}(\boldsymbol{X}+\boldsymbol{u}) = 1 + \nabla\boldsymbol{u} \tag{2.8}$$

From eq(2.7) w

$$\boldsymbol{\nabla}^{sym}\boldsymbol{u} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) & \frac{1}{2}(u_{1,3}+u_{3,1}) \\ \frac{1}{2}(u_{1,2}+u_{2,1}) & u_{2,2} & \frac{1}{2}(u_{2,3}+u_{3,2}) \\ \frac{1}{2}(u_{1,3}+u_{3,1}) & \frac{1}{2}(u_{2,3}+u_{3,2}) & u_{3,3} \end{bmatrix} \tag{2.9}$$

And

$$\boldsymbol{\nabla u} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \tag{2.10}$$

So from matrix multiplication we get

$$\frac{1}{2}\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{u}.\boldsymbol{\nabla u} = \begin{bmatrix} u_{k,1}u_{k,1} & u_{k,1}u_{k,2} & u_{k,1}u_{k,3} \\ u_{k,2}u_{k,1} & u_{k,2}u_{k,2} & u_{k,2}u_{k,3} \\ u_{k,3}u_{k,1} & u_{k,3}u_{k,2} & u_{k,3}u_{k,3} \end{bmatrix} \tag{2.11}$$

And the summation is performed over $k = 1, 2, 3$ respectively. The green lagrangean strain tensor finally represented as,

$$\boldsymbol{E_{ij}} = \frac{1}{2}(u_{i,j} + u_{j,i} + \ u_{k,i}u_{k,j}) \qquad \boldsymbol{E} = E_{ij}E_i{*}E_j \tag{2.12}$$

In vector form the above is represented as,

$$\boldsymbol{E(u)} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{13} \end{bmatrix} = \underbrace{\begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,1}+u_{1,2} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \end{bmatrix}}_{\mathrm{D}_\varepsilon u} + \underbrace{\begin{bmatrix} 1/2\,(u_{1,1}u_{1,1}+u_{2,1}u_{2,1}+u_{3,1}u_{3,1}) \\ 1/2\,(u_{1,2}u_{1,2}+u_{2,2}u_{2,2}+u_{3,2}u_{3,2}) \\ 1/2\,(u_{1,3}u_{1,3}+u_{2,3}u_{2,3}+u_{3,3}u_{3,3}) \\ u_{1,1}u_{1,2}+u_{2,1}u_{2,2}+u_{3,1}u_{3,2} \\ u_{1,2}u_{1,3}+u_{2,2}u_{2,3}+u_{3,2}u_{3,3} \\ u_{1,1}u_{1,3}+u_{2,1}u_{2,3}+u_{3,1}u_{3,3} \end{bmatrix}}_{E^{nl}(u)} \tag{2.13}$$

According to Eq. (2.13), the Green Lagrange strain tensor of geometrically non-linear deformations is calculated by addition of linear part $\mathbf{D}_{\boldsymbol{\varepsilon}}\boldsymbol{u}$ and the non-linear part $\boldsymbol{E^{nl}(u)}$,

$$E(u) = \mathrm{D}_\varepsilon u + E^{nl}(u) \tag{2.14}$$

2.3 Kinematics

It is the study of motion and deformation without reference to the cause.

2.3.1 Deformation

Deformation implies the change in shape or size of the body from an undeformed configuration $\boldsymbol{k_0}(\boldsymbol{\beta})$ to a deformed configuration $\boldsymbol{k_t}(\boldsymbol{\beta})$

A change in the configuration of a continuum body can be described by a displacement field. A displacement field is a vector field of all displacement vectors for all particles in the body which relates the deformed configuration with the undeformed configuration.

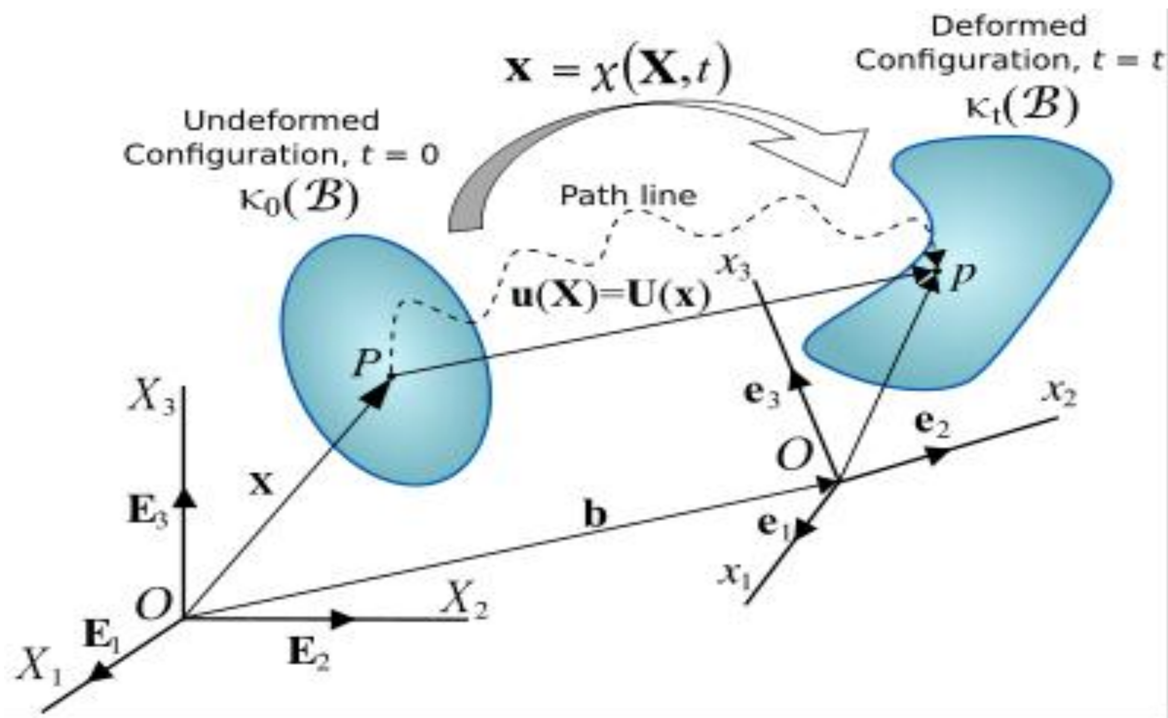

Fig 2.2 General motion of a deformable body

2.3.2 Material and Spatial Descriptions

2.3.2.1 Material Description

The positions and physical properties of the particles are described with respect to the original coordinate $\boldsymbol{X}$, occupied by the material paricle at time ***t*=0** for example the variation of density $\boldsymbol{\rho}$ over the body is describe as

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\boldsymbol{X}, \boldsymbol{t}) \tag{2.15}$$

2.3.2.2 Spatial Description

The positions and physical properties of the particles are described with respect to the position in space $\boldsymbol{x}$, occupied by the material particle at time $\boldsymbol{t}$ for example the variation of density $\boldsymbol{\rho}$ over the body is describe as

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\boldsymbol{X}, \boldsymbol{t}) \tag{2.16}$$

2.3.3 Deformation Gradient tensor

The deformation gradient tensor is the second order tensor which maps the line elements in the reference configuration to the line elements in the current configuration as shown in fig 2.3 is written as

$$\boldsymbol{F} = \boldsymbol{dx}/\boldsymbol{dX} \tag{2.17}$$

WheredX is the line element emanating from position X in the reference configuration and dx is the line element in the current configuration

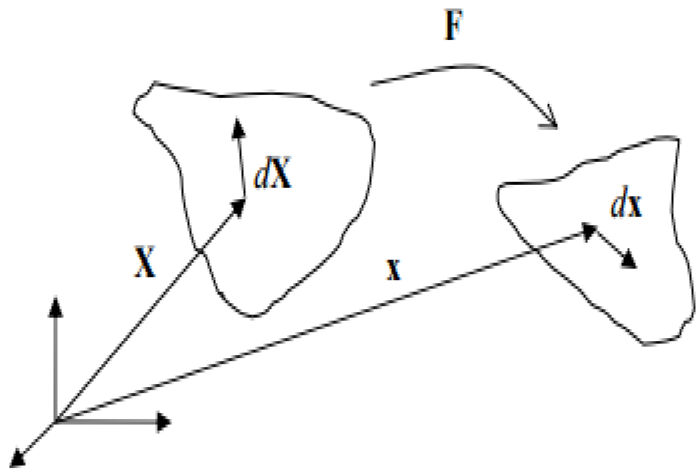

Fig 2.3 The Deformation gradient acting on the line element

In indical notation the deformation gradient tensor is expressed as

$$\boldsymbol{F} = \sum_{i,I=1}^{3} \boldsymbol{F}_{iI}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_I; \qquad \boldsymbol{F}_{iI} = \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{X}_I}; \qquad \boldsymbol{i}, \boldsymbol{I} = \mathbf{1}, \mathbf{2}, \mathbf{3} \tag{2.18}$$

Where e_i and E_I are the unit base vectors of initial geometry and deformed geormetry respectively.

2.3.4 The right Cauchy-Green deformation tensor

The right Cauchy Green deformation tensor is defined as

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \boldsymbol{U}^2 \tag{2.19}$$

Where $\boldsymbol{U}$ is the right stretch tensor

It can also be expressed in terms of eigen values and eigen vectors as

$$\boldsymbol{C} = \sum_{i=1}^{3} \boldsymbol{\lambda}_i^2 \boldsymbol{N}_i \otimes \boldsymbol{N}_i \tag{2.20}$$

Where $\boldsymbol{\lambda}_i^2$ are the eigen values(principal value) of $\boldsymbol{C}$ and $\boldsymbol{N}_i$ are the eigen vectors(principal direction) of $\boldsymbol{C}$

2.3.5 The left Cauchy-Green deformation tensor or finger tensor

The left Cauchy-Green deformation tensor is defined as

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{T} = \boldsymbol{V}^{2} \tag{2.21}$$

$\boldsymbol{V}$ is the left stretch tensor

It can also be expressed in terms of eigen value and eigen vector as

$$\boldsymbol{b} = \sum_{a=1}^{3} \lambda_a^2 \boldsymbol{n}_a \otimes \boldsymbol{n}_a \tag{2.22}$$

Where $\boldsymbol{\lambda_a^2}$ are the eigen values of $\boldsymbol{b}$ and $\boldsymbol{n_a}$are the eigen vectors of $\boldsymbol{b}$

2.3.6 Strain

Strain (Ɛ) is a measure of deformation representing the displacement between particles in the body relative to a reference length. A general deformation of a body can be expressed in the form $\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{X})$ where $\boldsymbol{X}$ is the reference position of material points in the body and $\boldsymbol{I}$ is the identity tensor

$$\mathcal{E} = \frac{\boldsymbol{\vartheta}(\boldsymbol{x} - \boldsymbol{X})}{\boldsymbol{\vartheta x}} = \boldsymbol{F}' - \boldsymbol{I} \tag{2.23}$$

2.3.7 Classification of strain

2.3.7.1 Engineering strain

The Cauchy strain or engineering strain$(\boldsymbol{H})$ is expressed as the ratio of change in dimension to the initial dimension of the material body where forces are applied. The body subjected to small deformation and strain is given as

$$\boldsymbol{H} = \frac{\Delta \boldsymbol{L}}{\boldsymbol{L}} = \frac{\boldsymbol{l} - \boldsymbol{L}}{\boldsymbol{L}} \tag{2.24}$$

where $\boldsymbol{L}$ is the original length of the fiber and $\boldsymbol{l}$ is the final length of the fiber

2.3.7.2 Logarithmic strain

It is also called true strain or hencky strain and it is obtained by integrating the incremental strain($\int_{L}^{l}\frac{\delta l}{l}$) when the stretching is taking place from its original length $\boldsymbol{L}$ to its final length $\boldsymbol{l}$. The logarithmic strain provides the correct measure of the final strain when deformation takes place in a series of increments. This strain is subjected to large deformation and expressed as

$$\boldsymbol{L} = \int_{L}^{l}\frac{\boldsymbol{\delta l}}{\boldsymbol{l}} = \ln\left(\frac{\boldsymbol{l}}{\boldsymbol{L}}\right) \qquad (2.25)$$

2.3.7.3 Green lagrangean strain

The green lagrangean strain($\boldsymbol{E}$) is a material tensor which gives information about the change in the squared length of the element.It deals with the large deformation and is defined as

$$\boldsymbol{E} = \frac{1}{2}\left(\frac{\boldsymbol{l}^2 - \boldsymbol{L}^2}{\boldsymbol{L}^2}\right) = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \qquad (2.26)$$

Where $\boldsymbol{C}$ is the left Cauchy green deformation tensor and $\boldsymbol{I}$ is the identity matrix

2.3.7.4 Eulerian almansi strain

The eulerian almansi strain($\boldsymbol{e}$) is a symmetric positive definite spatial tensor which give information about the change in the squared length of the element. It deals with large rotations and is defined as

$$\boldsymbol{e} = \frac{1}{2}\left(\frac{\boldsymbol{l}^2 - \boldsymbol{L}^2}{\boldsymbol{l}^2}\right) = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{b}^{-1}) \qquad (2.27)$$

Where $\boldsymbol{b}$ is the right Cauchy green deformation tensor.

2.3.7.5 Stretch ratio

The stretch ratio or extension ratio($\boldsymbol{\lambda}$) is a measure of the extension or normal strain of a differential line element. It is define as the ratio between the final length $\boldsymbol{l}$ and the initial length $\boldsymbol{L}$ of the material line.

$$\boldsymbol{\lambda} = \frac{\boldsymbol{l}}{\boldsymbol{L}} \qquad (2.28)$$

2.3.8 Polar decomposition

The polar decomposition of deformation gradient($\boldsymbol{F}$) is shown in fig 2.4 as,

$$\boldsymbol{F} = \boldsymbol{R}.\boldsymbol{U} = \boldsymbol{V}.\boldsymbol{R} \tag{2.29}$$

Where $\boldsymbol{R}$ is the othogonal rotation tensor, $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric tensors called the right stretch tensor and left stretch tensor respectively.

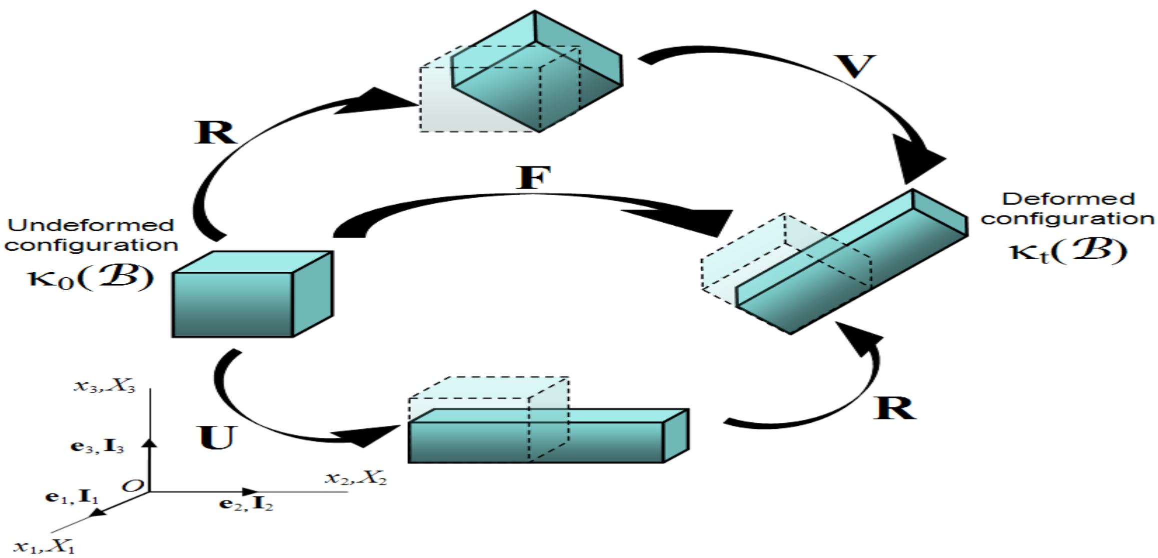

Fig 2.4 Polar decomposition of deformation gradient F

2.3.9 Volume change and area change

Consider an infinitesimal volume element in the material configuration with edges parallel to the Cartesian axes given by $d\boldsymbol{X_1} = dX_1\boldsymbol{E_1}, d\boldsymbol{X_2} = dX_2\boldsymbol{E_2}$ and $d\boldsymbol{X_3} = dX_3\boldsymbol{E_3}$, where $\boldsymbol{E_1}, \boldsymbol{E_2}$ and $\boldsymbol{E_3}$ are the orthogonal unit vectors(see Figure 3.5). The elemental material volume $\boldsymbol{dV}$ defined by these three vectors as

$$\boldsymbol{dV = dX_1 dX_2 dX_3} \tag{2.30}$$

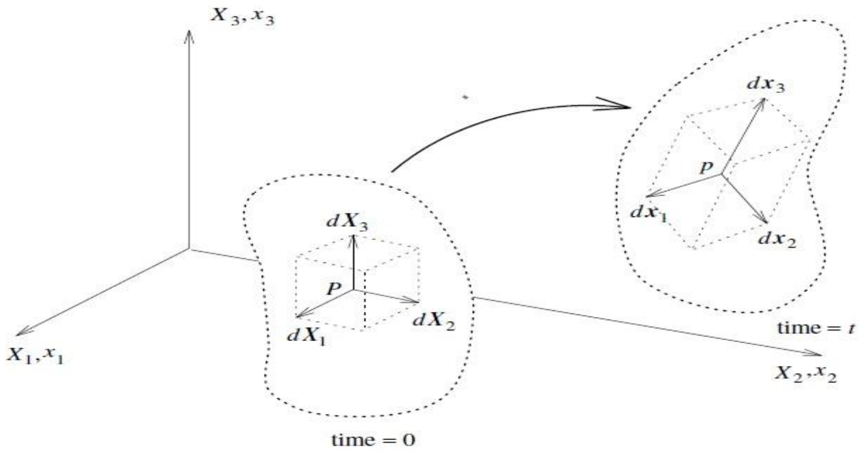

Fig 2.5 Volume change

After deformation the final volume ($\boldsymbol{dv}$) is shown as

$$\boldsymbol{dv = dX_1 dX_2 dX_3 \frac{\partial x}{\partial X_1} \cdot \left(\frac{\partial x}{\partial X_2} \times \frac{\partial x}{\partial X_3}\right) = J dV} \tag{2.31}$$

Where $\boldsymbol{J}$ is the Jacobian and is define as

$$\boldsymbol{J = detF} \tag{2.32}$$

Consider an element of area in the initial configuration $\boldsymbol{dA = dAN}$ which after deformation becomes $\boldsymbol{da = dan}$ as shown in fig 2.6. The relationship between deformed area and initial area is given as

$$\boldsymbol{dan = JDAF^{-T}.N} \tag{2.33}$$

Where $\boldsymbol{n}$ is the outward normal to the area element in the current configuration, N is the outward normal in the reference configuration, $\boldsymbol{F}$ is the deformation gradient and $\boldsymbol{J}$ is the Jacobian.

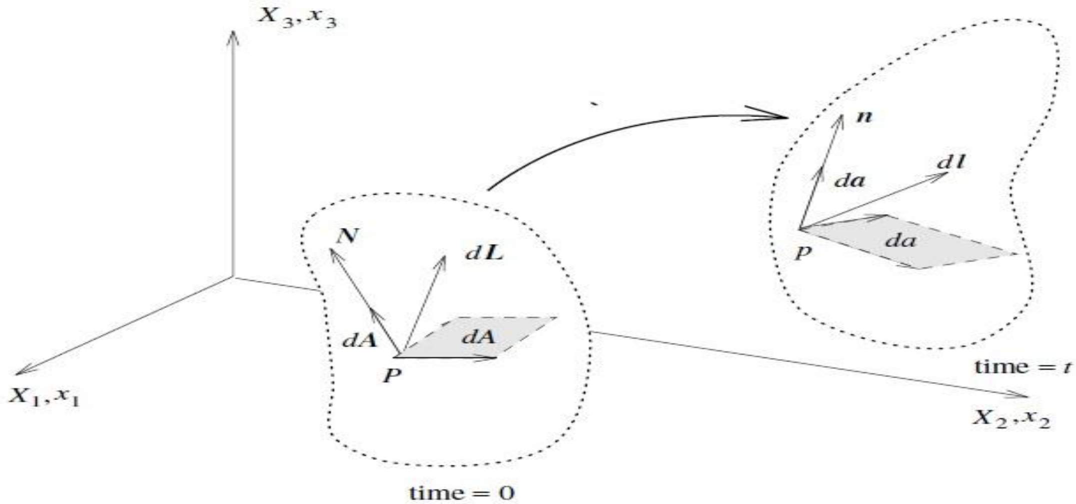

Fig 2.6 Area change

2.4 Stress and Equilibrium

2.4.1 Stress

Stress is defined as the intensity of internal resisting force developed at a point against the deformation caused due to load. It is generally expressed as

$$\boldsymbol{\sigma} = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{P}}{\Delta \boldsymbol{a}} \tag{2.34}$$

Where $\Delta \boldsymbol{a}$ is the elemental area to normal $\boldsymbol{n}$ in the neighborhood of point p, $\Delta \boldsymbol{P}$ is the internal resisting force against the load on the elemental area as shown in fig 2.7

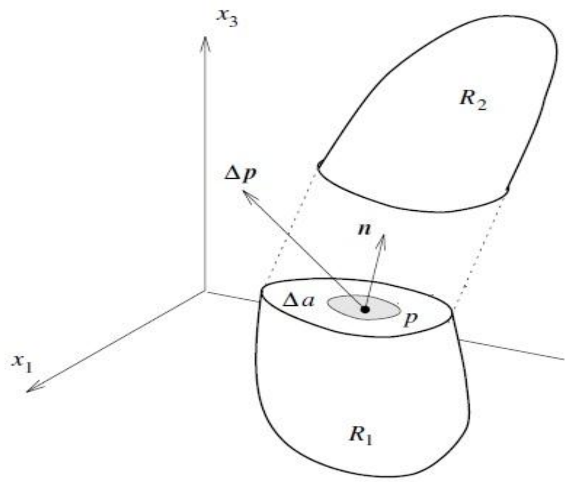

Fig 2.7 stress

2.4.2 Classification of stress tensor

There are three types of stress tensor in continuum mechanics Cauchy stress tensor, first-piola kirchoff stress tensor and second piola-kirchoff stress tensor

2.4.2.1 Cauchy stress tensor

The Cauchy stress tensor($\boldsymbol{\sigma}$) is a second order tensor that completely define the state of stress at a point inside a material in the deformed state. The tensor relates a unit-length direction vector $\boldsymbol{n}$ to the stress vector $\boldsymbol{T^n}$ across an imaginary surface perpendicular to $\boldsymbol{n}$ as shown in fig 2.8 and described as

$$\boldsymbol{T_j^n = \sigma_{ij} n_i} \qquad \boldsymbol{i, j = 1, 2, 3} \qquad (2.35)$$

Where $\boldsymbol{\sigma}$ is the Cauchy stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad (2.36)$$

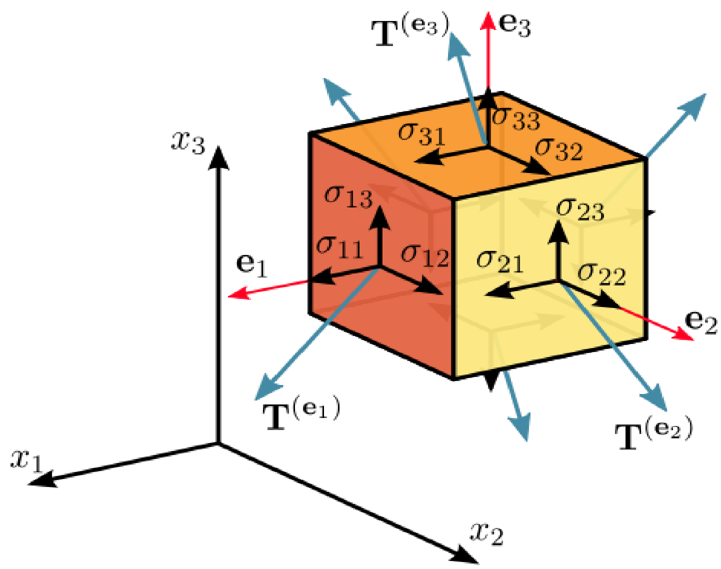

Fig 2.8 Component of stess in three dimension

2.4.2.2 First piola-kirchoff stress tensor

The first piola kirchoff stress tensor($\boldsymbol{P}$) is an unsymmetric two point tensor that relates forces in the current(spatial) configuration with area in the reference (material) configuration and expressed as

$$\boldsymbol{P} = \boldsymbol{J}\boldsymbol{\sigma}\boldsymbol{F}^{-\boldsymbol{T}} \tag{2.37}$$

2.4.2.3 Second piola-kirchoff stress tensor

The second piola-kirchoff stress tensor($\boldsymbol{S}$) is a symmetric one point tensor that relates forces in the reference configuration to area in the reference configuration. The force in the reference configuration is obtain via a mapping that preserves the relative relationship between the force direction and the area normal in the reference configuration. It is expressed as

$$\boldsymbol{S} = \boldsymbol{J}\boldsymbol{F}^{-1}.\boldsymbol{\sigma}.\boldsymbol{F}^{-\boldsymbol{T}} \tag{2.38}$$

2.4.3 Equilibrium

2.4.3.1 Translational equilibrium

Consider the spatial configuration of a general deformable body define by a volume $\boldsymbol{v}$ with boundary area $\boldsymbol{\partial v}$ as shown in Figure 3.9 is under the action of the body forces $\boldsymbol{f}$ per unit volume

$$\int_{\partial v} \boldsymbol{t}\,d\boldsymbol{a} + \int_{v} \boldsymbol{f}\,d\boldsymbol{v} = \boldsymbol{0} \tag{2.39}$$

and traction forces $\boldsymbol{t}$ per unit area acting on the boundary. Now the translational equilibrium implies that the sum of all the forces acting on the body vanishes

By using eq (2.35) and applying the gauss divergence theorem the above equation can be written as

$$\int_{v} (\bar{\boldsymbol{\sigma}} + \boldsymbol{f}) dv = \mathbf{0} \tag{2.40}$$

Where($\bar{\boldsymbol{\sigma}}$) is the vector which is obtained from the divergence of Cauchy stress tensor.

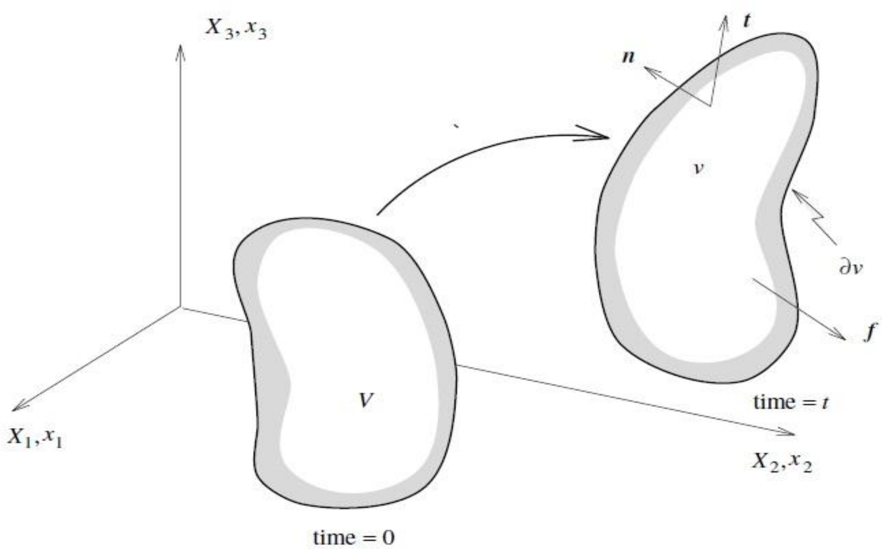

Fig 2.9 Equilibrium

2.4.3.2 Rotational equilibrium

Consider the rotational equilibrium of a body under the action of traction forces($\boldsymbol{t}$) and body forces($\boldsymbol{f}$)

$$\int_{dv} \boldsymbol{x} \times \boldsymbol{t} da + \int_{v} \boldsymbol{x} \times \boldsymbol{f} dv = \mathbf{0} \tag{2.41}$$

Where $\boldsymbol{x}$ is the position vector

By using the equation(2.35) and gauss divergence theorem the above equation can be written as

$$\int_{v} \boldsymbol{x} \times \bar{\boldsymbol{\sigma}}\, dv + \int_{v} \text{M}: \boldsymbol{\sigma}^{T}\, dv + \int_{v} \boldsymbol{x} \times \boldsymbol{f} dv = \mathbf{0} \tag{2.42}$$

Where M is the third order alterating tensor which is given as

$$M:\sigma^T = \begin{matrix}\sigma_{32} - & \sigma_{23} \\ \sigma_{13} - & \sigma_{31} \\ \sigma_{21} - & \sigma_{12}\end{matrix} = 0 \qquad (2.43)$$

2.4.3.3 Principle of virtual Displacement

It states that a body is in equilibrium if the internal virtual work is equal to the external virtual work for every kinematically admissible displacement field.

