A STUDY ON ACCELERATING COSMOLOGICAL MODELS OF THE UNIVERSE

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In

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By

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DECLARATION

I declare that the research work reported in this thesis entitled "**A Study on Accelerating Cosmological Models of the Universe**" for the award of the degree of *Doctor of Philosophy in Mathematics* has been carried out by me under the supervision of *Dr. Chandra Prakash Singh*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The research work embodied in this thesis, except where otherwise indicated, is my original research. This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. This thesis does not contain other person's data, graphs or other information, unless specifically acknowledged.

Date : **(Pankaj Kumar)**

CERTIFICATE

On the basis of declaration submitted by **Mr. Pankaj Kumar**, student of Ph.D., I hereby certify that the thesis titled **"A Study on Accelerating Cosmological Models of the Universe"** submitted to the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of the degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by him under my supervision.

I have read this thesis and that, in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any Degree or Diploma.

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Date : **(PANKAJ KUMAR)**

Place : Delhi, India.

Dedicated to My Parents

Sh. Meghpal Singh

&

Smt. Simla Devi

Contents

Preface

Einstein's general theory of relativity has changed our perception about the laws of the Universe remarkably and has developed a new branch of the science called Cosmology. General theory of relativity is one of the most beautiful and successful theory of modern science. The modern cosmology has entered into the era of observational precision which is changing the frontiers of our knowledge about the Universe very rapidly. The recent discovery of gravitational waves by LIGO probe is an example of it. At present, we have data coming from the probes like Tyep Ia supernova, Wilkinson Microwave Anisotropy Probe, Baryon Acoustic Oscillations, Sloan Digital Sky Survey, Planck collaboration, etc.

It was 1998 when the observations of Type Ia supernova changed our knowledge about the Universe dramatically. These observations suggested that the expansion of the Universe is accelerating whereas it was supposed to be decelerating according to general theory of relativity. In the recent years, the observations coming from different probes have confirmed the accelerated expansion of the Universe. It has been predicted that a mysterious type of energy content, which is uniformly distributed in the Universe and has negative pressure, is responsible for the accelerated expansion. This mysterious energy content is known as "Dark Energy" which contributes about 68.3% of the total energy budget of the Universe. Now, the great challenge in front of fundamental theories of cosmology is to understand the origin and nature of this so–called dark energy.

Since the discovery of accelerated expansion of the Universe, several theories have been proposed to explain the accelerated phenomena. The most natural and successful candidate of dark energy is the cosmological constant which was introduced by Einstein to obtain a static Universe. Many other dark energy candidates like scalar fields, chaplygin gas, holographic dark energy, new agegraphic dark energy, etc. have been proposed to explain the accelerated expansion of the Universe.

In the past two decades, a number of modified theories of gravity such as Gauss–

Bonnet $f(G)$ theory, $f(R)$ theories, $f(R,G)$ theory, $f(T)$ theory, $f(R,T)$ gravity etc. have also been proposed to explain the current epoch of cosmic acceleration. On the other hand, alternative theories like Brans–Dicke theory, Kaluza–Klein theory, loop quantum gravity, etc. have also been considered to explain the acceleration of the Universe. The study of dark energy models are of great interest in such modified gravity theories.

The thesis entitled **"A Study on Accelerating Cosmological Models of the Universe"** comprises seven chapters. The bibliography and the list of publications have been given at the end of the thesis.

Chapter 1 titled *"introduction"* provides a brief review of general theory of relativity, expanding Universe, inflationary Universe and accelerating Universe. It provides a short review on the candidates available in the literature to explain the accelerated expansion of the Universe. A brief survey of modified $f(R,T)$ gravity and Brans–Dicke theory have been carried out. The cosmological parameters, which play an important role in the study of cosmological models, have been discussed briefly. Thus, the present chapter creates a background, gives the motive of thesis work and provides the tools to achieve the goals.

Chapter 2 titled *"Viscous Cosmology in $f(R,T)$ Gravity"* deals with the modified $f(R,T)$ gravity in the framework of flat Friedmann-Robertson-Walker metric. The effect of bulk viscosity has been studied in modified $f(R,T)$ gravity theory. Viscous and non–viscous cases have been discussed. It has been shown that the accelerated expansion is possible in non–viscous fluid model. The phase transition is not possible in this model as a constant value of deceleration parameter has been obtained. However, in case of bulk viscous model, the phase transition has been observed. A detailed study of all possible evolution of the Universe has been discussed for all range of the coupling parameter between geometry and matter. The big–rip singularity has also been observed. The chapter is based on a published research paper "Friedmann Model with Viscous Cosmology in Modified $f(R,T)$ Gravity Theory, *The European Physical Journal C* **74**, 3070 (2014)".

In **Chapter 3** titled *"Viscous Cosmology with Matter Creation in $f(R,T)$ Gravity"*, we extend the work of previous chapter. In addition, the thermodynamics of particle creation has been included to study the evolution of the Universe. The exact solutions have been obtained for perfect and imperfect fluids with matter creation within the framework of flat Friedmann-Robertson-Walker model. The effect of particle creation with bulk viscosity has been studied extensively. It has been shown that both the processes have significant different effects on the evolution in the framework of $f(R,T)$ gravity, however, in the literature both have been treated as same physical phenomena. We have noticed that the big–rip singularity may be avoided for suitable values of the parameters. The early time inflation may be achieved for some range of parameters. The content of this chapter is based on a research paper entitled "Viscous Cosmology with Matter Creation in Modified $f(R,T)$ Gravity, *Astrophysics and Space Science* **357**, 120 (2015)".

In **chapter 4** titled *"Holographic Dark Energy in $f(R,T)$ Gravity"*, holographic dark energy model has been considered in $f(R,T)$ gravity to check the possibility of the accelerated expansion without taking interaction between holographic dark energy and dark matter. The behaviors of holographic dark energy as a perfect fluid and as a bulk viscous fluid have been discussed separately. It has been observed that the accelerated expansion is possible for both the models but the phase transition is possible for bulk viscous holographic dark energy model only. The statefinder diagnosis has been studied to discriminate HDE model with existing dark energy models. For bulk viscous holographic dark energy, it has been found that the model may behave like quintessence scalar field for a suitable range of parameters. The work presented in this chapter comprises the result of a published paper "Statefinder Diagnosis for Holographic Dark Energy Models in Modified $f(R,T)$ Gravity, *Astrophysics and Space Science* **361**, 157 (2016)".

In **chapter 5** titled *"Holographic Dark Energy in Brans-Dicke theory"*, an interacting holographic dark energy has been studied in the Brans–Dicke theory in the framework of flat Friedmann-Robertson-Walker Universe. We have proposed a logarithmic form of Brans–Dicke scalar field to discuss the evolution of the Universe. The equation of state parameter, deceleration parameter and energy density ratio have been obtained to describe the evolution of the Universe. We have presented a unified description of the evolution of the Universe. The long standing problem of cosmic coincidence has been resolved in this model. The chapter is based on a research paper "Holographic Dark Energy in Brans–