Consider the spatial configuration of a general deformable body define by a volume **v** with boundary area $\boldsymbol{\partial v}$ that is under the action of the body forces $\boldsymbol{f}$ per unit volume and traction forces $\boldsymbol{t}$ per unit area acting on the boundary.let $\boldsymbol{\delta}$υ denote the arbitrary virtual displacement as shown in fig 2.10. According to the principle of virtual work

$$\int_{\upsilon} \boldsymbol{\sigma\delta\varepsilon d\upsilon} = \int_{\upsilon} \boldsymbol{f.\delta\upsilon d\upsilon} + \int_{\partial\upsilon} \boldsymbol{t.\delta\upsilon da} \qquad (2.44)$$

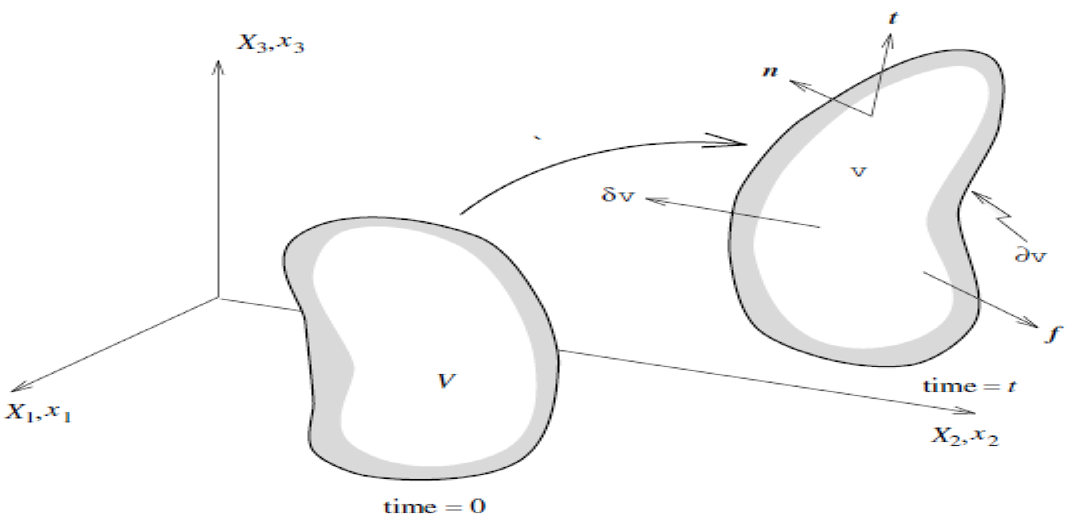

Fig 2.10 Principle of virtual work

In linear analysis we assume a small deformation which implies that the work is formulated over the original configuration. i.e.

$$\int_V \boldsymbol{\bar{\varepsilon}^T \tau dV} = \int_V \boldsymbol{\bar{U}^T f^B dV} + \int_S \boldsymbol{\bar{U}^{(s)} f^s dS} \qquad (2.45)$$

(internal virtual work) = (external virtual work = R)

where $\boldsymbol{\bar{U}}$, $\boldsymbol{\bar{\varepsilon}}$ are the virtual displacement and the corresponding virtual strain in the reference configuration respectively. $\boldsymbol{f^s}, \boldsymbol{f^B}$ are the external surface traction and the body force acting in the reference configuration respectively

In nonlinear analysis the aim is to evaluate the equilibrium positions of the complete body at discrete time points **0, Δt, 2Δt, 3Δt** …….. .

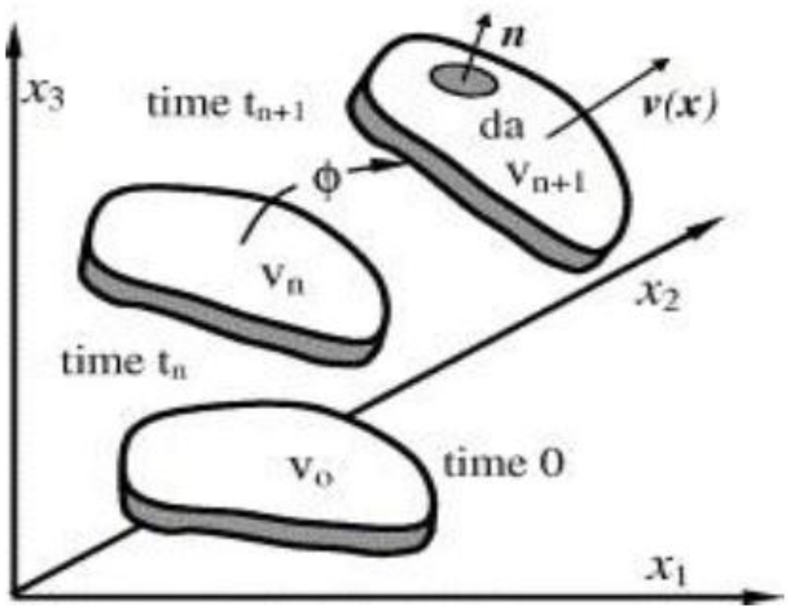

Fig 2.11 Element showing incremental motion

The equilibrium of the body corresponding to the t+Δt using the principle of virtual work is describe as

$$\int_{V}^{t+\Delta t} {}^{t+\Delta t}\tau_{ij}\; \delta\, {}_{t+\Delta t}\varepsilon_{ij}\; d^{t+\Delta t}V = \int_{V}^{t+\Delta t} {}^{t+\Delta t}f_i^B\; \delta u_i\; d^{t+\Delta t}V + \int_{S}^{t+\Delta t} {}^{t+\Delta t}f_i^S\; \delta u_i^S\; d^{t+\Delta t}S \qquad (2.46)$$

(internal virtual work) = (external virtual work=R)

where, ${}^{t+\Delta t}\tau_{ij}$ is the cartesian component of the Cauchy stress tensor (forces/area, in **deformed geometry**), ${}^{t+\Delta t}f_i^B$ is the component of externally applied forces per unit volume at time t+Δt, ${}^{t+\Delta t}f_i^S$ is the component of externally applied surface tractions per unit surface area at time t+Δt, ${}^{t+\Delta t}S$ is the surface at time t+Δt on which external tractions are applied, δu_i^S = δu_i evaluated on surface ${}^{t+\Delta t}S$ (since δu_i is zero initially on the real deformed surface at the time t+Δt) and $\delta_{t+\Delta t}\varepsilon_{ij}\; d^{t+\Delta t}V$ is the virtual strain which is describe as

$$\delta\, {}_{t+\Delta t}\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (2.47)$$

The equation (2.46) cannot be solved directly as the configuration of the body at time t+Δt is unknown. A solution is obtain by referring all the variables to a known previously calculated equilibrium configuration for the two approaches have been use i.e **Total lagrangean**

formulation and **Updated lagrangean formulation** which will be discussed in detail in the chapter 3

2.5 Hyperelasticity

Material is said to be hyperelastic for which the stress strain relationship is determine from a strain energy density function. Here the work done by the stresses during a deformation process is dependent only on the initial state at time t_0 and the final state at time t, the behaviour of the material is path independent.

A stored strain energy function per unit undeformed volume can be determine as the work done by the stresses from the initial to the final position as

$$\Psi(\boldsymbol{F}(\boldsymbol{X}),\boldsymbol{X}) = \int_{t_0}^{t} \boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X}),\boldsymbol{X}) : \dot{\boldsymbol{F}} dt; \qquad \dot{\Psi} = \boldsymbol{P} : \dot{\boldsymbol{F}} \tag{2.48}$$

Where $\dot{\boldsymbol{F}}$ is the rate of deformation gradient and $\dot{\Psi}$ is the rate of change of potential.

The rate of change of strain energy can also be expressed as

$$\dot{\Psi} = \sum_{i,j=1}^{3} \frac{\partial \Psi}{\partial \boldsymbol{F}_{ij}} \dot{\boldsymbol{F}}_{ij} \tag{2.49}$$

Comparing the above equation with equation(3.48) the components of two point tensor P are

$$\boldsymbol{P}_{ij} = \frac{\partial \Psi}{\partial \boldsymbol{F}_{ij}} \tag{2.50}$$

The above expression is written in more compact form as

$$\boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X}),\boldsymbol{X}) = \frac{\partial \Psi(\boldsymbol{F}(\boldsymbol{X}),\boldsymbol{X})}{\partial \boldsymbol{F}} \tag{2.51}$$

In terms of green lagrangean strain($\boldsymbol{E}$) and right cauhy green deformation tensor($\boldsymbol{F}$) the first piola kirchoff stress is calculated for compressible hyperelastic material as

$$\boldsymbol{P} = \boldsymbol{F}.\frac{\partial \Psi}{\partial \boldsymbol{E}} = 2\boldsymbol{F}.\frac{\partial \Psi}{\partial \boldsymbol{C}} \tag{2.52}$$

Similarly the second piola-kirchoff stress for compressible hyperelastic material is given by

$$\boldsymbol{S}(\boldsymbol{F}(\boldsymbol{X}),\boldsymbol{X}) = \boldsymbol{F}^{-1}.\frac{\partial \Psi}{\partial \boldsymbol{F}} \tag{2.53}$$

Where $\boldsymbol{F}^{-1}$ is the inverse of deformation gradient.

In terms of green lagrangean strain($\boldsymbol{E}$) and right cauhy green deformation tensor($\boldsymbol{F}$) the second piola kirchoff stress for compressible hyperelastic material shown as

$$\boldsymbol{S} = \frac{\partial \Psi}{\partial \boldsymbol{E}} = 2\frac{\partial \Psi}{\partial \boldsymbol{C}} \tag{2.54}$$

The Cauchy stress for compressible hyperselastic material in terms of deformation gradient, the green lagrangean strain and the right Cauchy green deformation graident is described as

$$\boldsymbol{\sigma} = \frac{1}{J}\frac{\partial \Psi}{\partial \boldsymbol{F}}.\boldsymbol{F}^{\boldsymbol{T}} = \frac{1}{J}\boldsymbol{F}.\frac{\partial \Psi}{\partial \boldsymbol{E}}.\boldsymbol{F}^{\boldsymbol{T}} = \frac{2}{J}\boldsymbol{F}.\frac{\partial \Psi}{\partial \boldsymbol{C}}.\boldsymbol{F}^{\boldsymbol{T}} \tag{2.55}$$

2.5.1 Elasticity tensor

2.5.1.1 The material or lagrangean elasticity tensor

The material or lagrangean elasticity tensor is a symmetric four order tensor which provides the relationship between the linearized stress $\boldsymbol{S}$ and the linearized strain $\boldsymbol{E}$ as

$$\boldsymbol{DS}[\boldsymbol{u}] = \boldsymbol{C}: \boldsymbol{DE}[\boldsymbol{u}] \tag{2.56}$$

Where $\boldsymbol{DS}[\boldsymbol{u}]$ is the directional derivative of second piola kirchoff stress in the direction of $\boldsymbol{u}$,$\boldsymbol{DE}[\boldsymbol{u}]$ is the directional derivative of green lagrangean strain in the direction of $\boldsymbol{u}$ and $\boldsymbol{K}$ is the lagrangean or material elasticity tensor which is defined by the partial derivatives as

$$\boldsymbol{C} = \sum_{I,J,K,L=1}^{3} \boldsymbol{C}_{IJKL}\boldsymbol{E}_{I} \otimes \boldsymbol{E}_{J} \otimes \boldsymbol{E}_{K} \otimes \boldsymbol{E}_{L} \tag{2.57}$$

Where $\boldsymbol{E}_{I}, \boldsymbol{E}_{J}, \boldsymbol{E}_{K}, \boldsymbol{E}_{L}$ are the unit base vector in the reference configuration and $\boldsymbol{H}_{IJKL}$ are the component of four order tensor $\boldsymbol{H}$ which is express as

$$\boldsymbol{C}_{IJKL} = \frac{\partial \boldsymbol{S}_{IJ}}{\partial \boldsymbol{E}_{KL}} = \frac{4\partial^{2}}{\partial \boldsymbol{H}_{IJ}\partial \boldsymbol{H}_{KL}} \tag{2.58}$$

2.5.1.2 The spatial or eulerian elasticity tensor

The spatial or eulerian elasticity tensor is a symmetric four order tensor which provide the relationship between the linearized Cauchy stress $\boldsymbol{\sigma}$ and the linearized strain($\boldsymbol{e}$) expressed as

$$\boldsymbol{c} = \sum_{\substack{ijkl=1 \\ IJKL=1}}^{3} \boldsymbol{J^{-1}F_{iI}F_{jJ}F_{kK}F_{lL}H_{IJKL}e_i \otimes e_j \otimes e_k \otimes e_l} \tag{2.59}$$

Where $\boldsymbol{e_i}, \boldsymbol{e_j}, \boldsymbol{e_k}, \boldsymbol{e_l}$ is the unit base vector of current configuration.

2.5.2 Isotropic hyperelastic material

2.5.2.1 Material description

The constitutive behaviour of the isotropic hyperelastic material is identical in every material direction. This implies that the hyperelastic potential $\boldsymbol{\Psi}$ is the invariant of the right Cauchy green deformation tensor $\boldsymbol{C}$ as

$$\Psi(\boldsymbol{C}(\boldsymbol{X}), \boldsymbol{X}) = \Psi(\boldsymbol{I_C}, \boldsymbol{II_C}, \boldsymbol{III_C}, \boldsymbol{X}) \tag{2.60}$$

Where the invariants of the right Cauchy green deformation tensor($\boldsymbol{C}$) are defined as

$$\boldsymbol{I_C = trC = C{:}I} \tag{2.61}$$

$$\boldsymbol{II_C = trCC = C{:}C} \tag{2.62}$$

$$\boldsymbol{III_C = detC = J^2} \tag{2.63}$$

Where $\boldsymbol{I}$ is the identity matrix.

The second piola-kirchoff stress is described for this material from equation (2.53) as

$$\boldsymbol{S} = \frac{\boldsymbol{2\partial}\Psi}{\boldsymbol{\partial C}} = \boldsymbol{2}\Psi_{\boldsymbol{I}}\boldsymbol{I} + \boldsymbol{4}\Psi_{\boldsymbol{II}}\boldsymbol{C} + \boldsymbol{2J^2}\Psi_{\boldsymbol{III}}\boldsymbol{C^{-1}} \tag{2.67}$$

Where$\Psi_{\boldsymbol{I}} = {}^{\boldsymbol{\partial}\Psi}/_{\boldsymbol{\partial I_C}}, \Psi_{II} = {}^{\boldsymbol{\partial}\Psi}/_{\boldsymbol{\partial II_C}}$ $\boldsymbol{and}\Psi_{III} = {}^{\boldsymbol{\partial}\Psi}/_{\boldsymbol{\partial III_C}}$

2.5.2.2 Spatial description

The cauchy stress for isotropic hyperelastic material is expressed as

$$\boldsymbol{\sigma} = \boldsymbol{2J^{-1}}\Psi_{\boldsymbol{I}}\boldsymbol{b} + \boldsymbol{4J^{-1}}\Psi_{\boldsymbol{II}}\boldsymbol{Ib^2} + \boldsymbol{2}\Psi_{\boldsymbol{III}}\boldsymbol{I} \tag{2.68}$$

where $\boldsymbol{b}$ is the left Cauchy green deformation tensor, $\boldsymbol{I}$ is the identity matrix and $(\Psi_{\boldsymbol{I}}\Psi_{\boldsymbol{II}}\Psi_{\boldsymbol{III}})$ are the derivative of strain energy function with respect to the invariants of spatial tensor $\boldsymbol{b}$.

2.5.3 Compressible neo hookean material.

A neo-hookean is a hyperelastic material model which is use for estimating the stress-strain behaviour of material withstanding large deformation. The energy function of the material is described as

$$\Psi = \frac{\boldsymbol{\mu}}{\mathbf{2}}(\boldsymbol{I_C} - \mathbf{3}) - \boldsymbol{\mu lnJ} + \frac{\boldsymbol{\lambda}}{\mathbf{2}}(\boldsymbol{lnJ})^{\mathbf{2}} \tag{2.69}$$

Where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the lame coefficients and $\boldsymbol{J}^{\mathbf{2}} = \boldsymbol{III_C}$

The second piola-kirchoff stress and cauchy stress for such material is expressed as

$$\boldsymbol{S} = \boldsymbol{\mu}(\boldsymbol{I} - \boldsymbol{C}^{-\mathbf{1}}) + \boldsymbol{\lambda}(\boldsymbol{lnJ})\boldsymbol{C}^{-\mathbf{1}} \tag{2.70}$$

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\mu}}{\boldsymbol{J}}(\boldsymbol{b} - \boldsymbol{I}) + \frac{\boldsymbol{\lambda}}{\boldsymbol{J}}(\mathbf{ln}\boldsymbol{J})\boldsymbol{I} \tag{2.71}$$

Where $\lambda \; and \; \mu$ are lame coefficients, J is jacobian and I is identity matrix.

The lagrangean elasticity tensor akin to the compressible neo-Hookean material is described as

$$\boldsymbol{H} = \boldsymbol{\lambda H}^{-\mathbf{1}} \otimes \boldsymbol{H}^{-\mathbf{1}} + \mathbf{2}(\boldsymbol{\mu} - \boldsymbol{\lambda lnJ})\boldsymbol{I} \tag{2.72}$$

Where $\boldsymbol{H}^{-\mathbf{1}} \otimes \boldsymbol{H}^{-\mathbf{1}} = \sum \boldsymbol{H}_{\boldsymbol{IJ}}^{-\mathbf{1}}\boldsymbol{H}\boldsymbol{E_I} \otimes \boldsymbol{E_J} \otimes \boldsymbol{E_K} \otimes \boldsymbol{E_L}$ and fourth order tensor $\boldsymbol{I_{IJKL}} = \frac{\boldsymbol{\partial H_{IJ}^{-1}}}{\boldsymbol{\partial H_{KL}}}$

Similarly the eulerian or spatial elasticity tensor is expressed as

$$\boldsymbol{h} = \frac{\boldsymbol{\lambda}}{\boldsymbol{J}}\boldsymbol{I} \otimes \boldsymbol{I} + \frac{\mathbf{2}}{\boldsymbol{J}}(\boldsymbol{\mu} - \boldsymbol{\lambda lnJ})\boldsymbol{i} \tag{2.73}$$

Where $\boldsymbol{i}$ is the fourth order tensor and it's component is given in terms of kroneker delta as

$$\boldsymbol{i} = \sum_{\boldsymbol{IJKL}}^{\mathbf{3}} \boldsymbol{F_{iI}F_{jJ}F_{kK}F_{lL}I_{IJKL}} = \boldsymbol{\delta_{IJ}\delta_{KL}} \tag{2.74}$$

2.5.4 Isotropic elasticity in principal directions

2.5.4.1 Material description

This is the case where the elastic potential is described in terms of $\boldsymbol{\lambda_a}$ rather than the invarients of $\boldsymbol{C}$, here the identity, the right Cauchy green tensor and its inverse are written as

$$\boldsymbol{I} = \sum_{a=1}^{3} \boldsymbol{N_a} \otimes \boldsymbol{N_a} \tag{2.75}$$

$$\boldsymbol{C} = \sum_{a=1}^{3} \lambda_a^2 \boldsymbol{N_a} \otimes \boldsymbol{N_a} \tag{2.76}$$

$$\boldsymbol{C}^{-1} = \sum_{a=1}^{3} \lambda_a^{-2} \boldsymbol{N_a N_a} \tag{2.77}$$

By substituting the equation(2.75), (2.76), (2.77) into equation(2.70) gives the second piola-kirchoff stress ($\boldsymbol{S}$) as,

$$\boldsymbol{S} = \sum_{I=1}^{3} (2\Psi_1 + 4\Psi_1 I\lambda_a^2 + 2III_C \Psi_{III} \lambda_a^{-2}) \boldsymbol{N_a} \otimes \boldsymbol{N_a} \tag{2.78}$$

Where $\Psi_I \Psi_{II} \Psi_{III}$ are the derivative of elastic potential with respect of invariant of $\boldsymbol{C}$, here it is important to transform into derivatives with respect to the stretches. For this purpose the invarients of C are described there in terms of squared stretches $\boldsymbol{\lambda_a^2}$ as

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{2.79}$$

$$II_C = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 \tag{2.80}$$

$$III_C = \lambda_1^2 \lambda_1^2 \lambda_3^2 \tag{2.81}$$

Differentiating these equations and substituting into equation (2.78) gives the second piola kirchoff stress $\boldsymbol{S}$ as,

$$\boldsymbol{S} = \sum_{a=1}^{3} \boldsymbol{S_{aa} N_a} \otimes \boldsymbol{N_a} \tag{2.82}$$

Where $\boldsymbol{S_{aa}}$ are the principal components of second piola kirchoff tensor and describe as the derivatives of Ψ with respect to the $\boldsymbol{\lambda_a^2}$

$$\boldsymbol{S_{aa}} = \frac{\boldsymbol{\partial}\Psi}{\boldsymbol{\partial\lambda_a^2}} \tag{2.83}$$

2.5.4.2 Spatial description

The Cauchy stress is described as,

$$\boldsymbol{\sigma} = \boldsymbol{J^{-1}FSF^T} = \sum_{a=1}^{3} \frac{\mathbf{2}}{\boldsymbol{J}} \frac{\boldsymbol{\partial}\Psi}{\boldsymbol{\partial\lambda_a^2}} \boldsymbol{FN_a} \otimes \boldsymbol{FN_a} \tag{2.84}$$

Where $\boldsymbol{FN_a}$=$\boldsymbol{\lambda_a n_a}$ provides the principal components of Cauchy stress as

$$\boldsymbol{\sigma} = \sum_{a=1}^{3} \boldsymbol{\sigma_{aa} n_a} \otimes \boldsymbol{n_a}; \qquad \boldsymbol{\sigma_{aa}} = \frac{\mathbf{1}}{\boldsymbol{J}} \frac{\boldsymbol{\partial}\Psi}{\boldsymbol{\partial ln\lambda_a}} \tag{2.85}$$

In Cartesian form the Cauchy stress express as

$$\boldsymbol{\sigma} = \sum_{j,k=1}^{3} \left(\sum_{a=1}^{3} \boldsymbol{\sigma_{aa} T_{aj} T_{ak}}\right) \boldsymbol{e_j} \otimes \boldsymbol{e_k} \tag{2.86}$$

Where $\boldsymbol{T_{aj}}$ are the Cartesian components of $\boldsymbol{n_a}$

2.5.4.3 Material elasticity tensor

The material elasticity tensor or lagrangean elasticity tensor (C) for a material is given in terms of the principal stretches as,

$$\boldsymbol{C} = \sum_{a,b=1}^{3} \mathbf{4} \frac{\boldsymbol{\partial}^2\Psi}{\boldsymbol{\partial\lambda_a^2 \partial\lambda_b^2}} \boldsymbol{N_a} \otimes \boldsymbol{N_a} \otimes \boldsymbol{N_b} \otimes \boldsymbol{N_b} + \sum \frac{\boldsymbol{S_{aa}} - \boldsymbol{S_{bb}}}{\boldsymbol{\lambda_a^2} - \boldsymbol{\lambda_b^2}} \boldsymbol{N} \otimes_{\boldsymbol{a}} \boldsymbol{N_b} \otimes \boldsymbol{N_a} \otimes \boldsymbol{N_b} \tag{2.87}$$

2.5.4.4 Spatial elasticity tensor

The spatial elasticity tensor is described as

$$\boldsymbol{c} = \sum_{a,b=1}^{3} \frac{1}{J} \frac{\partial^2 \Psi}{\partial ln\lambda_a \partial ln\lambda_b} \boldsymbol{n}_{a\otimes} \boldsymbol{n}_a \otimes \boldsymbol{n}_b \otimes \boldsymbol{n}_b - \sum_{a=1}^{3} 2\sigma_{aa} \boldsymbol{n}_a \otimes \boldsymbol{n}_a \otimes \boldsymbol{n}_a \otimes \boldsymbol{n}_a + \sum_{a,b=1}^{3} 2 \frac{\sigma_{aa}\lambda_b^2 - \sigma_{bb}\lambda_a^2}{\lambda_a^2 - \lambda_b^2} \boldsymbol{n}_a \otimes \boldsymbol{n}_b \otimes \boldsymbol{n}_a \otimes \boldsymbol{n}_b \quad (2.88)$$

2.5.4.5 A simple stretch based hyperelastic material

This material is defined by a hyperelastic potential function in terms of lograthmic stretches and lame coffecients λ and $\boldsymbol{\mu}$ as,

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \mu\left[(ln\lambda_1)^2 + (ln\lambda_2)^2 + (ln\lambda_3)^2\right] + \frac{\lambda}{2}(lnJ)^2 \quad (2.89)$$

Where because $J = \lambda_1\lambda_2\lambda_3$

$$lnJ = ln\lambda_1 + ln\lambda_2 + ln\lambda_3 \quad (2.90)$$

Using the equation(3.85) the principal Cauchy stress components emanate as,

$$\sigma_{aa} = \frac{1}{J}\frac{\partial\Psi}{\partial ln\lambda_a} = \frac{2\mu}{J} ln\lambda_a + \frac{\lambda}{J} lnJ \quad (2.91)$$

Where J is the jacobean and λ_a are the principal stretches.

The initial elastic tensor for this tye of material emanates as

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl} \quad (2.92)$$

It is also valid for all isotropic material at initial configuration.

2.5.4.6 Cases of plane strain and plane stress

In plane strain case the stretch in the third direction(λ_3) is equal to 1. Under these condition,the stored elastic potential becomes,

$$\Psi(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}) = \boldsymbol{\mu}\left[(\boldsymbol{ln\lambda_1})^2 + (\boldsymbol{ln\lambda_2})^2 \quad\right] + \frac{\boldsymbol{\lambda}}{\mathbf{2}}(\boldsymbol{lnj})^2 \tag{2.93}$$

Where $\boldsymbol{j}$ is the determinant of the componentof $\boldsymbol{F}$ in the n_1 and n_2 plane, $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are the lame coefficient.

Here the cauchy stress components emerges as,

$$\boldsymbol{\sigma_{aa}} = \frac{\mathbf{1}}{\boldsymbol{j}}\frac{\boldsymbol{\partial}\Psi}{\boldsymbol{\partial ln\lambda_a}} = \frac{\mathbf{2}\boldsymbol{\mu}}{\boldsymbol{j}}\boldsymbol{ln\lambda_a} + \frac{\boldsymbol{\lambda}}{\boldsymbol{j}}\boldsymbol{lnj} \tag{2.94}$$

In plane stress case the stress in n_3 is constrained .ie $\boldsymbol{\sigma_{33}} = \mathbf{0}$. Imposing this condition in equation(2.91), gives

$$\boldsymbol{ln\lambda_3} = -\frac{\boldsymbol{\lambda}}{\boldsymbol{\lambda} + \mathbf{2}\boldsymbol{\mu}}\boldsymbol{lnj} \tag{2.95}$$

By substituting the above expression in equation(2.89) and noting that $\boldsymbol{lnJ} = \boldsymbol{ln\lambda_3} + \boldsymbol{lnj}$ gives

$$\Psi(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}) = \boldsymbol{\mu}\left[(\boldsymbol{ln\lambda_1})^2 + (\boldsymbol{ln\lambda_2})^2 \quad\right] + \frac{\boldsymbol{\lambda}}{\mathbf{2}}(\boldsymbol{lnj})^2 \tag{2.96}$$

Where $\bar{\lambda}$ is the effective lame coefficient which is described as

$$\bar{\boldsymbol{\lambda}} = \boldsymbol{\gamma\lambda}; \qquad \frac{\mathbf{2}\boldsymbol{\mu}}{\boldsymbol{\lambda} + \mathbf{2}\boldsymbol{\mu}} \tag{2.97}$$

For this case the principal cauchy stress components are obtained as

$$\boldsymbol{\sigma_{aa}} = \frac{\mathbf{1}}{\boldsymbol{j}}\frac{\boldsymbol{\partial}\Psi}{\boldsymbol{\partial ln\lambda_a}} = \frac{\mathbf{2}\boldsymbol{\mu}}{\boldsymbol{j}}\boldsymbol{ln\lambda_a} + \frac{\boldsymbol{\lambda}}{\boldsymbol{j}}\boldsymbol{lnj} \tag{2.98}$$

2.4.5 Incompressible and nearly incompressible materials

Nearly incompressible materials are the materials that are incompressible but their numerical analysis consider a small measure of volumetric deformation.

2.4.5.1 Incompressible elasticity

The rate of volume change for a compressible material is expressed as

$$\dot{J} = \frac{1}{2} J C^{-1} : \dot{C} \tag{2.99}$$

Where $\dot{C}$ is the material rate tensor.

But in the incompressible case volume remain constant throughout the deformation i.e $J = 1$ and $\dot{J} = 0$. which gives the constraint on $\dot{C}$ as shown in fig 2.10

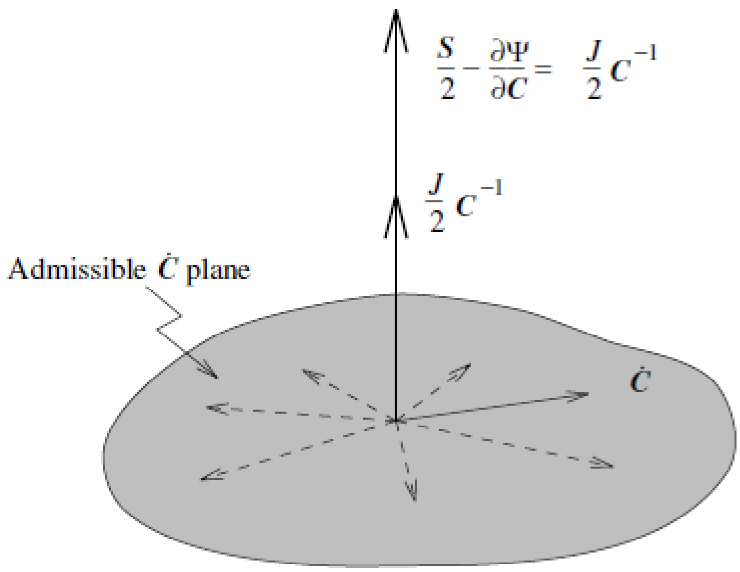

Fig 2.12 Incompressible constraint

$$\frac{1}{2} J C^{-1} : \dot{C} = 0 \tag{2.100}$$

The general incompressible and nearly compressible hyperelastic material constitutive equation is described as

$$S = 2 \frac{\partial \Psi(C)}{\partial C} + \gamma J C^{-1} \tag{2.101}$$

Where S is the second piola kirchoff stress, γ is constant and J is the jacobean which is equal to 1 in incompressible case.

The second piola kirchoff stress for incompressible material can also express as

$$\boldsymbol{S} = \mathbf{2}\frac{\boldsymbol{\partial}\widehat{\Psi}(\boldsymbol{C})}{\boldsymbol{\partial C}} + \text{Þ}\boldsymbol{J}\boldsymbol{C}^{-\mathbf{1}} \tag{2.102}$$

Where p is the hydrostatic pressure and $\widehat{\Psi}(\boldsymbol{C})$ is the distortional component of hyperelastic material.

2.4.5.2 Incompressible neo hookean material

The hyperelastic potential function for this material is described as

$$\Psi(\boldsymbol{C}) = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\mu}(\boldsymbol{trC} - \mathbf{3}) \tag{2.103}$$

Where trC is the trace of C

Now using equation (3.101) the second piola kirchoff stress S is obtained as

$$\boldsymbol{S} = \boldsymbol{\mu III}_{\boldsymbol{C}}^{-\mathbf{1/3}}\left(\boldsymbol{I} - \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{I}_{\boldsymbol{C}}\boldsymbol{C}^{-\mathbf{1}}\right) + \text{Þ}\boldsymbol{J}\boldsymbol{C}^{-\mathbf{1}} \tag{2.104}$$

The corresponding cauchy stress tensor is expressed as,

$$\boldsymbol{\sigma} = \boldsymbol{\mu J}^{-\mathbf{5/3}}\boldsymbol{F}\left(\boldsymbol{I} - \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{I}_{\boldsymbol{C}}\boldsymbol{C}^{-\mathbf{1}}\right)\boldsymbol{F}^{\boldsymbol{T}} + \text{p}\boldsymbol{F}\boldsymbol{C}^{-\mathbf{1}}\boldsymbol{F}^{\boldsymbol{T}} \tag{2.105}$$

2.4.5.3 Nearly incompressible hyperelastic material

The total hyperelastic potential function is defined as

$$\Psi(\boldsymbol{C}) = \widehat{\Psi}(\boldsymbol{C}) + \boldsymbol{U}(\boldsymbol{J}) \tag{2.106}$$

where $\widehat{\Psi}(\boldsymbol{C})$ is the distortional component of hyperelastic material and $\boldsymbol{U}(\boldsymbol{J})$ is the volumetric energy component which is define as

$$\boldsymbol{U}(\boldsymbol{J}) = \frac{\mathbf{1}}{\mathbf{2}}\kappa(\boldsymbol{J} - \mathbf{1})^{\mathbf{2}} \tag{2.107}$$

Where $\boldsymbol{k}$ is bulk modulus and it's value is very high typically$(10^3 - 10^4)$ for incompressible material.

Chapter-3

Finite Element Formulation For Non-Linear Analysis

3.1 Introduction

This chapter deals with the general procedure for solving Non linear finite element problem where first we describe the virtual work equilibrium equation which are valid for non linear material behaviour, large displacements and large strains. Then we discuss the two different approaches(total lagrangean formulation and updated lagrangean formulation) that have been used in incremental non linear finite element analysis then we describe the finite element matrices and equilibrium iteration process to solve the problem. Flow chart describing the solution procedure using total lagrange and updated lagrange is also shown at the end.

3.2 Formulation of continuum mechanics incremental equation

Consider the motion of a body in a Cartesian co-ordinate system as shown in Figure 3.1. The aim is to evaluate the equilibrium positions of the body at the discrete time points $\mathbf{0}, \Delta \boldsymbol{t}, \mathbf{2}\Delta \boldsymbol{t}, \mathbf{3}\Delta \boldsymbol{t}$…where $\Delta \boldsymbol{t}$ is an increment in time. Assume that the solution for the kinematic and static variables for all time steps from time 0 to time $\boldsymbol{t}$, inclusive, have been solved, and that the solution for time $\boldsymbol{t} + \Delta \boldsymbol{t}$ is required next. It is noted that the solution process for the next required equilibrium position is typical and would be applied repetitively until the complete solution path has been solved.

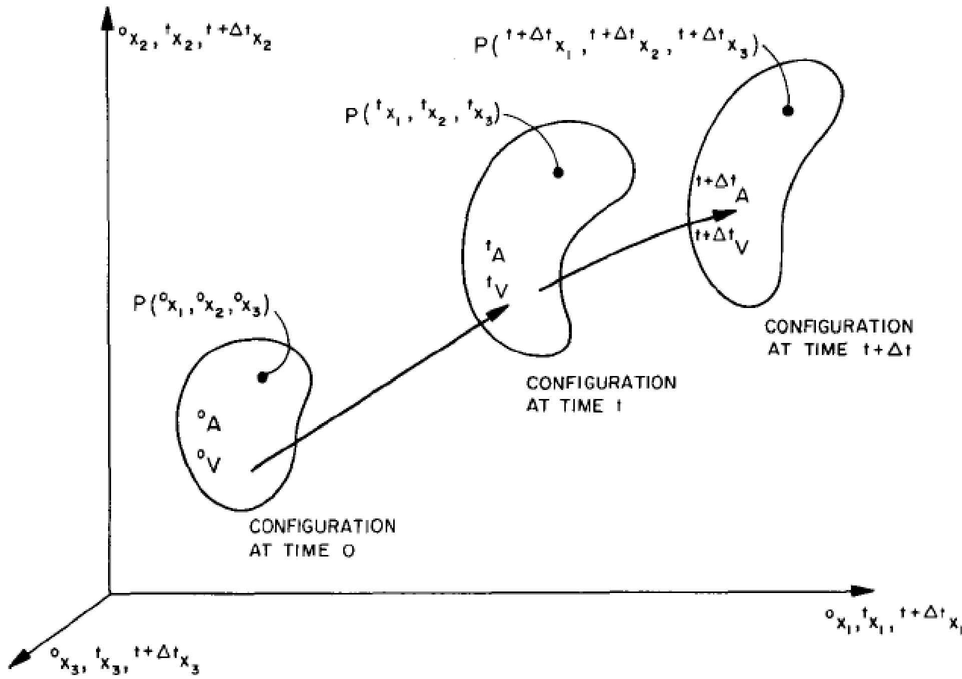

Fig 3.1 Motion of body in Cartesian co-ordinate system

Since the solution is known at all discrete points $\mathbf{0},\Delta \boldsymbol{t}, \mathbf{2}\Delta \boldsymbol{t} \dots\dots \boldsymbol{t}$, the basic aim of the formulation is to generate an equation of virtual work from which the unknown static and kinematic variables in the configuration at time $\boldsymbol{t}+\Delta\boldsymbol{t}$ can be solved. The principle of virtual work is used to express the equilibrium of the body in the configuration at the time $+\Delta\boldsymbol{t}$. It is stated as

$$\int_{t+\Delta t_v} {}^{t+\Delta t}\boldsymbol{\sigma_{ij}}\ \boldsymbol{\delta}\ {}_{t+\Delta t}\boldsymbol{e_{ij}}\ {}^{t+\Delta t}\boldsymbol{dv} = {}^{t+\Delta t}\boldsymbol{R} \tag{3.1}$$

Where ${}^{t+\Delta t}\boldsymbol{R}$ is the external virtual work expression and shown as,

$${}^{t+\Delta t}\boldsymbol{R} = \int_{t+\Delta t_v} {}^{t+\Delta t}_{t+\Delta t}\boldsymbol{f_k}\ \ \boldsymbol{\delta u_k}\ \ {}^{t+\Delta t}\boldsymbol{dv} + \int_{t+\Delta t_a} {}^{t+\Delta t}_{t+\Delta t}\boldsymbol{t_k}\ \ \boldsymbol{\delta u_k}\ {}^{t+\Delta t}\boldsymbol{dv} \qquad \boldsymbol{k}=\mathbf{1},\mathbf{2},\mathbf{3} \tag{3.2}$$

Where ${}^{t+\Delta t}\boldsymbol{f_k}$, ${}^{t+\Delta t}_{t+\Delta t}\boldsymbol{t_k}$ are the body force vector and surface traction vector in the configuration at the time $\boldsymbol{t}+\Delta\boldsymbol{t}$ respectively, $\boldsymbol{\delta u_k}$ is the (virtual) variation in the current displacement components ${}^{t+\Delta t}\boldsymbol{u_k}$and $\boldsymbol{\delta_{t+\Delta t}e_{ij}}$ are the corresponding(virtual) variation in strains. as

$$\boldsymbol{\delta_{t+\Delta t}e_{ij}} = \frac{1}{2}(\boldsymbol{u_{i,j}} + \boldsymbol{u_{j,i}}) \tag{3.3}$$

The difficulty here being that configuration of the body at time $\boldsymbol{t}+\boldsymbol{\Delta t}$ is unknown so the above equation cannot be solved directly. A solution can be obtained by referring all the variables to a known previously calculated equilibrium configuration for that two approaches have been used i.e **Total lagrangean formulation** and **Updated lagrangean formulation**

3.3 Total Lagrangean Formulation

In this formulation all variables in equations (3.1) and (3.2) are referred to the initial configuration at time 0 of the body. The external forces in equation (3.2) are calculated as

$$ {}^{t+\Delta t}\boldsymbol{f_k} \;\; {}^{t+\Delta t}\boldsymbol{dv} = {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{f_k} \;\; {}^{\mathbf{0}}\boldsymbol{dv} \tag{3.4} $$

$$ {}^{t+\Delta t}_{t+\Delta t}\boldsymbol{t_k} \;\; {}^{t+\Delta t}\boldsymbol{dv} = {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{t_k} \;\; {}^{\mathbf{0}}\boldsymbol{dv} \tag{3.5} $$

Where ${}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{f_k}$, ${}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{t_k}$ are the component of body force vector and surface traction vector in configuration at time $\boldsymbol{t+\Delta t}$ and referred to reference configuration(ie at time 0).
The volume integral of Cauchy stress time variation in strains in equation (3.1) is transformed

$$ \int_{t+\Delta t_v} {}^{t+\Delta t}\boldsymbol{\sigma_{ij}}\, \boldsymbol{\delta}\, {}_{t+\Delta t}\boldsymbol{e_{ij}} \; {}^{t+\Delta t}\boldsymbol{dv} \; = \; \int_{t+\Delta t_v} {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{S_{ij}}\, \boldsymbol{\delta}\, {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\varepsilon_{\boldsymbol{ij}} \;\; {}^{\mathbf{0}}\boldsymbol{dv} \tag{3.6} $$

Where ${}^{t+\Delta t}\boldsymbol{S_{ij}}$ is the Cartesian components of the 2nd Piola-Kirchhoff stress tensor corresponding to the configuration at time $\boldsymbol{t+\Delta t}$,referred to the configuration at time 0, $\boldsymbol{\delta}^{t+\Delta t}{}_{\mathbf{0}}\varepsilon_{\boldsymbol{ij}}$ is the variation in the Cartesian component of the green lagrange strain tensor in the configuration at time $\boldsymbol{t+\Delta t}$ but measured to the configuration at time 0 which is shown as,

$$ \boldsymbol{\delta}^{t+\Delta t}{}_{\mathbf{0}}\varepsilon_{\boldsymbol{ij}} = \boldsymbol{\delta}\frac{\mathbf{1}}{\mathbf{2}}({}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{i,j}} + {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{j,i}} + {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{k,i}}\, {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{k,j}} \tag{3.7} $$

Where ${}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{i,j}}$, ${}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{j,i}}$, ${}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{k,i}}$, ${}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{u_{k,j}}$ are the derivative of displacement componens to the configuration at time $t + \Delta t$ with respect to co-ordinates describing the configuration of the body at time 0(i.e ${}^{\mathbf{0}}\boldsymbol{x_i}$, ${}^{\mathbf{0}}\boldsymbol{x_j}$).

Comparing the equations (3.1) and (3.6), the equilibrium equation for the body in the configuration at time $\boldsymbol{t+\Delta t}$ but referred to the configuration at time $\boldsymbol{t+\Delta t}$ is obtained as

$$ \int_{\mathbf{0}_v} {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{S_{ij}}\, \boldsymbol{\delta}\, {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\varepsilon_{\boldsymbol{ij}} \;\; {}^{\mathbf{0}}\boldsymbol{dv} \; = \; {}^{t+\Delta t}\boldsymbol{R} \tag{3.8} $$

Where ${}^{t+\Delta t}\boldsymbol{R} = \int_{\mathbf{0}_v} {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{f_k} \;\; \boldsymbol{\delta u_k} \;\; {}^{\mathbf{0}}\boldsymbol{dv} + \int_{\mathbf{0}_a} {}^{t+\Delta t}_{\;\;\;\;\mathbf{0}}\boldsymbol{t_k} \;\; \boldsymbol{\delta u_k} \;\; {}^{\mathbf{0}}\boldsymbol{da}$

but the second piola kirchoff stresses and green lagrange strains in the configuration at time $\boldsymbol{t} + \Delta \boldsymbol{t}$ is unknown, so for the solution the following incremental decompositions are used.

$$ {}^{t+\Delta t}_{0}\boldsymbol{S}_{ij} = {}^{t}_{0}\boldsymbol{S}_{ij} + {}_{0}\boldsymbol{S}_{ij} \tag{3.9} $$

$$ {}^{t+\Delta t}_{0}\varepsilon_{ij} = {}^{t}_{0}\varepsilon_{ij} + {}_{0}\varepsilon_{ij} \tag{3.10} $$

Where ${}^{t}_{0}\boldsymbol{S}_{ij}$, ${}^{t}_{0}\varepsilon_{ij}$ are the known 2nd Piola-Kirchhoff stresses and Green-Lagrange strains in the configuration at time $\boldsymbol{t}$ and referred to configuration at time 0, ${}_{0}\boldsymbol{S}_{ij}$, ${}_{0}\varepsilon_{ij}$ are the incremental second piola kirchoff stress and incremental green lagrange strain components referred to configuration at time 0 respectively.

From the displacement definition of the green lagrange strain tensor $\boldsymbol{\delta}\,{}^{t+\Delta t}_{0}\varepsilon_{ij} = \boldsymbol{\delta}\ {}_{0}\varepsilon_{ij}$ and

$$ {}_{0}\varepsilon_{ij} = {}_{0}\boldsymbol{e}_{ij} + {}_{0}\boldsymbol{\eta}_{ij} \tag{3.11} $$

Where ${}_{0}\boldsymbol{e}_{ij}$, ${}_{0}\boldsymbol{\eta}_{ij}$ are the linear part and non- linear part of incremental green-lagrange strain and express as

$$ {}_{0}\boldsymbol{e}_{ij} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\left({}^{t+\Delta t}_{0}\boldsymbol{u}_{i,j} + {}^{t+\Delta t}_{0}\boldsymbol{u}_{j,i}\right) \tag{3.12} $$

$$ {}_{0}\boldsymbol{\eta}_{ij} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\left({}^{t+\Delta t}_{0}\boldsymbol{u}_{k,i}\,{}^{t+\Delta t}_{0}\boldsymbol{u}_{k,j}\right) \tag{3.13} $$

The incremental 2nd Piola-Kirchhoff stresses, are related to the incremental Green-Lagrange strains using the incremental constitutive tensor i.e,

$$ {}_{0}\boldsymbol{S}_{ij} = {}_{0}\boldsymbol{C}_{ijrs}\,{}_{0}\varepsilon_{rs} \tag{3.14} $$

Now by using the above relations a non linear equation for the incremental displacement $\boldsymbol{u}_{i}$ in total lagrangean can be written as

$$ \int_{0_v} {}_{0}\boldsymbol{C}_{ijrs}\ {}_{0}\varepsilon_{rs}\boldsymbol{\delta}\ {}_{0}\varepsilon_{ij}\ {}^{0}\boldsymbol{dv} + \int_{0_v} {}^{t}_{0}\boldsymbol{S}_{ij}\boldsymbol{\delta}\ {}_{0}\boldsymbol{\eta}_{ij}\ {}^{0}\boldsymbol{dv} = {}^{t+\Delta t}\boldsymbol{R} - \int_{0_v} {}^{t}_{0}\boldsymbol{S}_{ij}\boldsymbol{\delta}\ {}_{0}\boldsymbol{e}_{ij}\ {}^{0}\boldsymbol{dv} \tag{3.15} $$

3.4 Updated Lagrangean Formulation

In this formulation all variables in equations (3.1) and (3.2) are referred to the configuration at time $\boldsymbol{t}$ of the body which is the updated configuration of the body.By a same procedure to the derivation of the T.L formulation as discuss above, equation (3.1) is transformed to

$$\int_{{}^{t}v} {}^{t+\Delta t}_{t}\boldsymbol{S}_{ij}\, \boldsymbol{\delta}\, {}^{t+\Delta t}_{t}\varepsilon_{ij}\ \ {}^{t}\boldsymbol{dv} = {}^{t+\Delta t}\boldsymbol{R} \tag{3.16}$$

where${}^{t+\Delta t}_{t}\boldsymbol{S}_{ij}$ is the Cartesian component of the 2nd Piola-Kirchhoff stress tensor corresponding to the configuration at time $\boldsymbol{t}+\Delta\boldsymbol{t},$ referred to the configuration at time $\boldsymbol{t}, \boldsymbol{\delta}^{t+\Delta t}_{t}\varepsilon_{ij}$ is the variation in the Cartesian component of the green lagrange strain tensor in the configuration at time $\boldsymbol{t}+\Delta\boldsymbol{t}$ but measured to the configuration at time $\boldsymbol{t}$.

The incremental stress decomposition used in the U.L case is

$${}^{t+\Delta t}_{0}\boldsymbol{S}_{ij} = {}^{t}\boldsymbol{\sigma}_{ij} + {}_{t}\boldsymbol{S}_{ij} \tag{3.18}$$

Where ${}^{t}\boldsymbol{\sigma}_{ij}$ is the Cartesian component of Cauchy stress tensor in the configuration at time t and ${}_{t}\boldsymbol{S}_{ij}$ are the Cartesian component of 2nd Piola-Kirchhoff stress increment at time t.

By considering the strain increments ${}^{t+\Delta t}_{t}\varepsilon_{ij}$,the following relation given

$${}^{t+\Delta t}_{t}\varepsilon_{ij} = {}_{t}\varepsilon_{ij} \tag{3.18}$$

$${}_{t}\varepsilon_{ij} = {}_{t}\boldsymbol{e}_{ij} + {}_{t}\boldsymbol{\eta}_{ij} \tag{3.19}$$

Where ${}_{0}\boldsymbol{e}_{ij}$, ${}_{0}\boldsymbol{\eta}_{ij}$ are the linear part and non linear part of incremental green lagrange strains referred to the configuration at time t respectively and express as

$${}_{0}\boldsymbol{e}_{ij} = \frac{\boldsymbol{1}}{\boldsymbol{2}}({}^{t+\Delta t}_{0}\boldsymbol{u}_{i,j} + {}^{t+\Delta t}_{0}\boldsymbol{u}_{j,i}) \tag{3.20}$$

$${}_{0}\boldsymbol{\eta}_{ij} = \frac{\boldsymbol{1}}{\boldsymbol{2}}({}^{t+\Delta t}_{0}\boldsymbol{u}_{k,i}\,{}^{t+\Delta t}_{0}\boldsymbol{u}_{k,j}) \tag{3.21}$$

The constitutive relation between stress and strain increment in UL is given as

$${}_{t}\boldsymbol{S}_{ij} = {}_{t}\boldsymbol{C}_{ijrs}\ {}_{t}\varepsilon_{rs} \tag{3.22}$$

Where $_{t}C_{ijrs}$ are the components of constitutive tensor in the configuration at time t

In updated lagrangean the equation(3.15) can be rewritten as

$$\int_{t_v} {}_{t}C_{ijrs}\ {}_{t}\varepsilon_{rs}\delta\ {}_{t}\varepsilon_{ij}\ {}^{t}dv + \int_{t} {}^{t}\zeta_{ij}\delta\ {}_{t}\eta_{ij}\ {}^{t}dv = {}^{t+\Delta t}R - \int_{t_v} {}^{t}\zeta_{ij}\delta\ {}_{t}e_{ij}\ {}^{t}dv \quad (3.23)$$

Which represent a non linear equation for the incremental displacements u_i

3.5 Linearization of Internal virtual work.

The solution of equations (3.15) and (3.23) cannot be obtained directly,since they are non linear in displacement increments.So to obtain the solution we need to linearize the internal virtual work which has been done by assuming that in equations (3.15) $_{0}\varepsilon_{ij} = {}_{0}e_{ij}$ and in equation (4.23) $_{t}\varepsilon_{ij} = {}_{t}e_{ij}$ .It means that ,in addition to using $\delta\ {}_{0}\varepsilon_{ij} = \delta\ {}_{0}e_{ij}$ respectively,the incremental constitutive relation (explain in equation(3.14) and (3.15) employ are

$$_{0}S_{ij} = {}_{0}C_{ijrs}\ {}_{0}e_{rs} \quad (3.24)$$

$$_{t}S_{ij} = {}_{t}C_{ijrs}\ {}_{t}e_{rs} \quad (3.25)$$

So In the total lagrangean formulation the approximate equilibrium equation to be solved is

$$\int_{0_v} {}_{0}C_{ijrs}\ {}_{0}e_{rs}\delta\ {}_{0}e_{ij}\ {}^{0}dv + \int_{0_v} {}_{0}^{t}S_{ij}\delta\ {}_{0}\eta_{ij}\ {}^{0}dv = {}^{t+\Delta t}R - \int_{0_v} {}_{0}^{t}S_{ij}\delta\ {}_{0}e_{ij}\ {}^{0}dv \quad (3.26)$$

Whereas in the Updated lagrangean formulation the linearized equilibrium equation is

$$\int_{t_v} {}_{t}C_{ijrs}\ {}_{t}e_{rs}\delta\ {}_{t}e_{ij}\ {}^{t}dv + \int_{t} {}^{t}\zeta_{ij}\delta\ {}_{t}\eta_{ij}\ {}^{t}dv = {}^{t+\Delta t}R - \int_{t_v} {}^{t}\zeta_{ij}\delta\ {}_{t}e_{ij}\ {}^{t}dv \quad (3.27)$$

3.6 Finite element descritization

To solve the equations(3.26) and (3.27) the body is descritize into finite element whose boundaries is defined by the nodes as shown in fig 3.2

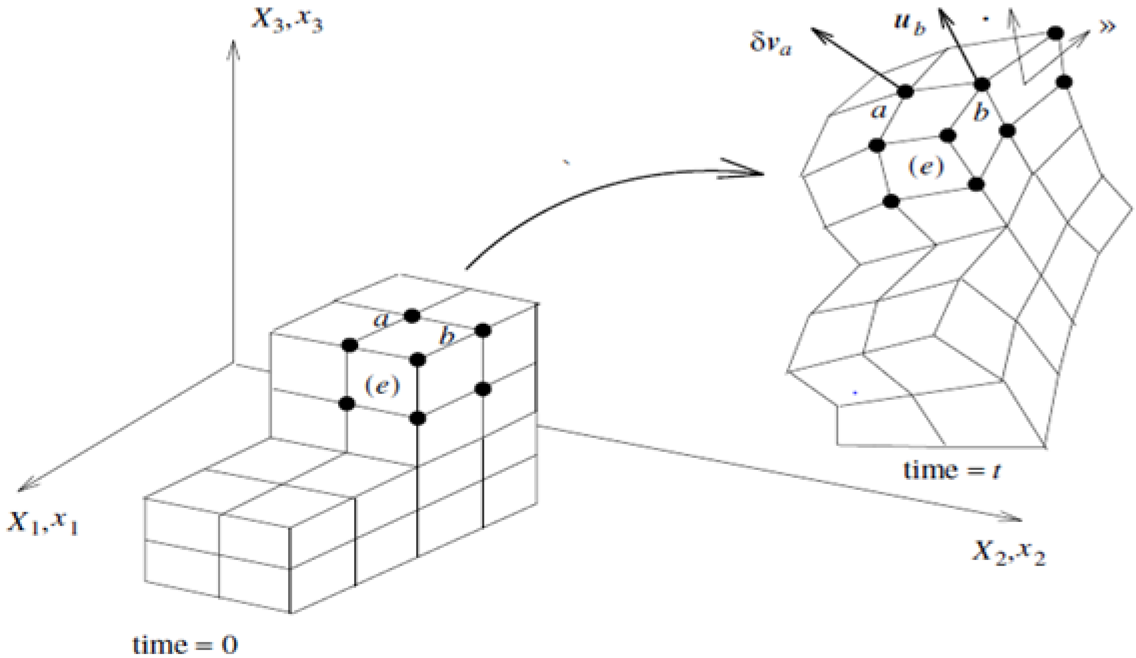

Fig3.2 Descritization

The co-ordinates and displacements in the configuration at time $\mathbf{0}, \boldsymbol{t}$ and $\boldsymbol{t} + \Delta\boldsymbol{t}$ are interpolated using the shape function

$$ {}^{0}\boldsymbol{x}_i = \sum_{k=1}^{T} \boldsymbol{N}_k \ {}^{0}\boldsymbol{x}_i^k \ ; \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3} \tag{3.28} $$

$$ {}^{t}\boldsymbol{x}_i = \sum_{k=1}^{T} \boldsymbol{N}_k \ {}^{t}\boldsymbol{x}_i^k \qquad {}^{t}\boldsymbol{u}_i = \sum_{k=1}^{T} \boldsymbol{N}_k \ {}^{t}\boldsymbol{u}_i^k \ ; \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3} \tag{3.29} $$

$$ {}^{t+\Delta t}\boldsymbol{x}_i = \sum_{k=1}^{T} \boldsymbol{N}_k {}^{t+\Delta t}\boldsymbol{x}_i^k \ ; \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3} \tag{3.30} $$

$$ \boldsymbol{u}_i = \sum_{k=1}^{T} \boldsymbol{N}_k \ \ \boldsymbol{u}_i^k \ ; \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3} \tag{3.31} $$

Where ${}^{0}\boldsymbol{x}_{i}^{k}$, ${}^{t}\boldsymbol{x}_{i}^{k}$, ${}^{t+\Delta t}\boldsymbol{x}_{i}^{k}$ are the coordinates of nodal point k corresponding to direction i at time $\boldsymbol{0}, \boldsymbol{t}, \boldsymbol{t}+\Delta\boldsymbol{t}$ respectively, $\boldsymbol{N_k}$ is the shape function corresponding to nodal point k, ${}^{t}\boldsymbol{u}_{i}^{k}$ is the displacement components of nodal point $\boldsymbol{k}$ in configuration at time t, $\boldsymbol{u_i}$ is the increment in displacement component and $\boldsymbol{T}$ is the number of nodal point in an element.

.Using the equations (3.28),(3.29),(3.30) and (3.31) to calculate the displacement derivatives required in the integrals, equation(4.15) becomes for an element is

$$({}_{0}^{t}\boldsymbol{K}_{L} + {}_{0}^{t}\boldsymbol{K}_{NL})\boldsymbol{u} = {}^{t+\Delta t}\boldsymbol{R} - {}_{0}^{t}\boldsymbol{F} \tag{3.32}$$

Where ${}_{0}^{t}\boldsymbol{K}_{NL}$, ${}_{,0}^{t}\boldsymbol{K}_{L}$ ***and*** ${}_{0}^{t}\boldsymbol{F}$ are obtained from finite element evalutaion of $\int_{{}^{0}v} {}_{0}^{t}\boldsymbol{S}_{ij}\boldsymbol{\delta}\ {}_{0}\boldsymbol{\eta}_{ij}\ {}^{0}\boldsymbol{dv}$, $\int_{{}^{0}v} {}_{0}\boldsymbol{C}_{ijrs}\ {}_{0}\boldsymbol{e}_{rs}\boldsymbol{\delta}\ {}_{0}\boldsymbol{e}_{ij}\ {}^{0}\boldsymbol{dv}$ and $\int_{{}^{0}v} {}_{0}^{t}\boldsymbol{S}_{ij}\boldsymbol{\delta}\ {}_{0}\boldsymbol{e}_{ij}\ {}^{0}\boldsymbol{dv}$, respectively, i.e

$${}_{0}^{t}\boldsymbol{K}_{L} = \int_{{}^{0}v} {}_{0}^{t}\boldsymbol{B}_{L}^{T}\ {}_{0}\boldsymbol{C}\,{}_{0}^{t}\boldsymbol{B}_{L}\ {}^{0}\boldsymbol{dv} \tag{3.33}$$

$${}_{0}^{t}\boldsymbol{K}_{NL} = \int_{{}^{0}v} {}_{0}^{t}\boldsymbol{B}_{NL}^{T}\ {}_{0}^{t}\boldsymbol{S}\,{}_{0}^{t}\boldsymbol{B}_{NL}\ {}^{0}\boldsymbol{dv} \tag{3.34}$$

$${}_{0}^{t}\boldsymbol{F} = \int_{{}^{0}v} {}_{0}^{t}\boldsymbol{B}_{L}^{T}\ {}_{0}\widehat{\boldsymbol{S}}\ {}^{0}\boldsymbol{dv} \tag{3.35}$$

Herein, ${}^{t+\Delta t}\boldsymbol{R}$ is the vector of external loads in configuration at time $\boldsymbol{t}+\Delta\boldsymbol{t}$, ${}_{0}\boldsymbol{C}$ is the tangent material property matrix at time t and referred to configuration at time 0, ${}_{0}^{t}\boldsymbol{F}$ is vector of nodal point forces due to stresses in configuration at time t with respect to configuration at time 0, ${}_{0}^{t}\boldsymbol{K}_{L}$, ${}_{0}^{t}\boldsymbol{K}_{NL}$ are the linear and non-linear strain stiffness matrix in configuration at time t referred to configuration at time 0 respectively, ${}_{0}^{t}\boldsymbol{S}$ and ${}_{0}\widehat{\boldsymbol{S}}$ are the second piola kirchoff stress matrix and vector in configuration at time $\boldsymbol{t}$ referred to configuration at time 0 , shown as

$${}_{0}^{t}S = \begin{bmatrix} {}_{0}^{t}S_{11} & {}_{0}^{t}S_{12} & 0 & 0 & 0 \\ {}_{0}^{t}S_{21} & {}_{0}^{t}S_{22} & 0 & 0 & 0 \\ 0 & 0 & {}_{0}^{t}S_{11} & {}_{0}^{t}S_{12} & 0 \\ 0 & 0 & {}_{0}^{t}S_{21} & {}_{0}^{t}S_{22} & 0 \\ 0 & 0 & 0 & 0 & {}_{0}^{t}S_{33} \end{bmatrix}; \qquad \hat{S} = \begin{bmatrix} {}_{0}^{t}S_{11} \\ {}_{0}^{t}S_{22} \\ {}_{0}^{t}S_{12} \\ {}_{0}^{t}S_{33} \end{bmatrix}$$

The linear strain displacement matrix ${}_{0}^{t}\boldsymbol{B}_{L}$ and non-linear strain displacement matrix ${}_{0}^{t}\boldsymbol{B}_{NL}$in configuration at time t with respect to configuration at time 0 is shown as

$$
{}_{0}^{t}\boldsymbol{B}_{\boldsymbol{L}} = \begin{bmatrix} {}_0N_{1,1} & 0 & {}_0N_{2,1} & 0 & \cdots & {}_0N_{T,1} & 0 \\ 0 & {}_0N_{1,2} & 0 & {}_0N_{2,2} & \cdots & 0 & {}_0N_{T,2} \\ {}_0N_{1,2} & {}_0N_{2,2} & {}_0N_{2,2} & {}_0N_{2,1} & \cdots & {}_0N_{T,2} & {}_0N_{T,1} \\ \frac{N_1}{{}^0\overline{x}_1} & 0 & \frac{N_2}{{}^0\overline{x}_1} & 0 & \cdots & \frac{N_T}{{}^0\overline{x}_1} & 0 \end{bmatrix}
$$

$$
{}_{0}^{t}\boldsymbol{B}_{\boldsymbol{NL}} = \begin{bmatrix} {}_0N_{1,1} & 0 & {}_0N_{2,1} & 0 & \cdots & {}_0N_{T,1} & 0 \\ {}_0N_{1,2} & 0 & {}_0N_{2,2} & 0 & \cdots & {}_0N_{T,2} & 0 \\ 0 & {}_0N_{1,1} & 0 & {}_0N_{2,1} & \cdots & 0 & {}_0N_{T,1} \\ 0 & {}_0N_{1,2} & 0 & {}_0N_{2,2} & \cdots & 0 & {}_0N_{T,2} \\ \frac{N_1}{{}^0\overline{x}_1} & 0 & \frac{N_2}{{}^0\overline{x}_1} & & \cdots & \frac{N_T}{{}^0\overline{x}_1} & 0 \end{bmatrix}
$$

Where ${}_{\boldsymbol{0}}\boldsymbol{N}_{\boldsymbol{k,j}}=\frac{\boldsymbol{\partial N_k}}{\boldsymbol{\partial {}^0x_j}}$; ${}^{\boldsymbol{0}}\overline{\boldsymbol{x}}_{\boldsymbol{1}} = \sum_{\boldsymbol{k=1}}^{\boldsymbol{T}} \boldsymbol{N_k}\ {}^{\boldsymbol{0}}\boldsymbol{x}_{\boldsymbol{1}}^{\boldsymbol{k}}$; $\boldsymbol{T}$ is the number of nodes.

Similarly the finite element solution of equation(3.27), which is obtained using the updated lagrangean formulation,results into

$$
({}_{t}^{t}\boldsymbol{K}_{\boldsymbol{L}} + {}_{t}^{t}\boldsymbol{K}_{\boldsymbol{NL}})\boldsymbol{u} = {}^{t+\Delta t}\boldsymbol{R} - {}_{t}^{t}\boldsymbol{F} \tag{3.36}
$$

Where ${}_{t}^{t}\boldsymbol{F}$ is the vector of nodal point forces due to stresses in configuration at time t with respect to configuration at time $\boldsymbol{t}$. ${}_{t}^{t}\boldsymbol{K}_{\boldsymbol{L}}$, ${}_{t}^{t}\boldsymbol{K}_{\boldsymbol{NL}}$ are the linear and non linear strain stiffness matrix in configuration at time t referred to configuration at time t respectively. It is described as,

$$
{}_{0}^{t}\boldsymbol{K}_{\boldsymbol{L}} = \int_{{}^t v} {}_{t}^{t}\boldsymbol{B}_{\boldsymbol{L}}^{\boldsymbol{T}}\ {}_{t}\boldsymbol{C}\,{}_{t}^{t}\boldsymbol{B}_{\boldsymbol{L}}\ {}^{t}\boldsymbol{dv} \tag{3.37}
$$

$$
{}_{0}^{t}\boldsymbol{K}_{\boldsymbol{NL}} = \int_{{}^t v} {}_{t}^{t}\boldsymbol{B}_{\boldsymbol{NL}}^{\boldsymbol{T}}\ {}^{t}\boldsymbol{\eta}\,{}_{t}^{t}\boldsymbol{B}_{\boldsymbol{NL}}\ {}^{t}\boldsymbol{dv} \tag{3.38}
$$

$$
{}_{t}^{t}\boldsymbol{F} = \int_{{}^t v} {}_{t}^{t}\boldsymbol{B}_{\boldsymbol{L}}^{\boldsymbol{T}}\ {}^{t}\hat{\boldsymbol{\sigma}}\ {}^{t}\boldsymbol{dv} \tag{3.39}
$$

Herein, the ${}_{t}\boldsymbol{C}$ symbolize the tangent material property matrix at time $\boldsymbol{t}$ and referred to configuration at time $\boldsymbol{t}$. ${}_{t}^{t}\boldsymbol{B}_{\boldsymbol{L}}$, ${}_{t}^{t}\boldsymbol{B}_{\boldsymbol{NL}}$ represent the linear and non linear strain displacement matrix at time $\boldsymbol{t}$ referred to the configuration at time $\boldsymbol{t}$ and shown as,

$$
{}_t^t\boldsymbol{B}_{\boldsymbol{L}} = \begin{bmatrix} {}_tN_{1,1} & 0 & {}_tN_{2,1} & 0 & \cdots & {}_tN_{T,1} & 0 \\ 0 & {}_tN_{1,2} & 0 & {}_tN_{2,2} & \cdots & 0 & {}_tN_{T,2} \\ {}_tN_{1,2} & {}_tN_{2,2} & {}_tN_{2,2} & {}_tN_{2,1} & \cdots & {}_tN_{T,2} & {}_tN_{T,1} \\ \dfrac{N_1}{{}^t\bar{x}_1} & 0 & \dfrac{N_2}{{}^t\bar{x}_1} & 0 & \cdots & \dfrac{N_T}{{}^t\bar{x}_1} & 0 \end{bmatrix}
$$

$$
{}_t^t\boldsymbol{B}_{\boldsymbol{NL}} = \begin{bmatrix} {}_tN_{1,1} & 0 & {}_tN_{2,1} & 0 & \cdots & {}_tN_{T,1} & 0 \\ {}_tN_{1,2} & 0 & {}_tN_{2,2} & 0 & \cdots & {}_tN_{T,2} & 0 \\ 0 & {}_tN_{1,1} & 0 & {}_tN_{2,1} & \cdots & 0 & {}_tN_{T,1} \\ 0 & {}_tN_{1,2} & 0 & {}_tN_{2,2} & \cdots & 0 & {}_tN_{T,2} \\ \dfrac{N_1}{{}^t\bar{x}_1} & 0 & \dfrac{N_2}{{}^t\bar{x}_1} & & \cdots & \dfrac{N_T}{{}^t\bar{x}_1} & 0 \end{bmatrix}
$$

The Cauchy stress matrix ${}^t\boldsymbol{\zeta}$ and stress vector ${}^t\hat{\boldsymbol{\zeta}}$ in the configuration at time t is described as,

$$
{}^t\boldsymbol{\zeta} = \begin{bmatrix} {}_0^t\tau_{11} & {}_0^t\tau_{12} & 0 & 0 & 0 \\ {}_0^t\tau_{21} & {}_0^t\tau_{22} & 0 & 0 & 0 \\ 0 & 0 & {}_0^t\tau_{11} & {}_0^t\tau_{12} & 0 \\ 0 & 0 & {}_0^t\tau_{21} & {}_0^t\tau_{22} & 0 \\ 0 & 0 & 0 & 0 & {}_0^t\tau_{33} \end{bmatrix}; \qquad {}^t\hat{\tau} = \begin{bmatrix} {}_0^t\tau_{11} \\ {}_0^t\tau_{22} \\ {}_0^t\tau_{12} \\ {}_0^t\tau_{33} \end{bmatrix}
$$

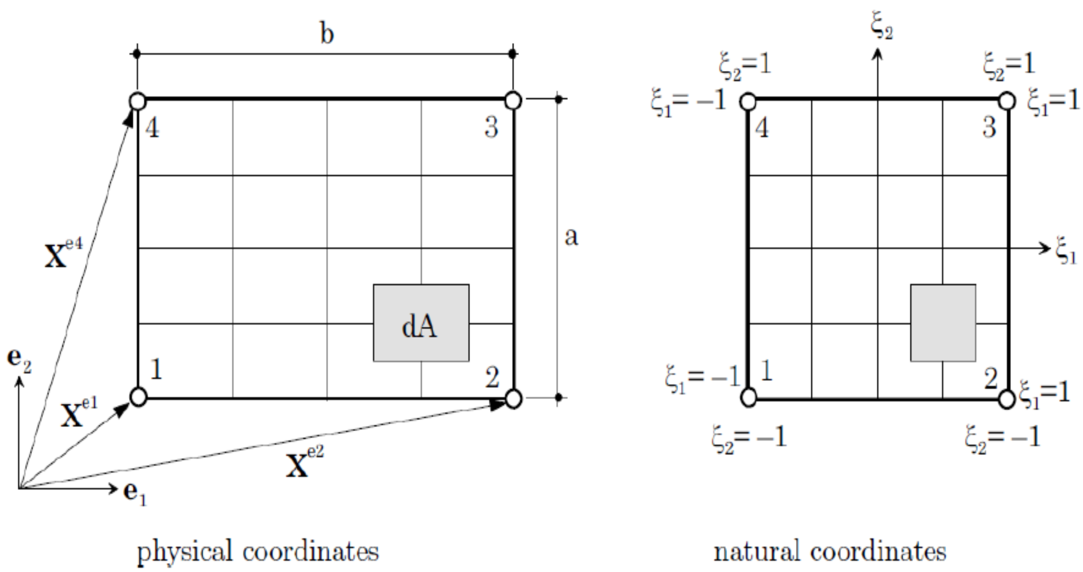

Fig 3.3 Coordinate system of 2 dimensional element

It should be noted that the elements of the matrices as discuss in this topic are function of natural coordinates (coordinates whose range lies between 0 and 1) as shown in fig 3.3 and that the volume integrations are performed using a coordinate change from cartesian to natural coordinate which is done by the jacobian matrix $\boldsymbol{J}$ and it's determinant i.e

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial X_1}{\partial \mathcal{E}_1} & \frac{\partial X_1}{\partial \mathcal{E}_2} & \frac{\partial X_1}{\partial \mathcal{E}_3} \\ \frac{\partial X_2}{\partial \mathcal{E}_1} & \frac{\partial X_2}{\partial \mathcal{E}_2} & \frac{\partial X_2}{\partial \mathcal{E}_3} \\ \frac{\partial X_3}{\partial \mathcal{E}_1} & \frac{\partial X_3}{\partial \mathcal{E}_2} & \frac{\partial X_3}{\partial \mathcal{E}_3} \end{bmatrix}$$

Where $\boldsymbol{X_1}, \boldsymbol{X_2}, \boldsymbol{X_3}$ are the Cartesian coordinate of three dimensional element and $\mathcal{E}_{\mathbf{1}}, \mathcal{E}_{\mathbf{2}}, \mathcal{E}_{\mathbf{3}}$ are the natural coordinates.

3.7 Equilibrium Iteration

The equations (3.32) and (3.36) are only approximations to the actual equations to be solved in each time step, i.e. equations (3.8) and (3.16), respectively. Depending on the non-linearities in the system, the linearized equilibrium equations (3.26) and (3.27) may introduce errors which ultimately result into solution instability. For this reason it may be necessary to iterate in each load step until, within the necessary assumptions on the variation of the material constants and the numerical time integration, equations (3.8) and (3.16) are satisfied to a required tolerance. The equation used in the Total lagrangean formulation and updated lagrangean formulation are

$$({}_0^tK_L + {}_0^tK_{NL})\Delta u^i = {}^{t+\Delta t}R - {}^{t+\Delta t}_{\ \ \ 0}F^{i-1} \tag{3.40}$$

$$({}_t^t\boldsymbol{K_L} + {}_t^t\boldsymbol{K_{NL}})\Delta \boldsymbol{u^i} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}_{\ \ \ t}\boldsymbol{F^{i-1}} \tag{3.41}$$

Where ${}^{t+\Delta t}\boldsymbol{u^i} = {}^{t+\Delta t}\boldsymbol{u^{i-1}} + \Delta \boldsymbol{u^i}$ and $\boldsymbol{i}$ is the number of iterations

The solution procedure of non linear problem using total lagrange formulation and updated lagrange formulation is explained in the fig(3.4) and fig(3.5) respectively

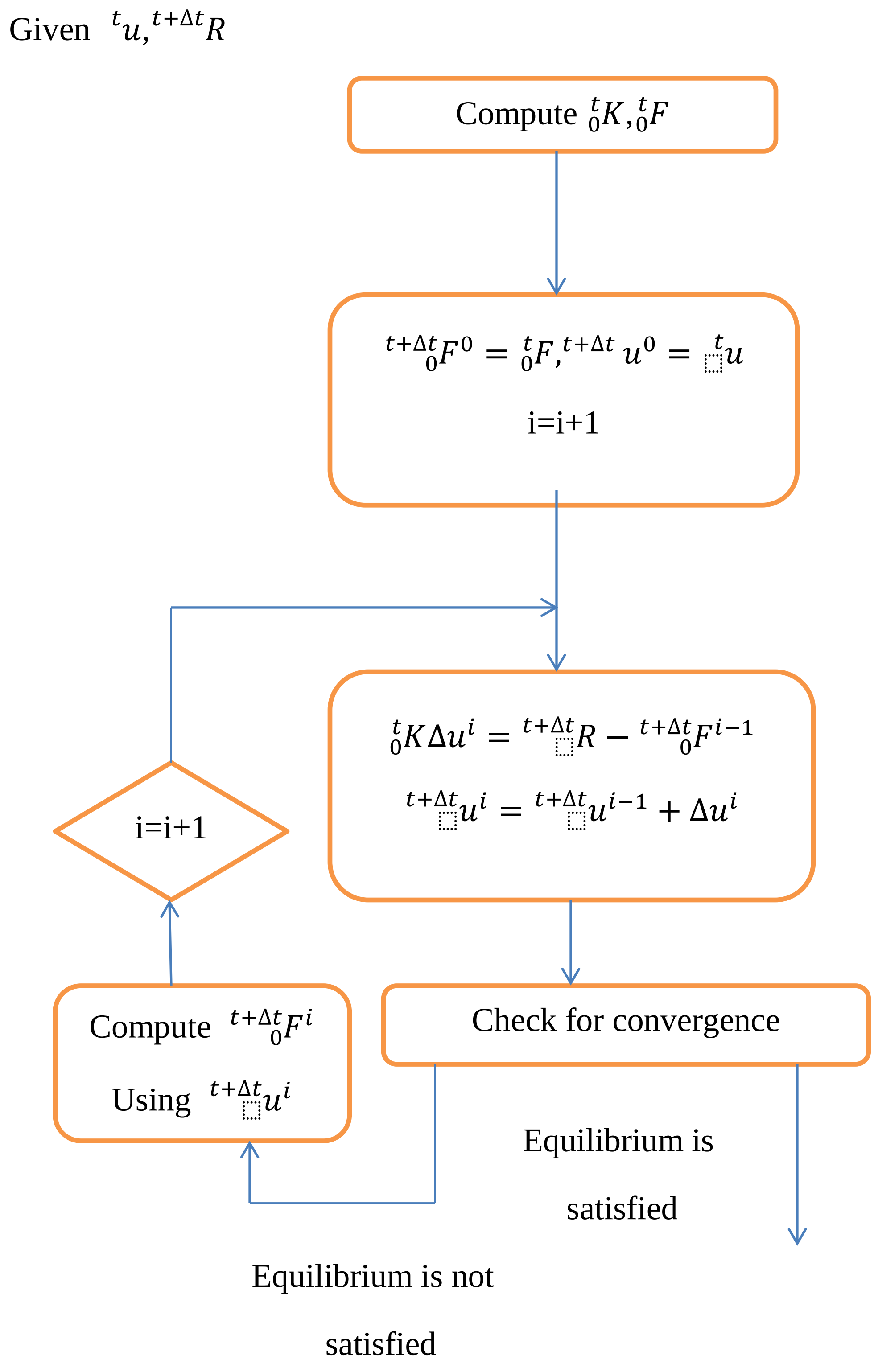

Fig3.4 Flow chart of solution procedure using Total lagrange formulation

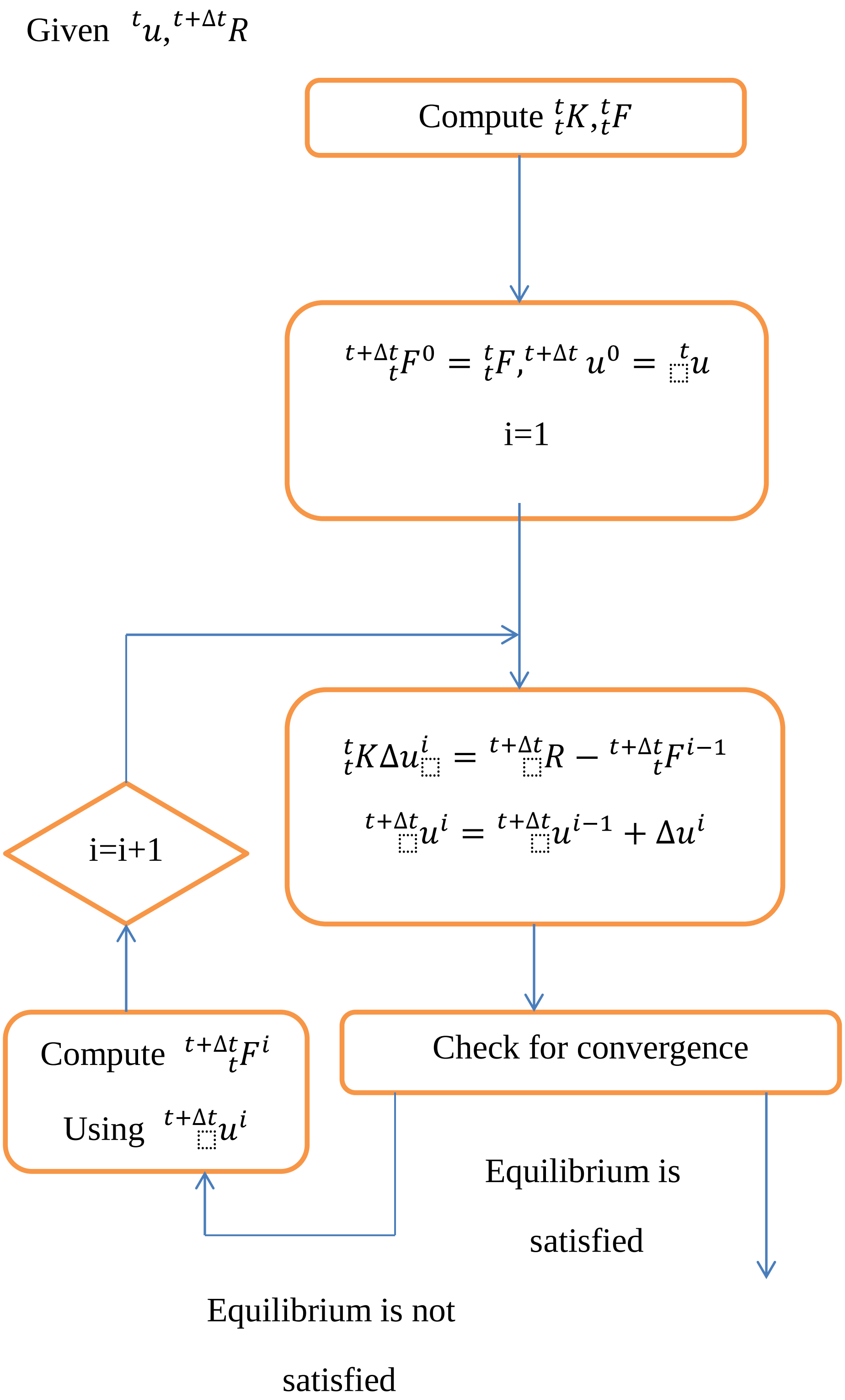

Fig3.5 Flow chart of solution procedure using updated lagrange formulation

Chapter-4

Solution Procedure Implemented in Code

4.1 Introduction

This chapter deals with solution procedure used in the program Flagshyp(written by Javier Bonet and Richard D.Wood and used in my thesis work for solving large strain hyperelastic problem) where first we describe the virtual work equilibrium equation. After that we discuss the updated lagrangean formulation followed by the finite element matrices and newton raphson iteration process implemented in the code. The flow chart describing the solution procedure is also shown at the end.

4.2 Virtual work equilibrium equation

Consider the motion of a body in a Cartesian co-ordinate system as shown in Figure 4.1.where **time 0, time** $\boldsymbol{t}$ and **time** $\boldsymbol{t+\Delta t}$ denotes the different **load levels** in the program . The aim is to evaluate the equilibrium positions of the body at the discrete time points $\mathbf{0}, \boldsymbol{\Delta t}, \mathbf{2}\boldsymbol{\Delta t}, \mathbf{3}\boldsymbol{\Delta t}$…where $\boldsymbol{\Delta t}$ is an increment in time (denotes increment of loads in the code). Assume that the variables for all time steps(load steps) from time 0 to time $\boldsymbol{t}$ have been solved, and that the solution for time $\boldsymbol{t} + \boldsymbol{\Delta t}$ is required next.

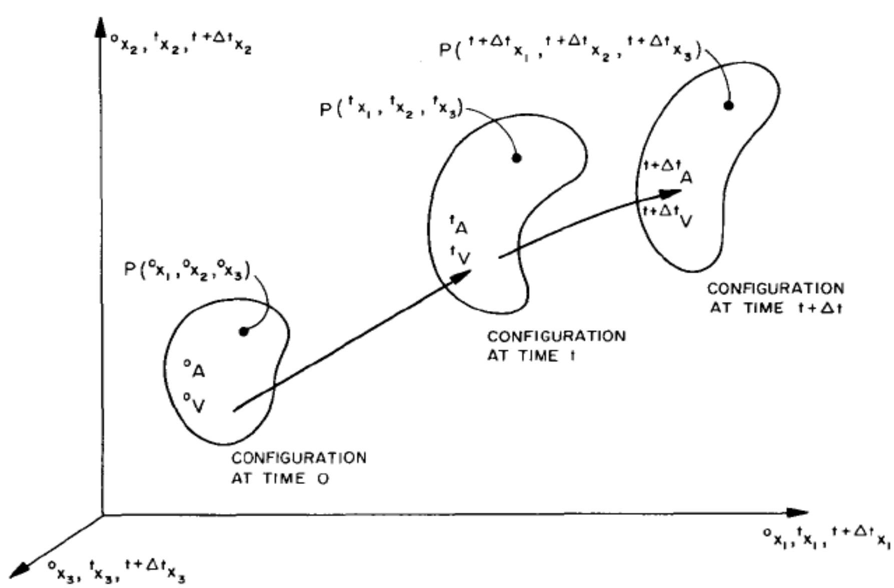

Fig 4.1 Motion of body in Cartesian coordinate system

The virtual work equilibrium equation in the configuration at time $\boldsymbol{t+\Delta t}$ is given as,

$$\int_{t+\Delta t_v} {}^{t+\Delta t}\boldsymbol{\sigma_{ij}}\ \boldsymbol{\delta}\ {}_{t+\Delta t}\boldsymbol{e_{ij}}\ {}^{t+\Delta t}\boldsymbol{dv} = {}^{t+\Delta t}\boldsymbol{R} \tag{4.1}$$

Where ${}^{t+\Delta t}\boldsymbol{R}$ is the external virtual work expression and shown as,

$$ {}^{t+\Delta t}\boldsymbol{R} = \int_{t+\Delta t_v} {}^{t+\Delta t}_{t+\Delta t}\boldsymbol{f_k}\ \ \boldsymbol{\delta u_k}\ {}^{t+\Delta t}\boldsymbol{dv} + \int_{t+\Delta t_a} {}^{t+\Delta t}_{t+\Delta t}\boldsymbol{t_k}\ \ \boldsymbol{\delta u_k}\ {}^{t+\Delta t}\boldsymbol{dv} \qquad \boldsymbol{k = 1, 2, 3} \tag{4.2}$$

Where ${}^{t+\Delta t}\boldsymbol{f_k}$, ${}^{t+\Delta t}_{t+\Delta t}\boldsymbol{t_k}$ are the body force vector and surface traction vector in the configuration at the time $\boldsymbol{t+\Delta t}$ respectively, $\boldsymbol{\delta u_k}$ is the(virtual) variation in the current displacement components ${}^{t+\Delta t}\boldsymbol{u_k}$ and $\boldsymbol{\delta_{t+\Delta t}e_{ij}}$ are the corresponding(virtual) variation in strains

$$\boldsymbol{\delta_{t+\Delta t}e_{ij}} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4.3}$$

The difficulty here being that configuration of the body at time $\boldsymbol{t+\Delta t}$ is unknown so the above equation cannot be solved directly for that wood use updated lagrangean formulation approach in the code.

4.3 Updated Lagrangean Formulation

In this formulation all variables in equations (4.1) and (4.2) are referred to the configuration at time $\boldsymbol{t}$ of the body which is the updated configuration of the body. By a same procedure to the derivation of the T.L formulation as discuss above, equation (4.1) is transformed to

$$\int_{t_v} {}^{t+\Delta t}_{t}\boldsymbol{S_{ij}}\ \boldsymbol{\delta}\ {}^{t+\Delta t}_{t}\varepsilon_{ij}\ \ {}^{t}\boldsymbol{dv} = {}^{t+\Delta t}\boldsymbol{R} \tag{4.4}$$

where ${}^{t+\Delta t}_{t}\boldsymbol{S_{ij}}$ is the Cartesian components of the 2nd Piola-Kirchhoff stress tensor corresponding to the configuration at time $\boldsymbol{t+\Delta t}$, referred to the configuration at time **t,** $\boldsymbol{\delta}^{t+\Delta t}_{t}\varepsilon_{ij}$ is the variation in the Cartesian component of the green lagrange strain tensor in the configuration at time $\boldsymbol{t+\Delta t}$ but measured to the configuration at time $\boldsymbol{t}$.

The incremental stress decomposition used in this case is

$$ {}^{t+\Delta t}_{0}\boldsymbol{S_{ij}} = {}^{t}\boldsymbol{\sigma_{ij}} + {}_{t}\boldsymbol{S_{ij}} \tag{4.5}$$

Where ${}^{t}\boldsymbol{\sigma_{ij}}$ is the Cartesian component of Cauchy stress tensor in the configuration at time t and ${}_{t}\boldsymbol{S_{ij}}$ are the Cartesian component of 2nd Piola-Kirchhoff stress increment at time $\boldsymbol{t}$.

By considering the strain increments ${}^{t+\Delta t}_{t}\varepsilon_{ij}$,the following relation is shown

$$ {}^{t+\Delta t}_{t}\varepsilon_{ij} = {}_{t}\varepsilon_{ij} \tag{4.6}$$

$$ {}_{t}\varepsilon_{ij} = {}_{t}e_{ij} + {}_{t}\eta_{ij} \tag{4.7}$$

Where ${}_{0}e_{ij}$, ${}_{0}\eta_{ij}$ are the linear part and non linear part of incremental green lagrange strains referred to the configuration at time t respectively. It is expressed as

$$ {}_{0}e_{ij} = \frac{1}{2}({}^{t+\Delta t}_{0}u_{i,j} + {}^{t+\Delta t}_{0}u_{j,i}) \tag{4.8}$$

$$ {}_{0}\eta_{ij} = \frac{1}{2}({}^{t+\Delta t}_{0}u_{k,i}\,{}^{t+\Delta t}_{0}u_{k,j}) \tag{4.9}$$

The constitutive relation between stress and strain increment in UL is given as

$$ {}_{t}S_{ij} = {}_{t}C_{ijrs}\;{}_{t}\varepsilon_{rs} \tag{4.10}$$

Where ${}_{t}C_{ijrs}$ are the components of constitutive tensor in the configuration at time t

Now by using the above relations a non linear equation for the incremental displacement u_i in updated lagrangean can be written as

$$ \int_{t_v} {}_{t}C_{ijrs}\;{}_{t}\varepsilon_{rs}\delta\;{}_{t}\varepsilon_{ij}\;{}^{t}dv + \int_{t} {}^{t}\sigma_{ij}\delta\;{}_{t}\eta_{ij}\;{}^{t}dv = {}^{t+\Delta t}R + \int_{t_v} {}^{t}\sigma_{ij}\delta\;{}_{t}e_{ij}\;{}^{t}dv \tag{4.11}$$

Which represent a non linear equation for the incremental displacements u_i

4.4 Internal virtual work linearization

The solution of equations (4.11) cannot be obtained directly, since they are non linear in displacement increments. So to obtain the solution we need to linearize the internal virtual work which has been done by assuming in equation (4.11) ${}_{t}\varepsilon_{ij} = {}_{t}e_{ij}$. It means that ,in addition to using $\delta\;{}_{0}\varepsilon_{ij} = \delta\;{}_{0}e_{ij}$ respectively,the incremental constitutive relation (explain in equation(4.10) employ are

$$ {}_{t}S_{ij} = {}_{t}C_{ijrs}\;{}_{t}e_{rs} \tag{4.12}$$

So In the total lagrangean formulation the approximate equilibrium equation to be solved is

$$\int_{t_v} {}_tC_{ijrs}\ {}_te_{rs}\delta\ {}_te_{ij}\ {}^tdv + \int_t\ {}^t\zeta_{ij}\delta\ {}_t\eta_{ij}\ {}^tdv \qquad = {}^{t+\Delta t}R - \int_{t_v}\ {}^t\zeta_{ij}\delta\ {}_te_{ij}\ {}^tdv \quad (4.13)$$

4.5 Finite element solution

To solve the equations(4.13) the body is descritize into finite element whose boundaries is defined by the nodes as shown in fig 4.2

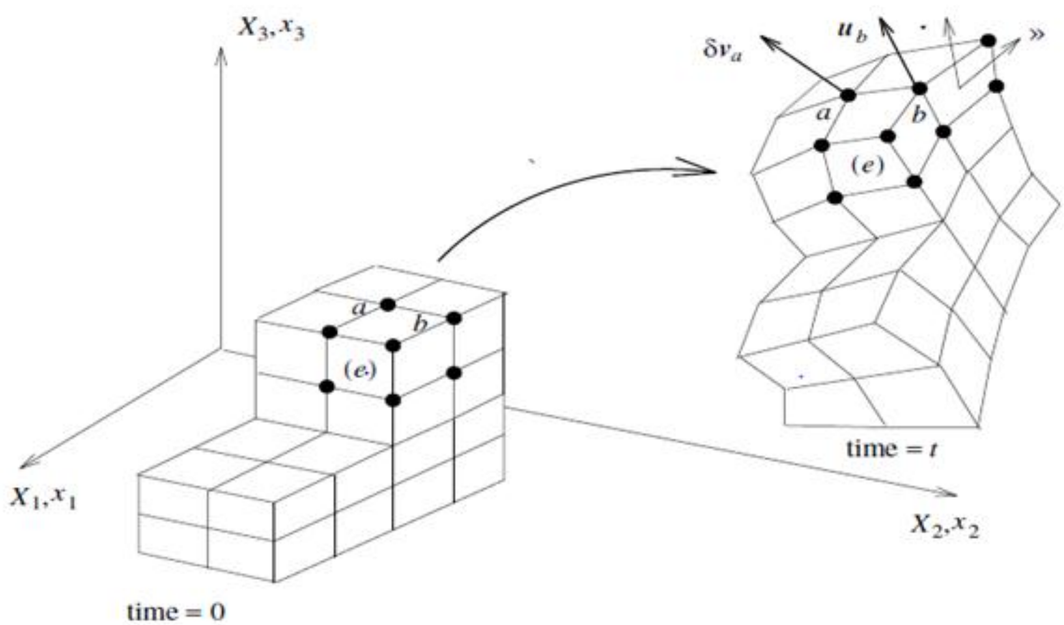

Fig 4.2 Descritization

The co-ordinates and displacements in the configuration at time $\mathbf{0}, \boldsymbol{t}\ \boldsymbol{and}\ \boldsymbol{t}+\Delta\boldsymbol{t}$ are interpolated using the shape function(N_k) for a node k

$$ {}^0x_i = \sum_{k=1}^{T} N_k\ {}^0x_i^k\ ; \qquad i = 1,2,3 \qquad (4.14)$$

$$ {}^tx_i = \sum_{k=1}^{T} N_k\ {}^tx_i^k \qquad {}^tu_i = \sum_{k=1}^{T} N_k\ {}^tu_i^k\ ; \qquad i = 1,2,3 \qquad (4.15)$$

$$^{t+\Delta t}\boldsymbol{x}_{\boldsymbol{i}} = \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{T}} \boldsymbol{N}_{\boldsymbol{k}}\,{}^{t+\Delta t}\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}\,; \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3} \qquad (4.16)$$

$$\Delta \boldsymbol{u}_{\boldsymbol{i}} = \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{T}} \boldsymbol{N}_{\boldsymbol{k}} \quad \boldsymbol{u}_{\boldsymbol{i}}^{\boldsymbol{k}}\,; \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3} \qquad (4.17)$$

Where ${}^{\mathbf{0}}\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}$, ${}^{t}\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}$, ${}^{t+\Delta t}\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{k}}$ are the coordinates of nodal point k corresponding to direction $\boldsymbol{i}$ at time $\mathbf{0}, \boldsymbol{t}, \boldsymbol{t} + \Delta \boldsymbol{t}$ respectively, $\boldsymbol{N}_{\boldsymbol{k}}$ is the shape function corresponding to nodal point k, ${}^{t}\boldsymbol{u}_{\boldsymbol{i}}^{\boldsymbol{k}}$ is the displacement components of nodal point k in configuration at time t, $\Delta \boldsymbol{u}_{\boldsymbol{i}}$ is the increment in displacement component and $\boldsymbol{T}$ is the number of nodal point in an element.

By using the equations the finite element solution of equation(4.13), results into

$$\left({}_{t}^{t}\boldsymbol{K}_{\boldsymbol{k},\boldsymbol{b}}\right)\Delta \boldsymbol{u} = {}^{t+\Delta t}\boldsymbol{R} - {}_{t}^{t}\boldsymbol{F} \qquad (4.17)$$

Herein,

$$^{t+\Delta t}\boldsymbol{R} = \int_{{}^{t}\boldsymbol{v}} \boldsymbol{N}_{\boldsymbol{k}}\boldsymbol{f}\ {}^{t}\boldsymbol{dv} + \int_{{}^{t}\boldsymbol{a}} \boldsymbol{N}_{\boldsymbol{k}}\boldsymbol{t}\ {}^{t}\boldsymbol{da} \qquad (4.18)$$

$${}_{t}^{t}\boldsymbol{F} = \int_{{}^{t}\boldsymbol{v}} \boldsymbol{\sigma}\boldsymbol{\nabla}\boldsymbol{N}_{\boldsymbol{k}}\ {}^{t}\boldsymbol{dv}\,; \qquad {}_{t}^{t}\boldsymbol{F}_{\boldsymbol{k},\boldsymbol{i}} = \sum_{\boldsymbol{j}=\mathbf{1}}^{\mathbf{3}} \int_{{}^{t}\boldsymbol{v}} \boldsymbol{\sigma}_{\boldsymbol{ij}} \frac{\partial \boldsymbol{N}_{\boldsymbol{k}}}{\partial \boldsymbol{x}_{\boldsymbol{j}}}\ {}^{t}\boldsymbol{dv} \qquad (4.19)$$

Where $\boldsymbol{\nabla}\boldsymbol{N}_{\boldsymbol{k}}$ is the Cartesian derivate of shape function($\boldsymbol{N}_{\boldsymbol{k}}$) and is derive as

$$\frac{\partial \boldsymbol{N}_{\boldsymbol{k}}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{X}}{\partial \mathcal{E}}^{-\boldsymbol{T}} \frac{\partial \boldsymbol{N}_{\boldsymbol{k}}}{\partial \mathcal{E}}\,; \qquad \frac{\partial \boldsymbol{X}}{\partial \mathcal{E}} = \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{X}_{\boldsymbol{k}} \otimes \boldsymbol{\nabla}_{\mathcal{E}}\boldsymbol{N}_{\boldsymbol{k}} \qquad (4.20)$$

4.5.1 The tangent stiffness matrix

The tangent stiffness matrix ((${}_{t}^{t}\boldsymbol{K}_{\boldsymbol{kb}}$) provide the relationship between changes in forces at node k due to changes in current postion of node b and is shown as

$${}_{t}^{t}\boldsymbol{K}_{\boldsymbol{kb}} = {}_{t}^{t}\boldsymbol{K}_{\boldsymbol{c},\boldsymbol{kb}} + {}_{t}^{t}\boldsymbol{K}_{\boldsymbol{\sigma},\boldsymbol{kb}} + {}_{t}^{t}\boldsymbol{K}_{\boldsymbol{g},\boldsymbol{kb}} + {}_{t}^{t}\boldsymbol{K}_{\boldsymbol{p},\boldsymbol{kb}} \qquad (4.21)$$

Where ${}_{t}^{t}\boldsymbol{K}_{c,kb}$, ${}_{t}^{t}\boldsymbol{K}_{\sigma,kb}$, ${}_{t}^{t}\boldsymbol{K}_{g,kb}$, ${}_{t}^{t}\boldsymbol{K}_{p,kb}$ is the constitutive component, the stress component, the dilation component and the external pressure component of tangent stiffness matrix respectively.

4.5.1.1 Constitutive component of tangent stiffness matrix

The constitutive component of tangent stiffness matrix is given as

$$[{}_{t}^{t}\boldsymbol{k}_{c,kb}]_{ij} = \int_{{}^{t}v} \sum_{k,l=1}^{3} \frac{\partial N_k}{\partial x_k} c_{ijkl}^{sym} \frac{\partial N_k}{\partial x_l} \; {}^{t}dv \qquad i, j = 1, 2, 3 \tag{4.22}$$

Where c_{ijkl}^{sym} is the symmetrized constitutive tensor

$$c_{ijkl}^{sym} = \frac{1}{4}(c_{ikjl} + c_{iklj} + c_{kijl} + c_{kilj}) \tag{4.23}$$

4.5.1.2 Stress Component of Tangent Stiffness Matrix

The stress component of tangent stiffness matrix relating node k to node b is given as

$$[{}_{t}^{t}\boldsymbol{k}_{c,kb}]_{ij} = \int_{{}^{t}v} \sum_{k,l=1}^{3} \frac{\partial N_k}{\partial x_k} \sigma_{kl} \frac{\partial N_b}{\partial x_l} \delta_{ij} \; {}^{t}dv \qquad i, j = 1, 2, 3 \tag{4.24}$$

Where δ_{ij} is the kroneker delta

4.5.1.3 Dilation Component of Tangent Stiffness Matrix

This component of tangent stiffness matrix is compute by the mean dilation method and is added when the material is incompressible

$${}_{t}^{t}\boldsymbol{k}_{g,kb} = \bar{\boldsymbol{k}} \; {}^{t}v \;\; \bar{\nabla} N_k \otimes \bar{\nabla} N_b \tag{4.25}$$

Where $\bar{\boldsymbol{k}}$ is the bulk modulus, $\bar{\nabla} N_k, \bar{\nabla} N_b$ are the average Cartesian derivative of the shape function

4.5.1.4 External Pressure Component of Tangent Matrix

The external pressure component of the tangent matrix is added when the pressure is acting on the body and is expressed as

$$ {}_{t}^{t}\boldsymbol{k}_{p,kb} = \sum_{o=1}^{3} \mathcal{E}_{ijk}\,[\boldsymbol{k}_{p.kb}]_{o} \qquad \boldsymbol{i},\boldsymbol{j} = \mathbf{1},\mathbf{2},\mathbf{3} \tag{4.26} $$

Where $\mathcal{E}_{ijk}$ is the third order alternating tensor, $\boldsymbol{k}_{p,ab}$ is the vector of stiffness coefficient.

$$ \boldsymbol{k}_{p,kb} = \frac{1}{2}\int_{{}^{t}a} \boldsymbol{p}\frac{\partial \boldsymbol{x}}{\partial \mathcal{E}}\left(\frac{\partial \boldsymbol{N}_{k}}{\partial \boldsymbol{\eta}}\boldsymbol{N}_{b} - \frac{\partial \boldsymbol{N}_{b}}{\partial \boldsymbol{\eta}}\boldsymbol{N}_{b}\right) \boldsymbol{d}\mathcal{E}\boldsymbol{d\eta} + \frac{1}{2}\int_{{}^{t}a} \boldsymbol{p}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\eta}}\left(\frac{\partial \boldsymbol{N}_{k}}{\partial \mathcal{E}}\boldsymbol{N}_{b} - \frac{\partial \boldsymbol{N}_{b}}{\partial \mathcal{E}}\boldsymbol{N}_{b}\right) \boldsymbol{d}\mathcal{E}\boldsymbol{d\eta} \tag{4.27} $$

It should be noted that all the derivation discuss in section 5.5 is for the single node(k) of an element. Assembling of the matrices need to be performed which is done by the loop function in the code.

4.6 Newton Raphson Iteration

The equations (4.17) is the only approximation to the actual equation to be solved in each load step, i.e. equation (4.4). The linearized equilibrium equation(4.13) may introduce error . For this reason it become necessary to iterate in each load step until, within the necessary assumptions on the variation of the material constants equation (4.17) are satisfied to a required tolerance. The equation used in the updated lagrangean formulation is

$$ ({}_{t}^{t}\boldsymbol{K}_{L} + {}_{t}^{t}\boldsymbol{K}_{NL})\Delta \boldsymbol{u}^{i} = {}^{t+\Delta t}\boldsymbol{R} - {}_{t}^{t+\Delta t}\boldsymbol{F}^{i-1} \tag{4.28} $$

Where ${}^{t+\Delta t}\boldsymbol{u}^{i} = {}^{t+\Delta t}\boldsymbol{u}^{i-1} + \Delta \boldsymbol{u}^{i}$ and i is the number of iterations

The solution procedure as described in the chapter is shown in the fig 4.3

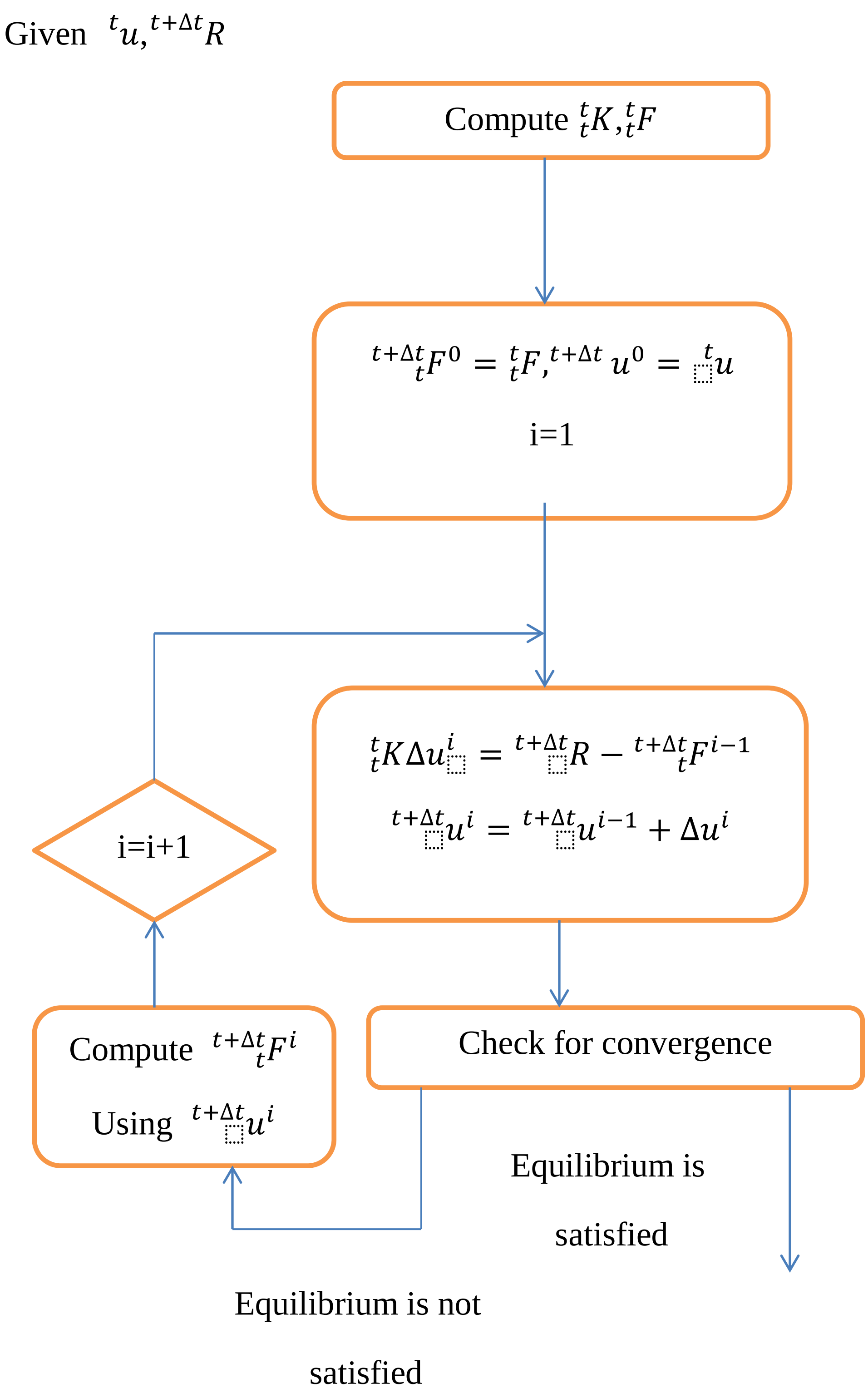

Fig4.3 Flow chart of solution procedure using updated lagrange formulation

Chapter-5

Computer Implementation

5.1 Introduction

This chapter describe the program structure of the solution procedure(discussed in previuos chapter) for solving the large strain hyperelastic problem written by wood []. First of all, we discuss the master routine which controls the overall organization of the program and describe the solution procedure. After that we discuss the various subroutines which compute the nodal forces,stresses, stiffness matrix, behaviour of material etc. Description of output and input file are also describe here.

5.2 Description of input file

The input file required by the program is described below.

Table 5.1: Input file of program

Input	Lines	Comments
1-title(1:80)	1	Problem title
2-eltype	1	Element type (note 1)
3-npoin	1	Number of mesh points
4-ip,icode(ip),x(i,ip), i=1,ndime	Node	Coordinate data: ip:node number icode(ip):boundary code(note 2) x(i,ip):x,y,z coordinates ndime:No of nodes per element
5-nelem	1	Number of elements
6-ie,matno(ie), Lnods(i,ie),i=1,node	nelem	ie:element number matno(ie):material number lnods(i,ie):nodal connectivities nnods:No of nodes per element
7-nmats	1	Number of different materials
8-im,matyp(im)	2×nmats	Material data: im: material number matyp(im):constitutive equation props(i,im):properties(note 3)
9-nplds,nprs,nbpel (gravt(i),i=1,ndime)	1	Loading data: Nplds: no of loaded nodes Nprs: no of non zero individual prescribed displacement

10-ip,(force(i), i=1,ndime	nplds	Point loads: ip:node number force:force vector
11-ip,id,presc	nprs	Prescribed displacements ip:node number id:spatial direction prescx:total nominal prescribed displ
13-nincr,xlmax,dlamb Miter,cnorm,search,arcln	1	Solution control parameters nincr:number of load/displacement increment xlmax:max value of load scaling parameter dlamb:load parameter increment miter: maximum allowed no of iterations per increment cnorm:convergence tolerance searc: line search parameter(if 0.0 not in use) arcln:arc length parameter(if 0.0 not in use)

Note 1. The element types which havs been recognized as tria3-3 noded linear triangle,tria6-6 noded quadratic triangle,quad4-4-noded bilinear triangle,tetr4-4 noded linear tetrahedron,tetr10-10 noded quadratic tetrahedron,hexa8-8 noded trilinear hexahedron.

Note 2. The boundary codes are given as 0-free,1-x prescribed,2-y prescribed,3-x,y prescribed,4-z prescribed,5-x,z prescribed,6-y,z prescribed,7-x,y,z prescribed.

Note 3. The constitutive equations have been implemented

1-Plain strain or three-dimensional compressible neo-hookean,

2-not defined,

3-plain strain or three dimensional hyperelastic in principle direction

4-plane stress hyperelastic in principle direction

5-plane strain or three dimensional nearly incompressible neo hookean

6-plane stress incompressible neo-hookean

7-plane stress incompressible hyperelasticity in principal directions.

The list of properties to be used in input line 8 are shown below in which $\boldsymbol{\rho}$ is the density,$\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the lame coefficients,$\boldsymbol{k}$ is the bulk modulus and $\boldsymbol{h}$ is the thickness for plane stress case

Table 5.2- List of properties read by the hyperelastic material

Constitutive Equation type	Props (1)	Props(2)	Props(3)	Props(4)
1	ρ	μ	λ	------
3	ρ	μ	λ	------
4	ρ	μ	λ	h
5	ρ	μ	k	-------
6	ρ	μ	h	-------
7	ρ	μ	k	-------

5.3 Description of output file

The content and structure of output file produce the program is described below

Table 5.3-Output file generated by the program

Output	Lines	Comments
1-Title,’ at increment:’,incrm,’,load: ‘,xlamb	1	Title line: Title:partial problem title Incrm:increment number Xlamb:current load parameter
2-elytp	1	Element type
3-npoin	1	Number of mesh nodes
4-ip,icode(ip),(x(i,ip), I=1,ndime)(force(i),i=1,ndime	npoin	Coordinates and force data: ip:node number icode(ip):boundary code x(i,ip):x,y,z coordinates force(i):total external force or reaction vector ndime:No of dimensions
5-nelem	1	Number of elements
6-ie,matno(ie),(lnods(i,ie), i=1,node)	nelem	Same as input line 6
7-(stress(i,ig,ie),i=1,nstrs	ngaus × ngaus	Cauchy stresses for each gauss point (ig=1,ngaus) of each mesh element(ie=1,nelem (2-D) stress:$\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$(plane strain) (2-D) stress:$\sigma_{xx}, \sigma_{xy}\sigma_{yy}, h$(plane stress)

5.4 Solution algorithm

Updated lagrangian formulation followd by newton raphson iteration technique has been used in the program flagshyp to solve the large strain hyperelastic problem. In this topic we will discuss the algortihm,the main routine which follow the algorithm and the various subroutines used in the program.

5.4.1 Algorithm

1. Input the geometry of problem, material properties and solution parameters (segment 1).

2. Initialize the external force $\boldsymbol{R} = \boldsymbol{0}$, (segment1)

3. Initialize the residual $\boldsymbol{H} = \boldsymbol{0}$, $\boldsymbol{x} = \boldsymbol{X}$ (initial geometry) and Find initial $\boldsymbol{K}$ (segment 2)

4. Loop over load increments (Segment 4)

i.Consider the increment parameter $\Delta\lambda$(segment 4)

ii.Set $\boldsymbol{\lambda} = \boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}$, $\boldsymbol{R} = \boldsymbol{\lambda}\overline{\boldsymbol{R}}$,$\boldsymbol{H} = \boldsymbol{H} + \Delta\overline{\boldsymbol{\lambda R}}$ (segment 4))

If there is prescribed displacement (segment 4)

i.Update the geometry by adding the prescribed displacement in the initial coordinates

ii.Find the $\boldsymbol{F}, \boldsymbol{R}, \boldsymbol{K}, \boldsymbol{H} = \boldsymbol{\lambda R} - \boldsymbol{F}$(Segment 4)

End IF (Segment 4)

i.Do while the residual norm $\left(\frac{//\boldsymbol{H}//}{//\boldsymbol{R}//}\right)$> convergence tolerance (segment 5)

(From here newton raphson iteration start)

i.Solve **Ku=H** (Segment 5)

ii Loop over line search(if line search parameter(η) is given then it update the geometry by scaling the displacement u before each iteration and if η value is 1 then it follow the standard newton raphson method. (segment6)

i.Update Geometry **x=x+u** (segment 6)

ii.Find $\boldsymbol{F}, \boldsymbol{R}, \boldsymbol{K}, \boldsymbol{H} = \boldsymbol{\lambda R} - \boldsymbol{F}$ (segment 6)

End Loop

End Loop

ii.Output increment results (current coordinates of geometry or reactions).(segment7)

End loop

Where $(\frac{//\boldsymbol{H}//}{//\boldsymbol{R}//})$ is the residual norm in which R =reaction component+force component, F is the internal force and u is the displacement and $\boldsymbol{K}$ is the stiffness matrix

The external load F is applied in a series of increments because it is easier to find the converged solution for each individual load step.

$$\boldsymbol{R} = \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{l}} \Delta\boldsymbol{\lambda}\boldsymbol{R}_{\boldsymbol{i}} \tag{5.1}$$

Where $\Delta\boldsymbol{\lambda}$ is the load increment $\boldsymbol{l}$ is the total number of load increment.

The implementation of the steps of algorithm in the program is describe by the program segments.

5.4.2 Program segment 1-Input

```
      call welcome(title,rest)
      if(.not.rest) then
         call elinfo(mnode,mgaus,ndime,nnode,ngaus,nnodb,
     &                     ngaub,eledb,elbdb,eltyp)
         call innodes(mpoin,ndime,npoin,x,icode,ldgof)
         call inelems(melem,nelem,nnode,lnods,matno,
     &                      nmats)
         nstrs=4
```

```
      call degfrm (mdgof,npoin,ndime,stifp,ndgof,negdf,
&                  ldgof,stifd)
      call profile(mprof,ndgof,nelem,nnode,ndime,lnods,
&                 ldgof,nprof,kprof)
      call matprop(ndime,nmats,matyp,props)
      call inloads(ndime,npoin,ndgof,negdf,nnodb,mbpel,
&                  ldgof,eload,pdisp,gravt,nprs,nbpel,
&                  lbnod,press)
      call incontr(nincr,xlmax,dlamb,miter,cnorm,searc,
&                 ,1)
```

The program will start either from an input data file or, when convenient, using data from a restart file written during a previous incomplete analysis. The mode of operation is controlled by the user interactively from the screen and is submitted in the logical variable rest which is read by the welcome routine. In the following it has been assumed that the data are read for the first time.Subroutine **elinfo** reads the element type eltyp as discussed in note1 and establishes the arrays **eledb** and **elbdb** which stores the shape functions and their non dimensional derivatives for each gauss point of the chosen element type as shown below

Eledb(1:ndime+1,1:nnode+1,1:ngaus)

$$\begin{bmatrix} N_1 \varepsilon_i \eta_i \zeta_i & \ldots\ldots & N_n \varepsilon_i \eta_i \zeta_i & W_i \\ \frac{\partial N_i}{\partial \varepsilon} \varepsilon_i \eta_i \zeta_i & \ldots\ldots. & \frac{\partial N_i}{\partial \varepsilon} \varepsilon_i \eta_i \zeta_i & \varepsilon_i \\ \frac{\partial N_i}{\partial \eta} \varepsilon_i \eta_i \zeta_i & \ldots\ldots & \frac{\partial N_i}{\partial \eta} \varepsilon_i \eta_i \zeta_i & \eta_i \\ \frac{\partial N_i}{\partial \zeta} \varepsilon_i \eta_i \zeta_i & \ldots\ldots & \frac{\partial N_i}{\partial \zeta} \varepsilon_i \eta_i \zeta_i & \zeta_i \end{bmatrix} \quad \text{for } i=1,2,3$$

Where $\boldsymbol{\varepsilon_i, \eta_i, \zeta_i}$ are the coordinates of isoparametric plane and $\boldsymbol{W_i}$ is the Gaussian weight
Subroutine **innodes** and subroutine inelems reads the nodal input data and element input data as shown in input file. Suboutine **nodecon** determines the node to node connectivities used in degfrm subroutine, which at the same time put the degree of freedom numbers to nodes and stores them in ldgof. In profile subroutine the profile pointers are obtained and stored them in array kprof which describe the height of each profile column and then addresses in stifp of each column as shown in fig 5.1

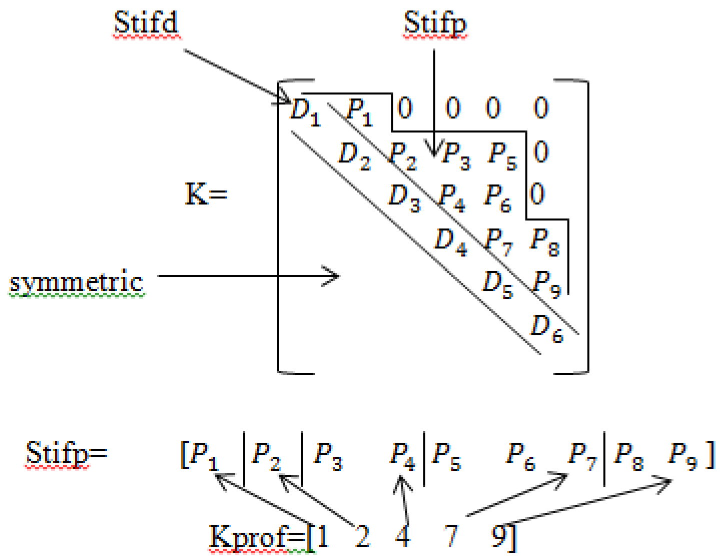

Fig 5.1 Profile storage pointers

The material property depending upon the constitutive equations(see note 2) is read from the input file in subroutine **matprop**. The loads and displacement apply in the problem is read from the input file in the subroutine **inloads**. This subroutine also initialize the force vector and prescribed displacement. Subroutine **Incontr** reads the solution control parameter (number of load/displacement increments, maximum value of load scaling parameter, load parameter increment,maximum allowed no of iteration per increment, convergence tolerance, line search parameter) from the input file

5.4.3 Program Segment-2 Initialization and dump

```
      xlamb=0.0
      incrm=0
      call initno(ndime,npoin,ndgof,nprof,x,x0,stifd,
  &                 stifp,resid)
      call initel(ndime,nnode,ngaus,nelem,gravt,x,eledb,
  &                lnods,matno,matyp,props,ldgof,eload,
  &                kprof,stifd,stifp,vinc,elecd,elacd,vol0)
      call dump(title,eltyp,ndime,npoin,nnode,ngaus,nstrs,
  &                 nelem,ndgof,negdf,nprs,nprof,nmats,incrm,
  &                 xlamb,nbpel,nnodb,ngaub,matyp,props,matno,
  &                 lnods,x,x0,kprof,stifd,stifp,resid,eload,
  &                 ldgof,icode,eledb,pdisp,vol0,elbdb,lbnod,pres
```

The residuals $\boldsymbol{R}$, the stiffness matrix component is set to zero in the subroutine **initno**. It also define the initial geometry $\boldsymbol{x0}$ as the current geometry $\boldsymbol{x}$ which is the 3rd step of algorithm .The equivalent nodal forces due to gravity and the tangent stiffness matrix at the unstressed configuration(it coincides with the small strain linear elastic stiffness matrix) is evaluate in the subroutine **initel**. It also calculate the total mesh volume and print on the screen.In **initel** subroutine there are subroutines called **gradsh, kvolume, cisotp** and **kconst**. **Gradsh** subroutine first evaluate the jacobian matrix at each gauss point using equation (4.20) and then the cartesian derivative of shape function at each gauss point according to the equation(4.20). Subroutine **kvolume** evaluate the mean dilation component(volumetric component) of the stiffness matrix by using the equation (4.24) and assemble them in the tangent matrix. This subroutine is used for incompressible type of matreials.Subroutine **cisotp** calculate the isotropic elasticity tensor by using the equation (2.92) and the subroutine **kconst** calculate the constitutive component of stiffness matrix by using the equation (4.22) and assemble them in the tangent matrix after that the program dumped all the information as discuss in segment1 and segment 2 to a restart file by using the subroutine **dump.**

.

5.4.4 Program segment 3-Restart

```
    else
        call restar1(title,eltyp,ndime,npoin,nnode,ngaus,
 &                      nstrs,nelem,ndgof,negdf,nprs,nprof,
 &                      nmats,incrm,xlamb,nbpel,nnodb,ngaub)
        call restar2(ndime,npoin, nnode, ngaus,nelem,ndgof,
 &                      negdf,nprof,nmats,nnodb,ngaub,nbpel,
 &                      matyp,props,matno,lnods,x,x0,kprof,
 &                      stifd,stifp,resid,eload,ldgof,icode,
 &                      eledb,pdisp,vol0,elbdb,lbnod,press)
        call incontr(nincr,xlmax,dlamb,miter,cnorm,searc,
                        & arcln,5)
    endif
```

If the solution is not converged and the maximum allowed number of iteration is reached,in that case the program will restart from the previous incomplete solution the restart file is read and the solution control parameters are reset by reading them interactively from the screen.this allow the user to reset the convergence tolerance, the maximum number of iteration per step, number of loads, displacement increment, maximum value of load scaling parameter, load parameter increment, line search parameter.

5.4.5 Program segment 4 –Incrementation Loop

```
   do while((xlamb.lt.xlmax).and.(incrm.lt.nincr))
       incrm=incrm+1
       xlamb=xlamb+dlamb
       call force(ndgof,dlamb,eload,tload,resid)
       call bpress(ndime,nnodb,ngaub,nbpel,dlamb,elbdb,
&                     lbnod,press,x,ldgof,tload,react,
&                     resid,kprof,stifd,stifp)
       if(nprs.gt.0) then
       call prescx(ndime,npoin,ldgof,pdisp,x0,x,xlamb)
       call initrk(ndgof,nprof,negdf,xlamb,eload,tload,
&                    resid,react,stifd,stifp)
       call bpress(ndime,nnodb,ngaub,nbpel,xlamb,elbdb,
&                     lbnod,press,x,ldgof,tload,react,
&                     resid,kprof,stifd,stifp)
       call elemtk(ndime,nnode,ngaus,nstrs,nelem,x,x0,
&                     eledb,lnods,matno,matyp,props,ldgof,
&                     stres,resid,kprof,stifd, stifp,
&                     react,vinc,elecd,elacd,vol0)
     endif
```

The do while loop control the load or prescribed displacement incrementation. This remain active till the load scaling parameter xlamb is less than the maximum value of load scaling parameter xlmax and the increment number is smaller than the total number of increments nincr. The current value of external point load is evaluate in the subroutine **force**. The small residual carried over from the previous increment is also added by the subroutine **force** this prevents errors in the converged solution from propagating throughout the solution process. Subroutine **bpress** evaluates the addition to the tangent stiffness matrix due to the change in its initial pressure component that results from the increment in the magnitude of the pressure. In the subroutine **bpress** there is a subroutine called **pforce** and **kpress**. Subroutine **pforce** evaluate the nodal forces due to pressure component and add it to the total load vector(tload). Subroutine **kpress** evaluate the pressure component of the stiffness matrix by using the equation (4.27) and assemble them in the tangent matrix.

If there is a prescribed displacement in the problem then the current increment in their value will change the current geometry and necessiate complete re-evaluation of equivalent internal and surface pressure forces and the tangent stiffness matrix (this is effectively a geometry update).

For that subroutine **prescx** will reset the current geometry for those nodes with prescribed displacements based on their initial coordinates, the nominal value of the prescribed displacement, and the current load scaling parameter xlamb. Subroutine **initrk** initializes the tangent matrix to zero and the residuals to the current value of external and point loads. Subroutine bpress adds the total value of the nodal forces due to surface pressure and obtains the corresponding initial surface pressure tangent matrix component. Elemtk subroutine evaluates the current equivalent nodal forces due to stresses, subtracts these from the residual vector and computes the constitutive and initial stress components of the tangent matrix. Subroutine elemtk discuss in detail in the setion 5.4.7.

5.4.6 Program Segment 5-Newton Raphson Loop

```
    niter=0
    rnorm=2.*cnorm
    do while((rnorm.gt.cnorm).and.(niter.lt.miter))
       niter=niter+1
    call datri(stifp,stifp,stifd,kprof,
&             ndgof,.false.,6)
    call dasol(stifp,stifp,stifd,resid,displ,
&              kprof,ndgof,6,)
      endif
```

The do while loop controls the Newton–Raphson iteration process. This will continues until the residual norm rnorm is smaller than the tolerance cnorm and while the iteration number niter is smaller than the maximum allowed miter (Note that rnorm is initialized in such a way that thisloop is completed at least once.). The first step of the process is the factorization((LDU decomposition) of the current tangent matrix which is performed in the subroutine **datri**, note that for the first iteration of the first increment, the tangent stiffness matrix is available either from **initel** in Segment 2 or from **elemtk** in Segment 4. Subsequently, the current matrix is that evaluated at the previous iteration unless nonzero displacements have been prescribed, in which case it is re-evaluated in Segment 4 after that the subroutine **dasol** perfrom the back substitution from which the displacements $\boldsymbol{u}$ is obtained.

5.4.7 Program Segment 6-Solution Update and Equilibrium Check

```
      eta0=0.
      eta=1.
      nsear=0
      rtu=rtu0*searc*2.
     do while((abs(rtu).gt.abs(rtu0*searc)).and.
&                (nsear.lt.5))
         nsear=nsear+1
         call update(ndime,npoin,ldgof,x,displ,eta-eta0)
         call initrk(ndgof,nprof,negdf,xlamb,eload,tload,
&                         resid,react,stifd,stifp)
         call bpress(ndime,nnodb,ngaub,nbpel,xlamb,elbdb,
&                          lbnod,press,x,ldgof,tload,react,resid,
&                          kprof,stifd,stifp)
         call elemtk(ndime,nnode,ngaus,nstrs,nelem,x,x0,
&                          eledb,lnods,matno,matyp,props,ldgof,
&                          stres,resid,kprof,stifd,stifp,react,
&                          vinc,elecd,elacd,vol0)
         call search(ndgof,resid,displ,eta0,eta,rtu0,rtu)
      enddo
      call checkr(incrm,niter,ndgof,negdf,xlamb,resid,
&                    tload,react,rnorm)
    enddo
```

If the line search parameter searc is input as 0.0, then the do while loop is perform only once per newton raphson iteration in order to update the solution and check for the equilibrium. If the line search parameter is not 0.0, the line search is active and the inner line search loop may be perform up to five times. The routine update adds the displacements calculated in Segment 5, scaled by the line search factor $\boldsymbol{\eta}$ (initially set to 1) to the current nodal positions. Now from the new geometry Subroutines **initrk,bpress**, and **elemtk** serve the same purpose as discussed in Segment 4. The subroutine search calculate a new line search factor ´ if required. Finally,**checkr** subroutine compute the residual norm rnorm.

5.4.8 Program Segment 7-Retry or Output Result

```
        if(niter.ge.miter) then
           write(6,100)
           call restar1(title,eltyp,ndime,npoin,nnode,
&                     ngaus,nstrs,nelem,ndgof,negdf,
&                     nprs,nprof,nmats,incrm,xlamb,
&                     nbpel,nnodb,ngaub)
           callrestar2(ndime,npoin,nnode,ngaus,nelem,
&                     ndgof,negdf,nprof,nmats,nnodb,
&                     ngaub,nbpel,matyp,props,matno,
&                     lnods,x,x0,kprof,stifd,stifp,
&                     resid,eload,ldgof,icode,eledb,
&                     pdisp,vol0,elbdb,lbnod,press)
            call incontr(nincr,xlmax,dlamb,miter,cnorm,
&                       searc,arcln,5)
         else
            call output(ndime,nnode,ngaus,nstrs,npoin,
&                      nelem,eltyp,title,icode,incrm,
&                      xlamb,x,lnods,ldgof,matno,stres,
&                      tload,react)
            call dump(title,eltyp,ndime,npoin,nnode,
&                     ngaus,nstrs,nelem,ndgof,negdf,
&                     nprs,nprof,nmats,incrm,xlamb,
&                     nbpel,nnodb,ngaub,matyp,props,
&                     matno,lnods,x,x0,kprof,stifd,
&                     stifp,resid,eload,ldgof,icode,
&                     eledb,pdisp,vol0,elbdb,lbnod,
&                     press)
        endif
```

If the newton raphson iteration not converge then the program provide the opportunity to the user to interact with it. In that case the program will restart from the previous converged increment with revised control parameter. Subroutine **restar1** read all scaler information from input file required to restart the program from last converged position and subroutine **restar2** read all vector information from input file required to restart the program from last converged position. Subroutine **output** write the output file describe in section 5.3 while subroutine **dump** create a binary file with the information to enable future restart.

5.5 Subroutine elemtk

This subroutine of the program calculate the equivalent nodal forces due to stresses,constitutive and initial stress component of the tangent stifness matrix.the coefficients of the items is assembled during computation.

5.5.1 elemtk Segment 1-Element Loop

```
      subroutine elemtk(ndime,nnode,ngaus,nstrs,nelem,x,x0,
&                              eledb,lnods,matno,matyp,props,ldgof,
&                              stres,resid,kprof,stifd,stifp,react,
&                              vinc,elecd,elacd,vol0)
      implicit double precision (a-h,o-z)
      dimension lnods(nnode,*),props(8,*),ldgof(ndime,*),
&                    kprof(*),stifd(*),stifp(*),x(ndime,*),
&                    x0(ndime,*),matno(*),
&                    eledb(ndime+1,nnode+1,ngaus),
&                    elacd(ndime,*),resid(*),ctens(3,3,3,3),
&                    sigma(3,3),btens(3,3),matyp(*),react(*),
&                    stres(nstrs,ngaus,*),stret(3),princ(3,3),
&                    sprin(3),vol0(*),elecd(ndime,nnode,*),
&                    vinc(*)
      data ((btens(i,j),i=1,3),j=1,3)
&             /0.,0.,0.,0.,0.,0.,0.,0.,0./
      do ie=1,nelem
          im=matno(ie)
         mat=matyp(im)
         call gradsh(ie,ndime,nnode,ngaus,eledb,lnods,x,
&               elecd,vinc)
```

Subroutine **gradsh** first evaluate the Jacobian matrix by using the equation (3.20 b) and then the current Cartesian derivatives$\partial N_a / \partial x$ of the shape from the equation (3.20 a). Both the jacobian and the Cartesian derivative are evaluated at all gauss points of the elements.

5.5.2 elemtk segment 2-Mean Dilation Implementation

```
   if(mat.eq.5) then
      call gethata(ngaus,vol0(ie),vinc,barj)
      xkapp=props(4,im)
      press=xkapp*(barj-1.)
      xkapp=xkapp*barj
      call kvolume(ndime,nnode,ngaus,xkapp,vinc,elecd,
&                       elacd,lnods(1,ie),ldgof,kprof,stifd,
&                       stifp)
   else if (mat.eq.7) then
       call gethata(ngaus,vol0(ie),vinc,barj)
       xkapp=props(4,im)
       press=xkapp*log(barj)/barj
       xkapp=(xkapp/barj)-press
      call kvolume(ndime,nnode,ngaus,xkapp,vinc,elecd,
&                       elacd,lnods(1,ie),ldgof,kprof,stifd,
&                       stifp)
   endif
```

For nearly incompressible material 5 and 7 (see note 3) the mean dilation technique is used to calculate the volume ratio between current and initial volume using in subroutine **getheta**. Subroutine **kvolume** evaluate the dilation component of the stiffness matrix assemble it to tangent stiffness matrix by using the equation (4.25)

5.5.3 elemtk segment-3 Evaluation of Deformation Gradient Tensor and Left Cauchy Green Tensor

```
do ig=1,ngaus
   call leftcg(ndime,nnode,lnods(1,ie),x0,
&              elecd(1,1,ig),detf,btens)
```

Subroutine leftcg first compute the inverse deformation gradient as

$$F^{-1} = \partial X/\partial x = \sum_{a} X_a \otimes \nabla N_a \tag{5.2}$$

Then obtain the $\boldsymbol{F}$, and finally the right Cauchy green deformation tensor by using the equation (2.21).(we can also calculate the green lagrangean strain and eulerian almansi strain by using the equation (2.26) and (2.27) as we compute the F here).This subroutine also the point wise volume ratio $\boldsymbol{J}$ (or area ratio in plane strain case).

5.5.4 elemtk Segment-4 Compressible Neo-Hookean Material

```
if(mat.eq.1) then
   xlamb=props(3,im)/detf
   xmu=props(2,im)/detf
   xme=xmu-xlamb*log(detf)
   call stress1(ndime,xmu,xme,btens,sigma)
   call cisotp(ndime,xlamb,xme,ctens)
```

This constitutive equation is described in section 2.4.3. Subroutine **stress1** evaluate the Cauchy stress components in terms of b by using the equation (2.71).whereas subroutine **cisotp** evaluate the coefficient of elasticity tensor by using the equation (2.92)

5.5.5 elemtk Segment-5 Hyperelastic material in principal direction

```
      else if(mat.eq.3) then
         xlam=props(3,im)/detf
         xmu=props(2,im)/detf
         call jacobi(btens,stret,princ)
         call stress3(ndime,xmu,xlam,detf,stret,princ,sigma,
&        sprin)
         call cprinc(ndime,xlam,xmu,stret,princ,sprin,ctens)
```

This material is described in section 2.4.4.5.first the subroutine **Jacobi** evaluate the principal directions ($\boldsymbol{n_a}$) and principal stretches ($\boldsymbol{\lambda}$) and stored in princ and stret. Respectively.Using this stretches and directions the subroutine stress3 compute the Cartesian component of the Cauchy stress tensor with the help of equation (2.86) and equation (2.91). Subroutine **cprinc** compute the corresponding elasticity tensor using equation (2.88).

5.5.6 elemtk Segment 6- Plane Stress Hyperelastic Material in Principal Direction

```
   else if(mat.eq.4) then
        det3d=detf**props(5,im)
        xlam=props(3,im)/det3d
        xmu=props(2,im)/det3d
        call jacobi(btens,stret,princ)
        call stress3(ndime,xmu,xlam,detf,stret,princ,sigma,
&                      sprin)
        call cprinc(ndime,xlam,xmu,stret,princ,sprin,ctens)
        vinc(ig)=vinc(ig)*props(4,im)*det3d/detf
        stres(4,ig,ie)=props(4,im)*det3d/detf
```

This material is described in section 2.4.4.6 and is execute in the same way to previous equation.except the parameter $\boldsymbol{\gamma}$ which has been obtain in the subroutine **matprop** with the help of equation (2.97) after that by using this parameter the current thickness is update in terms of volume ratio,area ratio and initial thickness.

5.5.7 elemtk Segment 7-Nearly Incompressible Neo Hookean

```
else if(mat.eq.5) then
    xmu=props(2,im)
    call stress5(ndime,xmu,detf,btens,sigma)
    call cdevia(ndime,xmu,detf,btens,ctens)
    call cvolum(ndime,press,ctens)
```

This material is described in section(2.4.5.3) and (2.4.5.3). The component of Cauchy stress are evaluated in subroutine stress5 by using the equation(2.105). From Subroutine **cdevia** and **cvolum** obtain the deviatoric and volume component the elasticity tensor.

5.5.8 elemtk Segment 8-Plane Stress Incompressible Neo-Hookean

```
else if(mat.eq.6) then
    xmu=props(2,im)
    xme=xmu/(detf*detf)
    xla=2.*xme
    call stress6(ndime,xmu,detf,btens,sigma)
    call cisotp(ndime,xla,xme,ctens)
    vinc(ig)=vinc(ig)*props(4,im)/detf
    stres(4,ig,ie)=props(4,im)/detf
```

First the effective shear coefficient $\boldsymbol{\mu'}$ is calculated as

$$\boldsymbol{\mu'} = \frac{\boldsymbol{\mu}}{\boldsymbol{j^2}} \tag{5.3}$$

Where $\boldsymbol{j^2} = \boldsymbol{det_{2\times2} b}$.

After that subroutine stess6 evaluate the components of Cauchy stress tensor as

$$\boldsymbol{\sigma_{ij}} = \boldsymbol{\mu b_{ij}} - \frac{\boldsymbol{\mu}}{\boldsymbol{j^2}} \tag{5.4}$$

Finally the coefficients of elasticity tensor are obtain by using the equation(2.92)

5.5.9 elemtk Segment 9- Plane stress incompressible in principal direction

```
else if(mat.eq.8) then
    xmu=props(2,im)
    xlam=2.*xmu
    call jacobi(btens,stret,princ)
```

```
      call stress3(ndime,xmu,xlam,detf,stret,princ,
&                      sigma,sprin)
      call cprinc(ndime,xlam,xmu,stret,princ,sprin,
&                     ctens)
      vinc(ig)=vinc(ig)*props(4,im)/detf
      stres(4,ig,ie)=props(4,im)/detf
endif
```

This material is identical to the material explained in segment 6, but because of incompressiblity the lame coefficient lambda $\boldsymbol{\lambda} \rightarrow \infty$ due to which $\boldsymbol{J}$=1,the parameter gamma($\boldsymbol{\gamma}$)=0.

5.5.10 elemk Segment 10-Internal Forces and Tangent Stiffness

```
        call internal(ndime,nnode,lnods(1,ie),ldgof,
&                         sigma,elecd(1,1,ig),vinc(ig),
&                         resid,react)
        call kconst(ndime,nnode,lnods(1,ie),ldgof,
&                       ctens,elecd(1,1,ig),vinc(ig),kprof,
&                       stifd,stifp)
        call ksigma(ndime,nnode,lnods(1,ie),ldgof,
&                        sigma,elecd(1,1,ig),vinc(ig),kprof,
&                        stifd,stifp)
        ks=1
        do id=1,ndime
           do jd=id,ndime
              stres(ks,ig,ie)=sigma(id,jd)
              ks=ks+1
           enddo
         enddo
       enddo
enddo
return
end
```

The Components of Cauchy stress tensor and the Cartesian derivative of shape function evaluateated in **gradsh** subroutine are used in the subroutine **internal** to evaluate and assemble the equivalent nodal forces due to stresses $\boldsymbol{F}$ by using the equation(4.19)**.** This subroutine also evaluate the residual($\boldsymbol{H} = \boldsymbol{R} - \boldsymbol{F}$). Subroutine **kconst** evaluate and assemble the constitutive component of the tangent stiffness matrix by using the equations (4.22) and (4.23).The stress component of the tangent stiffness matrix is obtain and assemble by using the equation(4.24) in subroutine **ksigma**.

Chapter-6

Results and Discussion

6.1 Introduction

The solutions of large strain hyperelastic problems obtained through Flagshyp Code (written by Javier Bonet and Richard D.Wood) and abaqus are discussed.

6.2 Plate With Hole

6.2.1 Problem Statement

A plate of dimension $6.5mm \times 6.5mm \times 0.079\ mm$ with a hole in the centre of diameter $0.5\ mm$ has been chosen . It is constrained in vertical direction and is free to move in horizontal direction as shown in the fig 6.1. Rubber (Incompressible elastomer) is used with shear modulus $\mu = 0.4225\ \ N/mm^2$. The pate is stretched to six times of its original length (i.e stretched to the length of 39 mm)

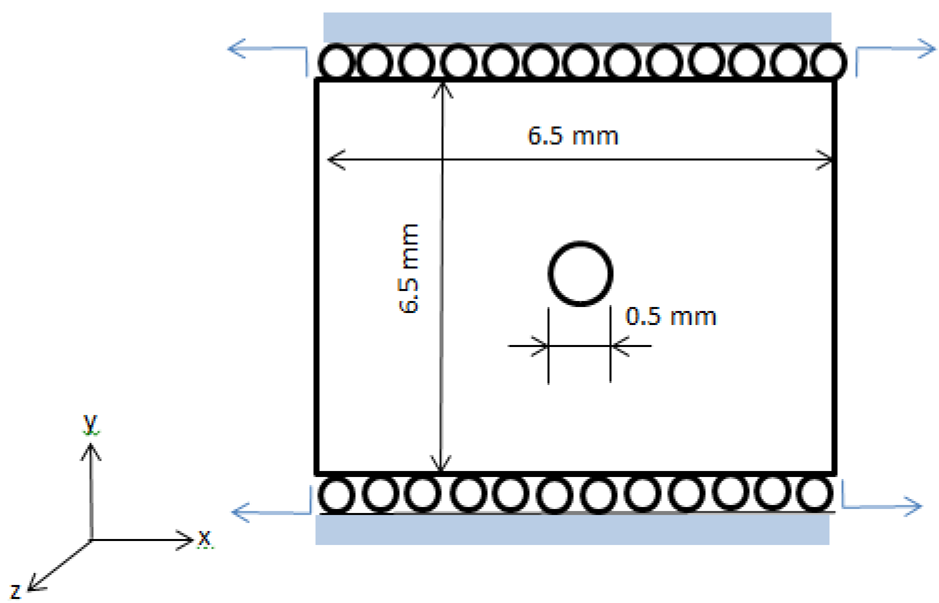

Fig 6.1 Plate with Hole

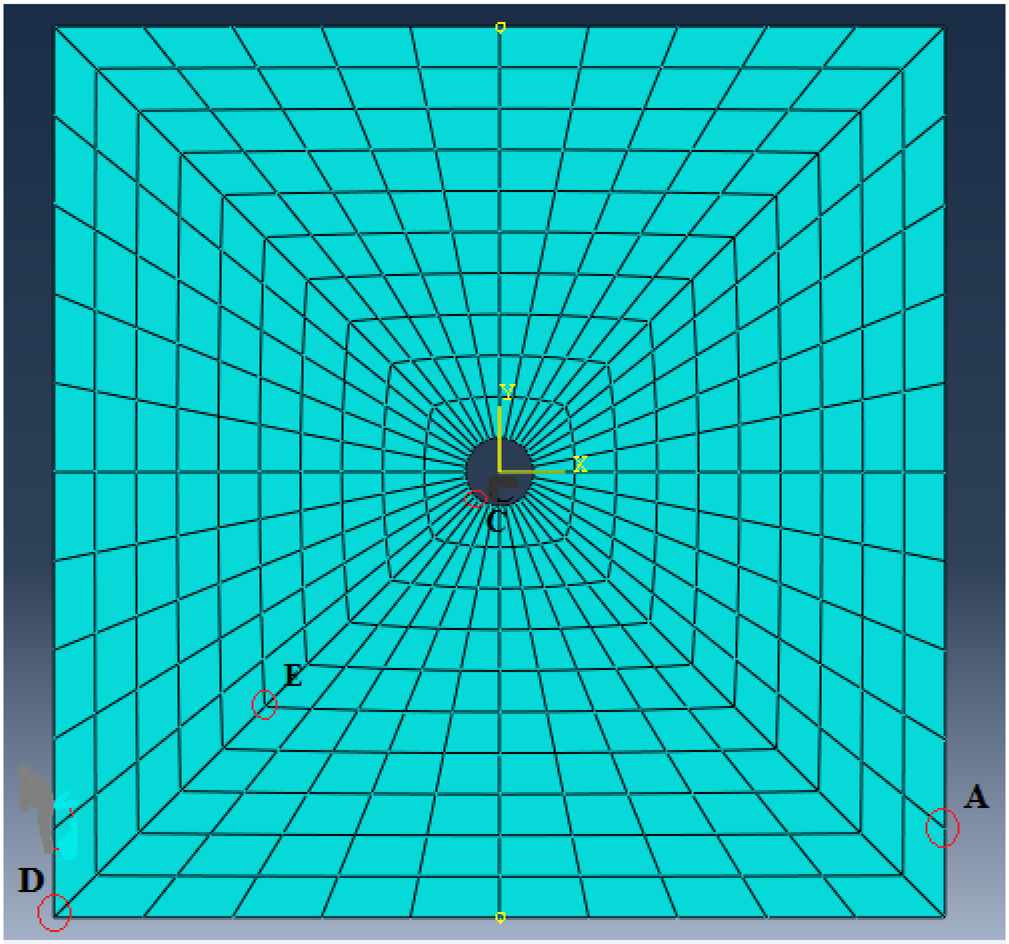

Fig 6.2 Meshed Geometry with nodes where analysis is performed

6.2.2 Solution

A plane stress incompressible neo-hookean model(discussed in chapter-3) is used to solve the problem. Plate is divided into 400 four-noded quadrilateral elements as shown in fig 6.2. The problem is solved using 5 increments viz 3.25 mm increment on both side of length in every increment step as shown in fig 6.1. The convergence criterion is set to 1.e-10 in the code with a maximum 25 iterations in every step.

The Resulting horizontal force vs displacement graphs of node A and B (shown in fig 6.2) obtained from Flagshyp code and abaqus are shown in fig 6.3 and fig6.4 respectively.

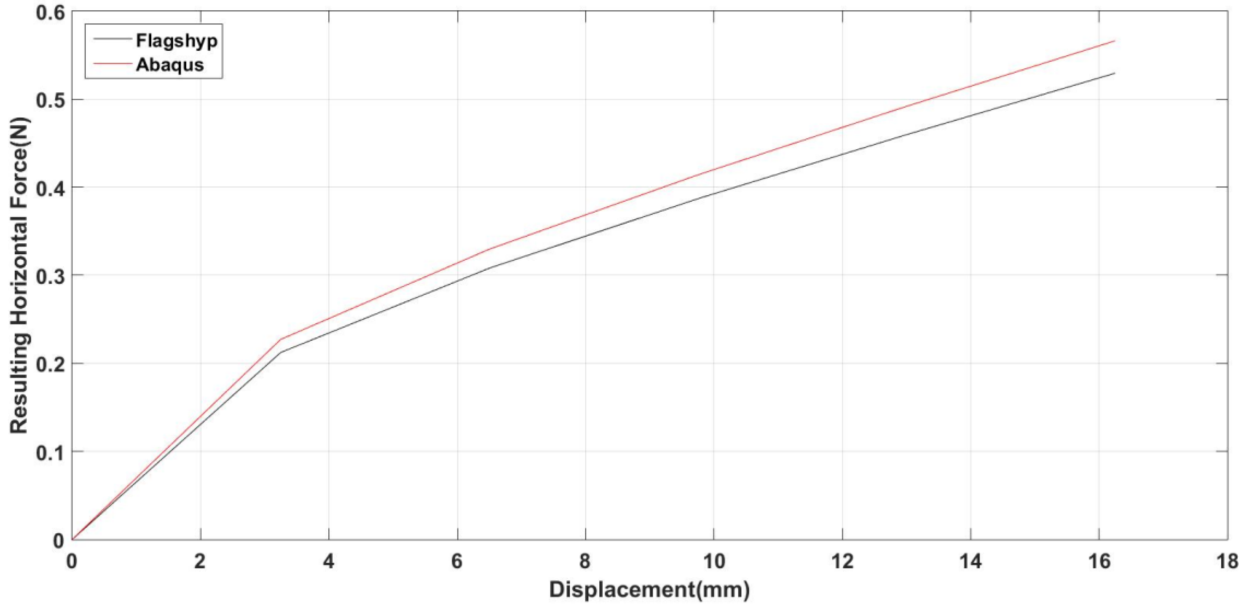

Fig 6.3 Displacement v/s Resulting Horizontal Force of node A

From figure 6.3, it can be observed that the material behaviour is highly non-linear elastic and there is an error of 5% between abaqus and Flagshyp results

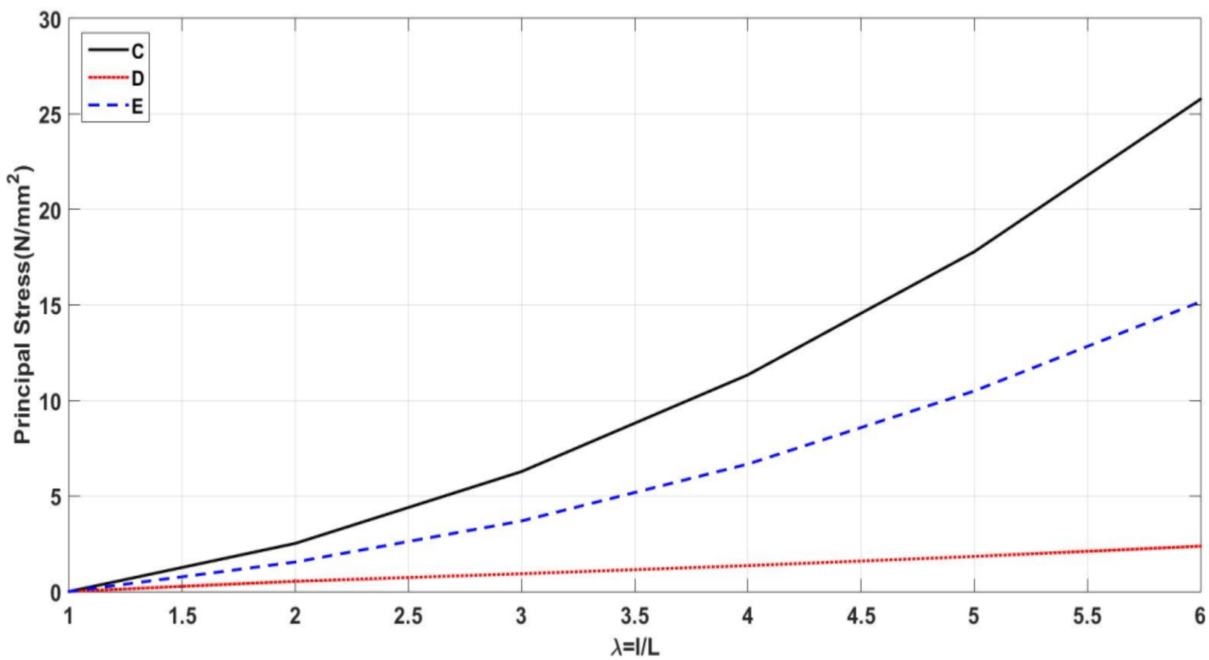

Fig 6.4 Maximum Principal stress v/s Stretch ratio of node C,D and E

In Fig 6.4, l is the length of the plate in every increment; L is the initial length of the plate; C, D and E are the nodes of the element as shown in fig 6.2. It can be observed from fig 6.4 that the behaviour of material is non-linear and maximum stress occur at A because of the stress concentration near the hole whereas minimum stress occur at B because forces are evenly distributed here.

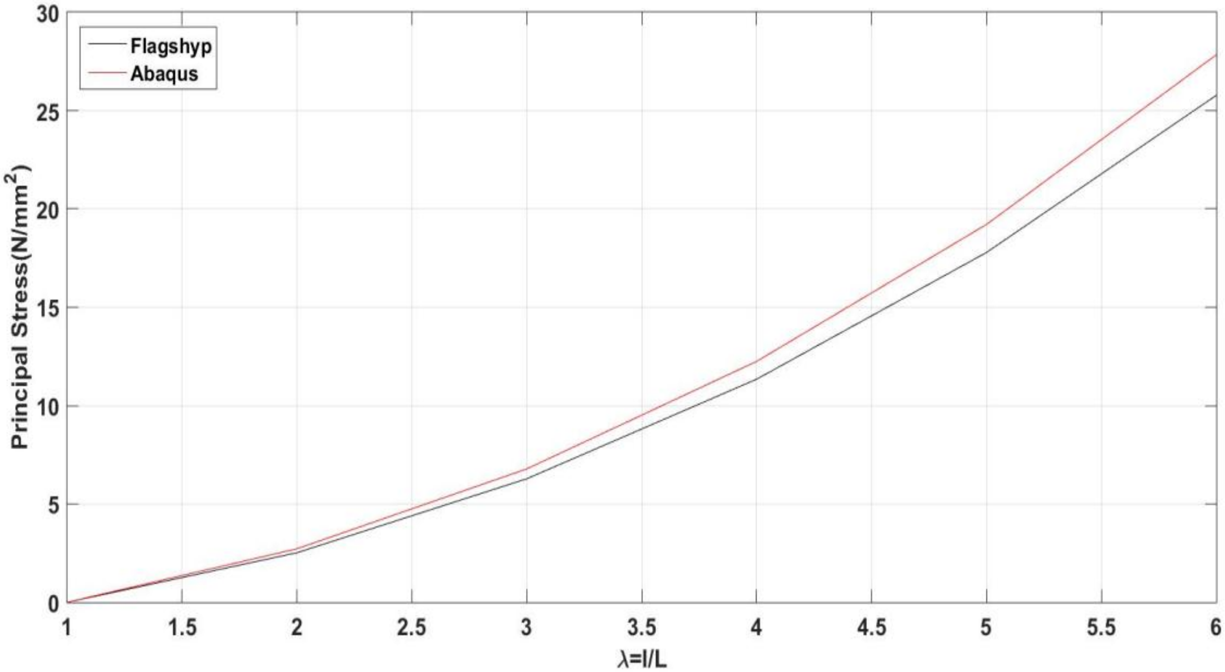

Fig 6.5 Maximum Principal stress v/s Stretch ratio at node C obtained from Flagshyp and abaqus

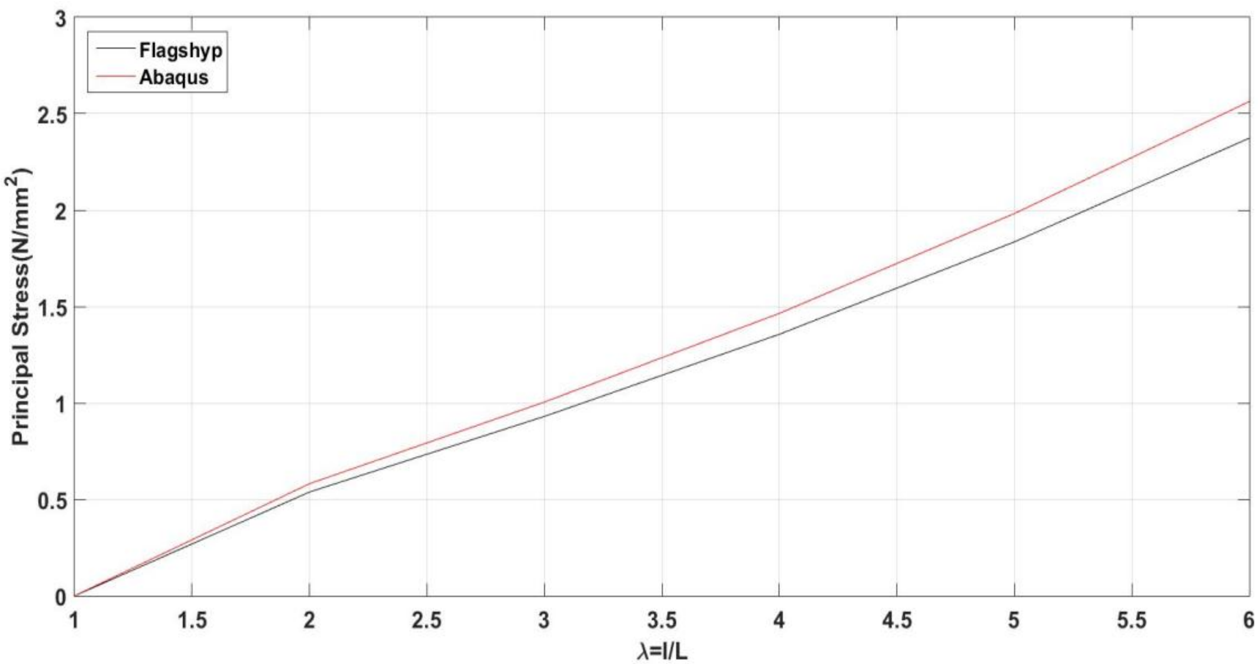

Fig 6.6 Maximum Principal stress v/s Stretch ratio at node D obtained from Flagshyp and abaqus

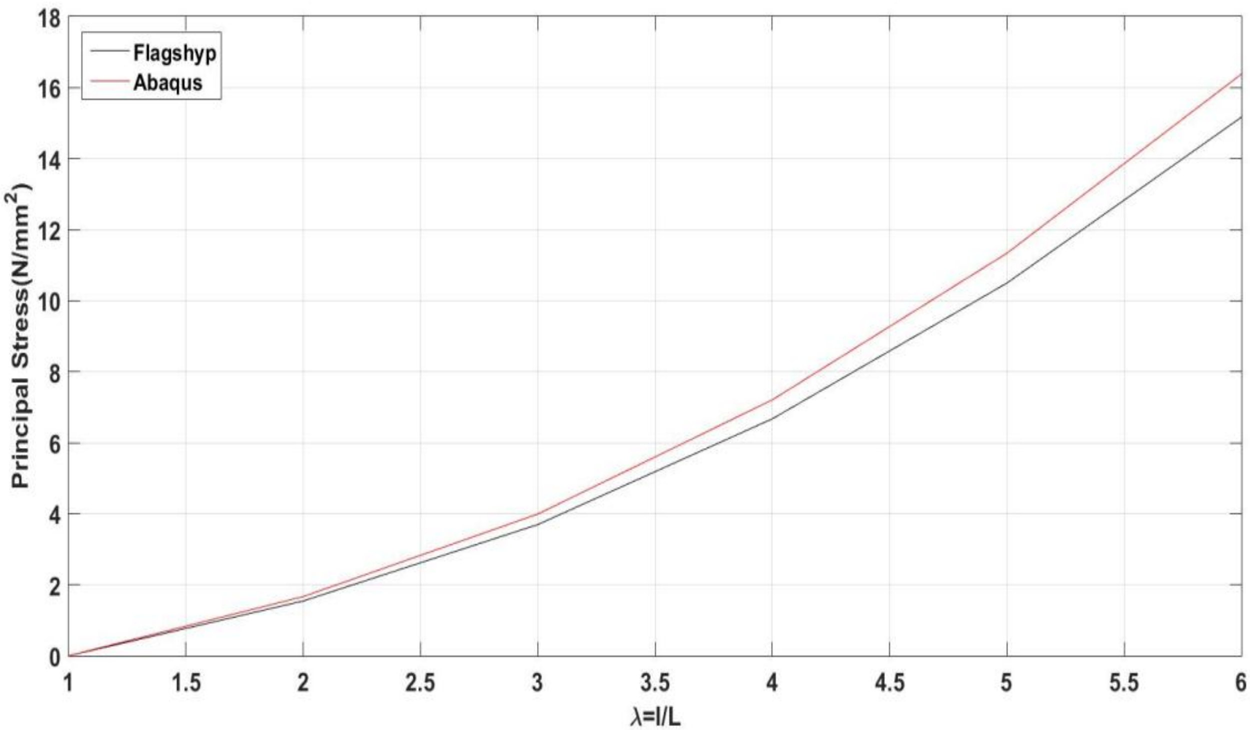

Fig 6.7 Maximum Principal stress v/s Stretch ratio at node E obtained from Flagshyp and abaqus.

From (6.5-6.7), It can be observed that there is an error of 7% between Flagshyp and abaqus results.

6.3 Truss

6.3.1 Problem Statement

A four noded truss of dimension $140mm \times 1.414mm \times 0.707\ mm$ has been chosen. The initial orientation angle is 45 degree. There is a hinge support on node 1 and roller support on node 2 which constrained the motion in x direction and allow to move in y direction as shown in fig 6.4. Rubber (Incompressible elastomer) is used with shear modulus $\mu = 0.4225\ N/mm^2$ and $v = 0.5$. The displacement of the top node is prescribed downward in increment of $1mm$ (see fig 6.9)

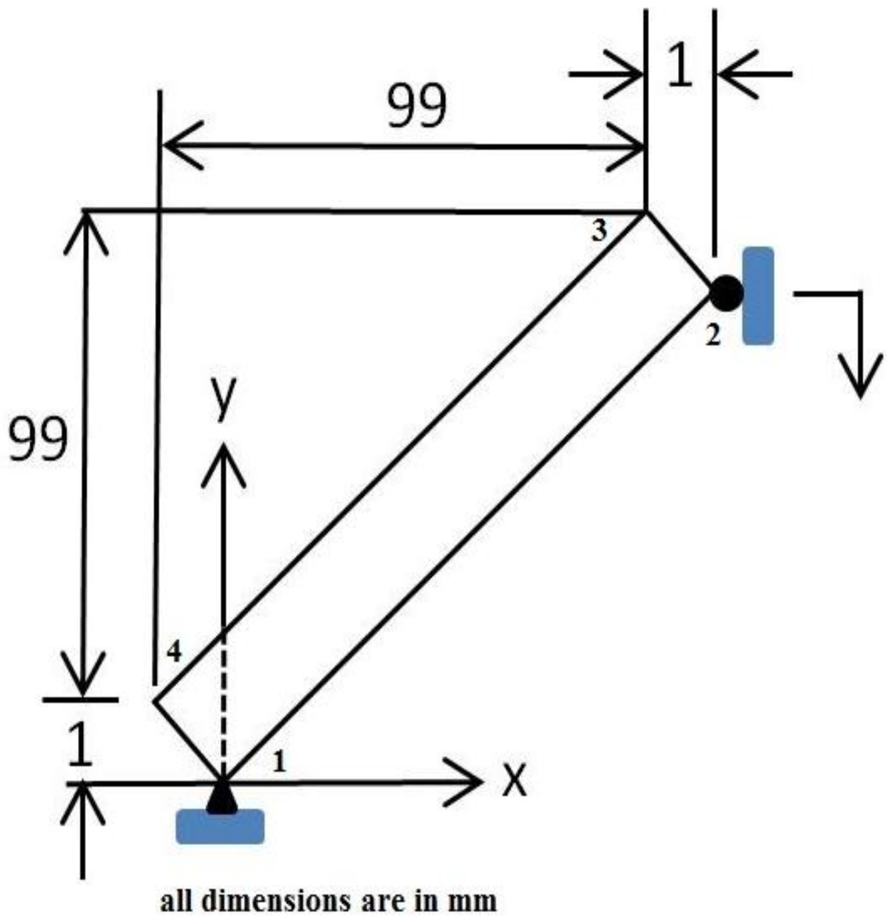

Fig 6.8 Truss with prescribed displacement in downward direction

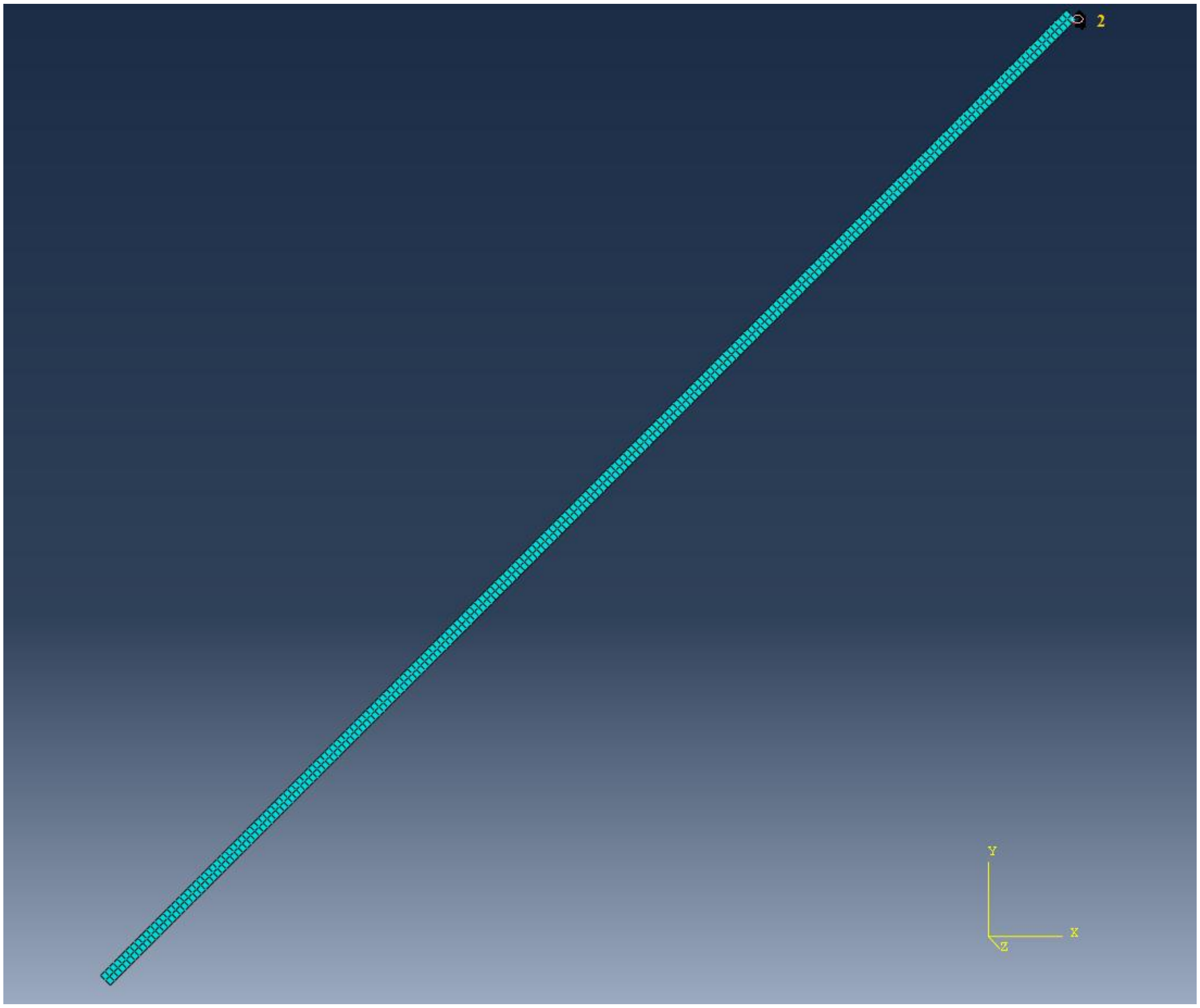

Fig 6.9 Meshed Geometry with node where analalysis is performed

6.3.2 Solution

A plane stress case, Ogden Model(Incompressible hyperelsticity in principal directions) is used to solve the problem using 198 increments (1 mm increment in every increment step). A truss is divided into 700 element as shown in fig 6.9. The convergence criterion is set to 1.e-10 in the code with a maximum 25 iterations in every step.

The vertical Reaction vs displacement graph of node 2 obtained from Flagshyp code is shown in fig 6.10.

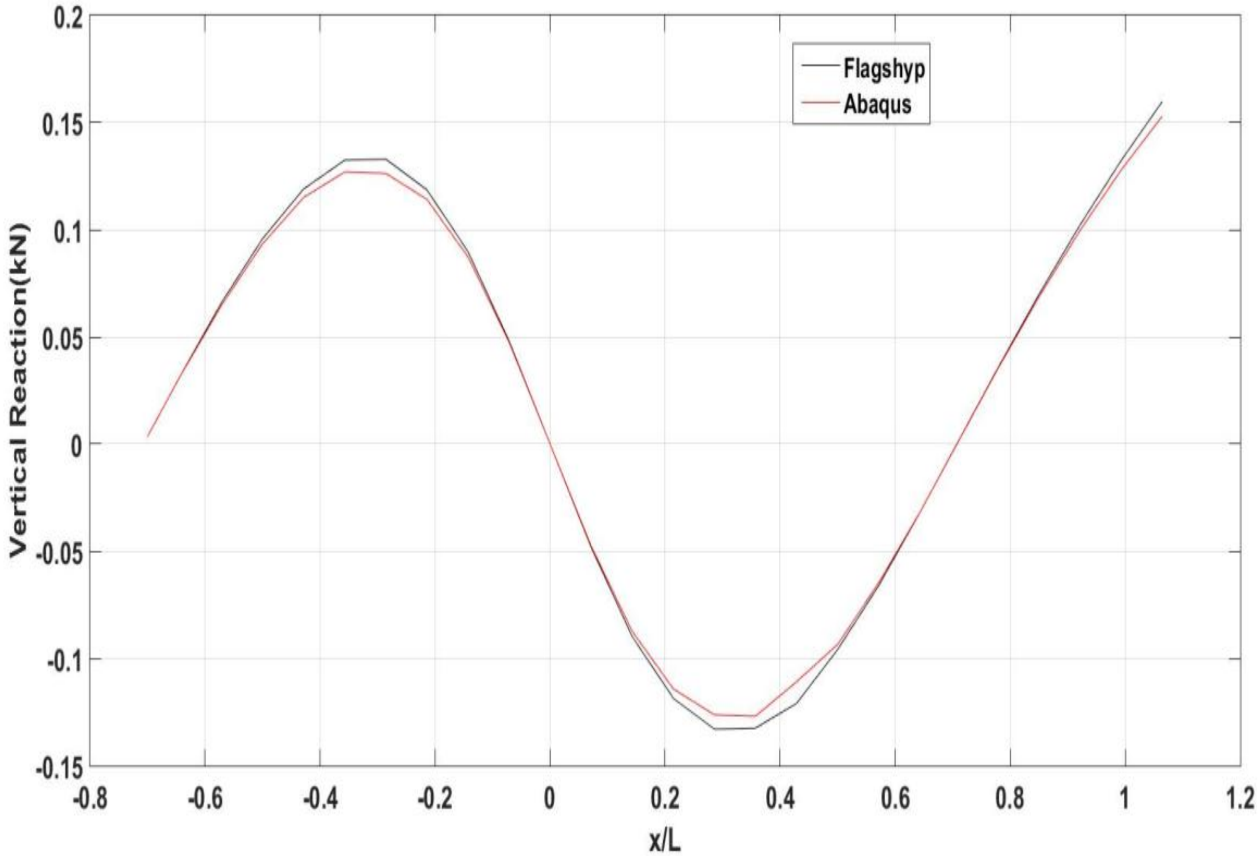

Fig 6.10 Vertical Reaction v/s Displacement of node 2

In fig 6.10, $\boldsymbol{x}$ is the vertical position of node 2 in every increment step and $\boldsymbol{L}$ is the length of the truss. It can be observed that the truss exhibit non-linear snap through behaviour (also see fig 6.11). In this non-linear instability region the equilibrium path goes from one stable point (at which force is $0.1322KN$) to another stable point(at which force is $0.1329\ KN$).

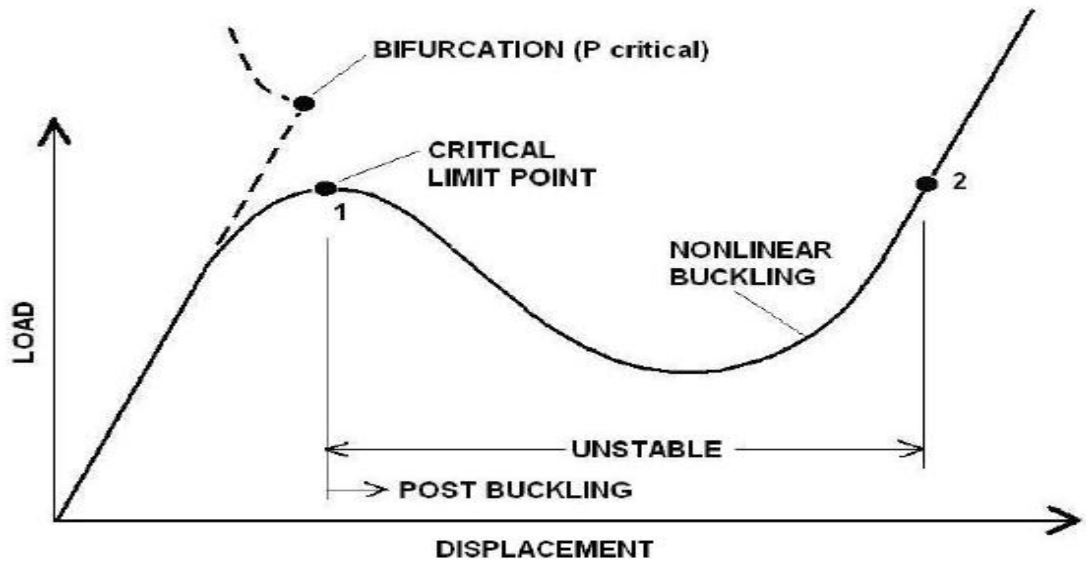

Fig 6.11 Non-Linear Snap through behaviour of Truss[13]

6.4 Pressurised Cylinder

6.4.1 Problem Statement

A quarter part of the pressurised cylinder of radius 15mm and outer radius 25mm has been choosen. The boundary conditions are shown in fig 6.12. Rubber (Incompressible elastomer) is used with shear modulus $\mu = 0.62\ N/mm^2$. Internal Pressure of 0.8 MPa is acting on the cylinder.

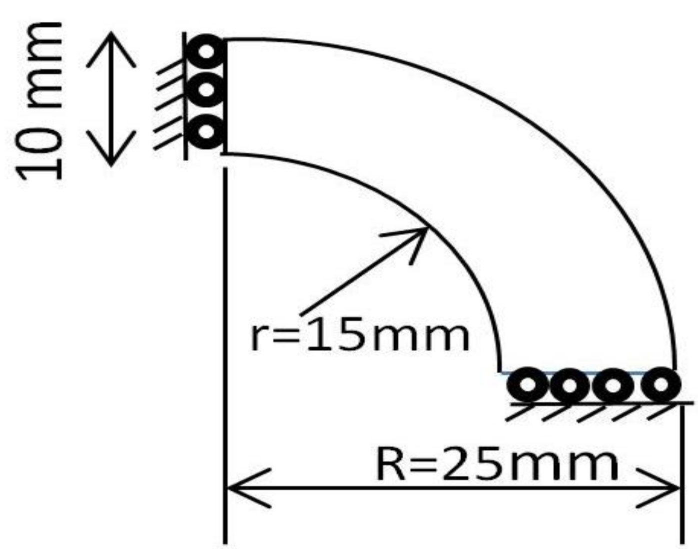

Fig 6.12 Quarter part of Pressurised cylinder

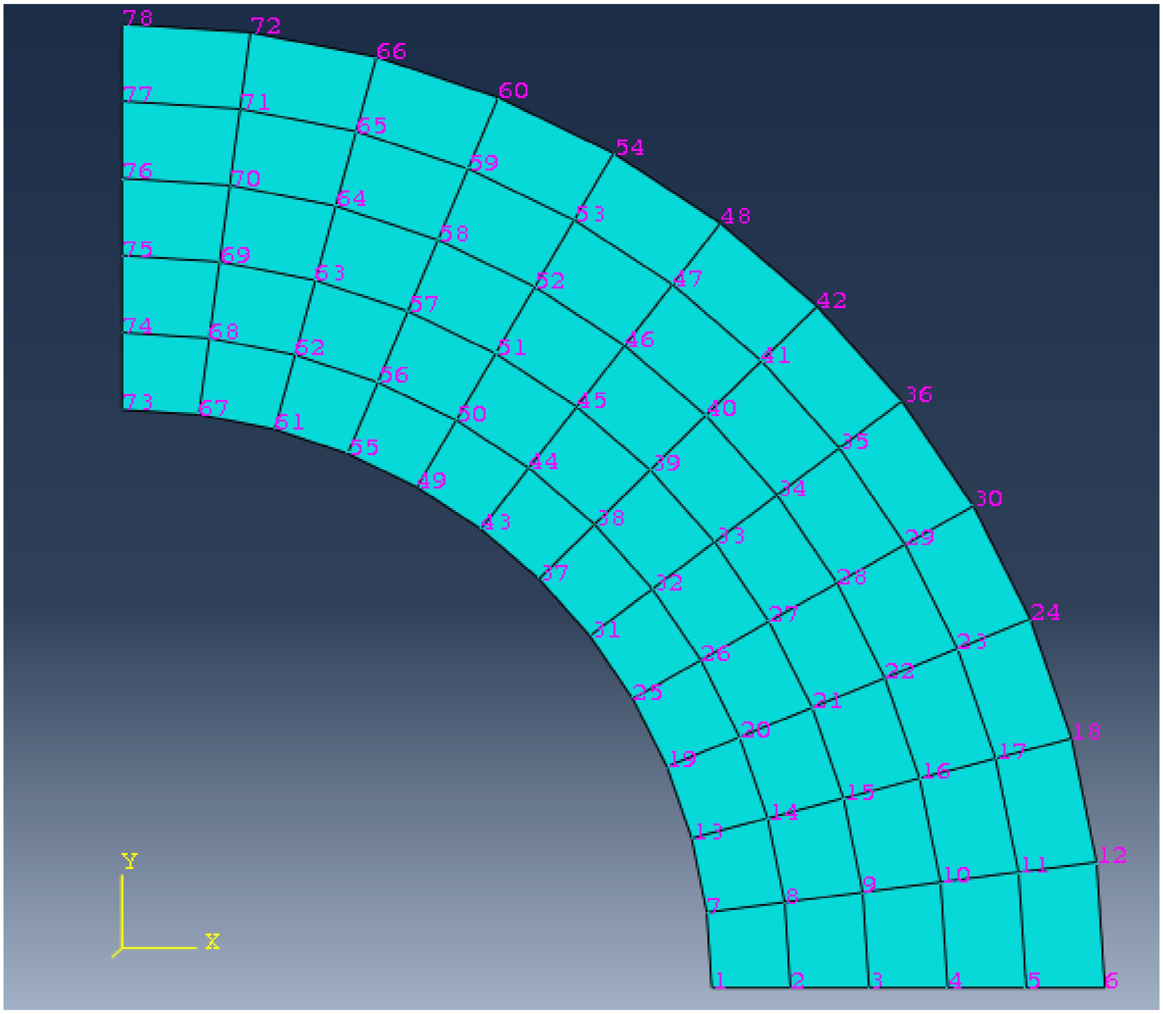

Fig 6.13 Meshed Geometry

6.4.2 Solution

A plane stress case, Mooney-Rivlin model is used to solve the problem. Cylinder is divided into 60 four-noded quadrilateral elements as shown in fig 6.13. The problem is solved using 4 increment (0.2 MPa is acting in 1st increment with an increment of 0.2 MPa in every increment step). The convergence criterion is set to 1.e-10 in the code with a maximum 25 iterations in every step.

All the results discussed below are calculated on the node 1,2,3,4,5 and 6

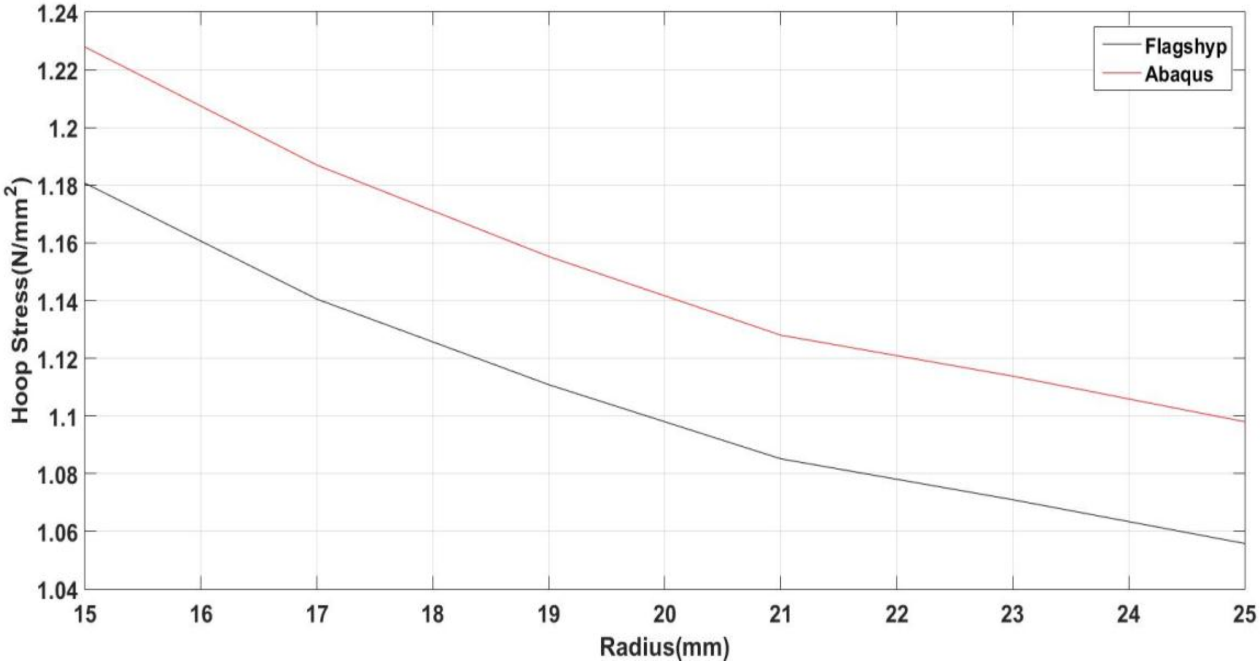

Fig 6.14 Hoop Stress v/s Radius at 0.8 MPa (Internal Pressure)

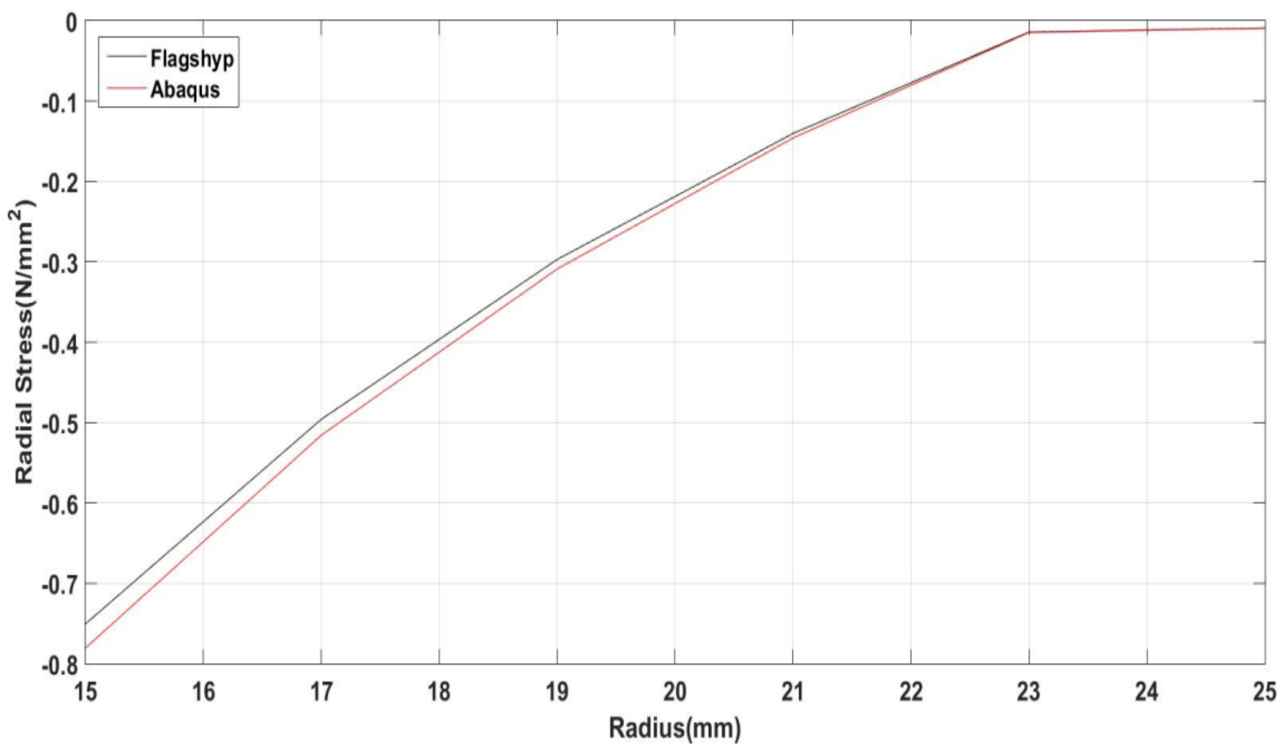

Fig 6.15 Radial stress v/s Radius at 0.8 MPa (Internal Pressure)

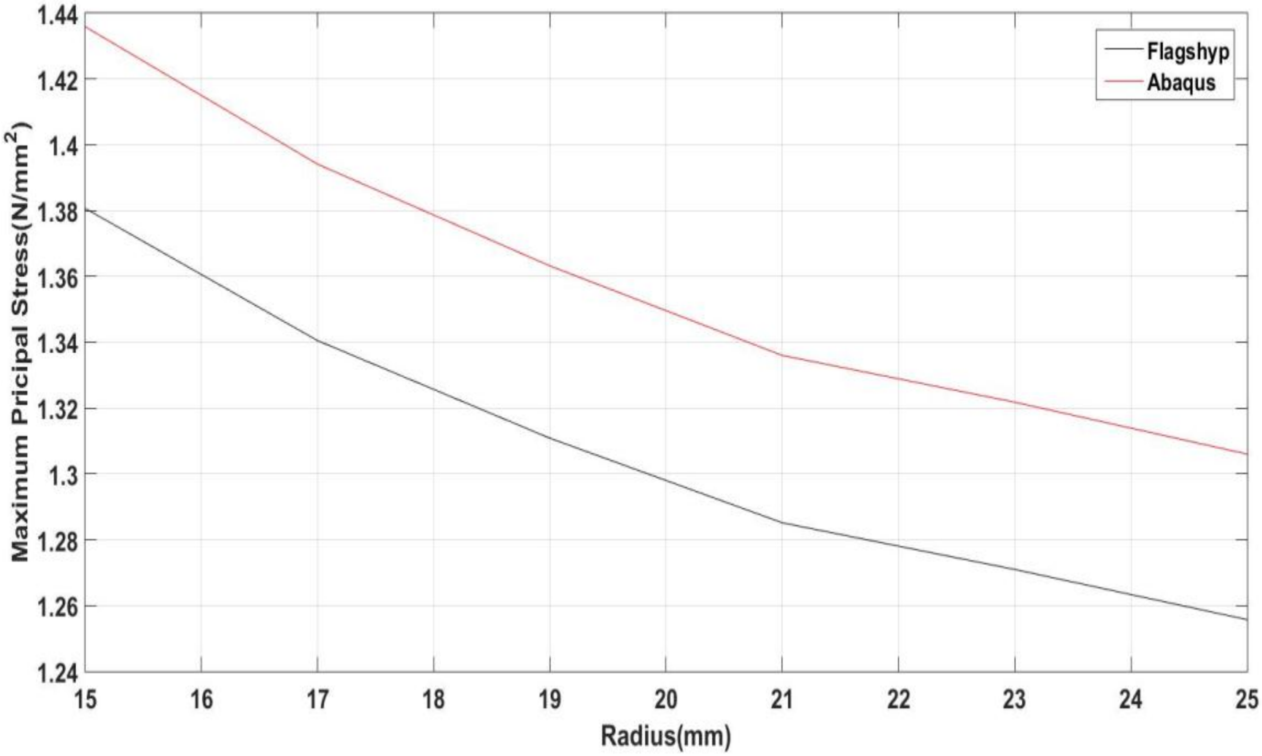

Fig 6.16 Maximum principal stress v/s Radius at 0.8 MPa (Internal Pressure)

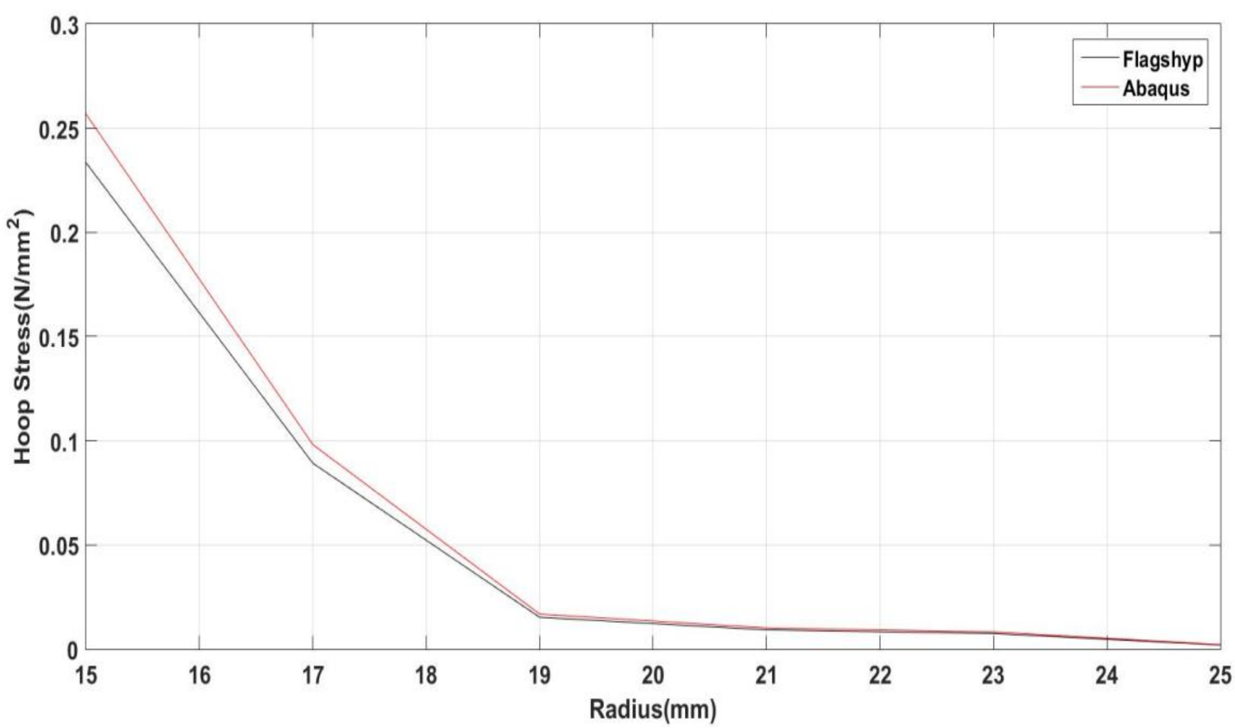

Fig 6.17 Hoop Stress v/s Radius at 0.2 MPa (Internal Pressure)

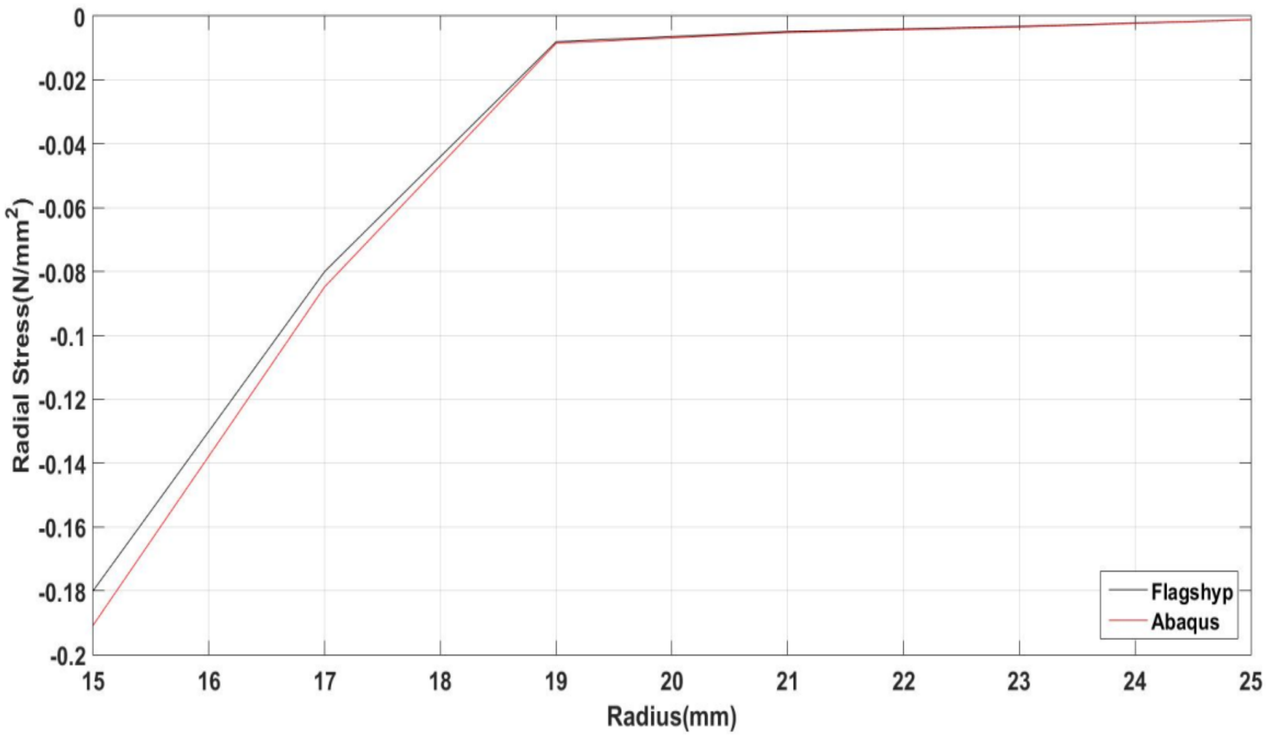

Fig 6.18 Radial Stress v/s Radius at 0.2 MPa (Internal Pressure)

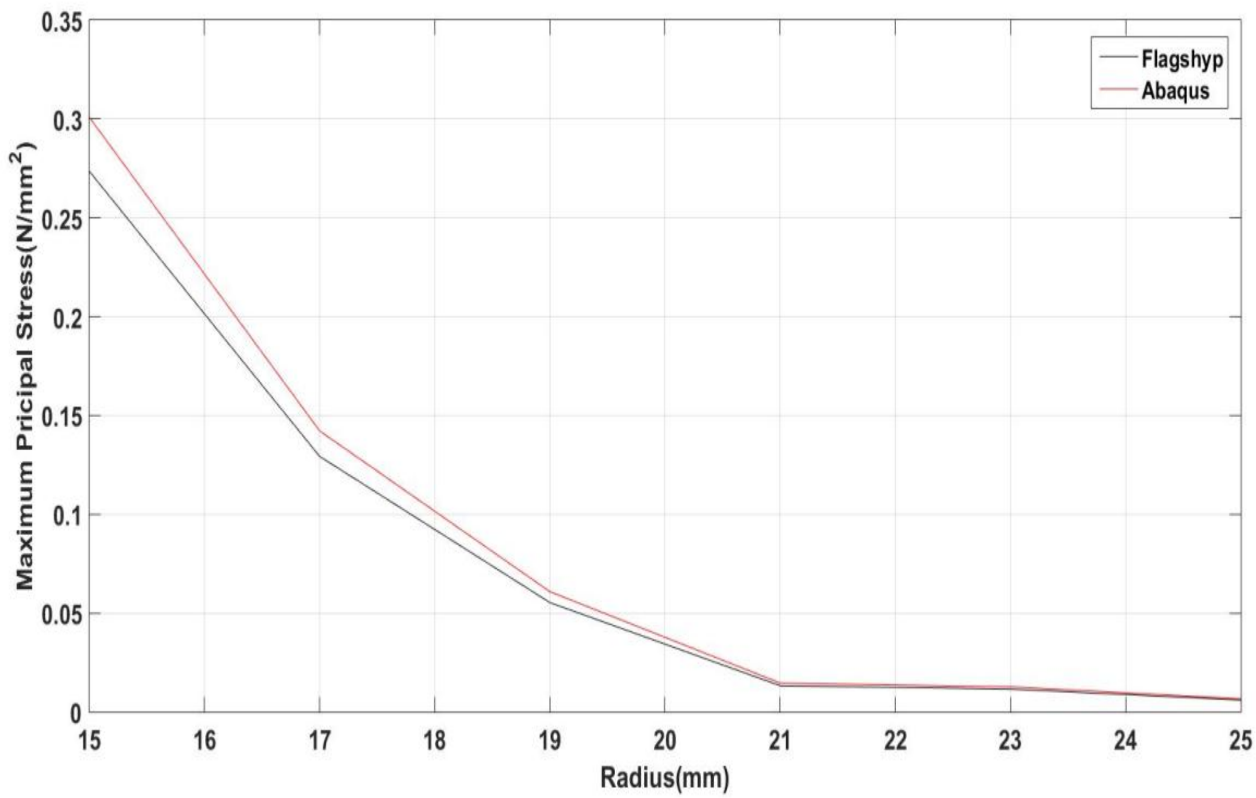

Fig 6.19 Maximum Principal Stress v/s Radius at 0.2 MPa (Internal Pressure)

It can be observed from the Figures(6.14 to 6.19) that the stresses are maximum in the inner radius where cylinder subjected to internal pressure.It decreases through the thickness and minimum in the outer radius.

Chapter 7

Conclusion and Future Scope

7.1 Conclusions

In the work, the behaviour of hyperelastic material (incompressible elastomer) has been studied for large strains through the Finite element large strain hyperelasticity program (Flagshyp) written by Javier Bonet and Richard D.Wood. The results have been scrutinized to describe the behaviour of the material in different boundary and load conditions. The following conclusions are illustrated in this work.

1. The three hyperelastic constitutive model odgen, neo-hookean and mooney-rivlin are used for the analysis of truss problem, plate with hole problem and pressurized cylinder problem.

2. Only material non-linearity is considered as the program is structured using Updated lagrangean formulation.

3. The results depict that the material is highly non linear elastic.

4. The non-linear snap through behaviour is obtained in truss problem

5. In plate with hole problem it is that intensity of stresses are high near the hole because of the stress concentration in this region whereas the stresses are minimum on the edge of the plate because forces are evenly distributed here.

7.2 Future Scope

1. This work can be extended to viscoelastic materials and elasto-plastic materials.

2. The software on total lagrangean method can be developed and results compared with the updated lagrangean method.

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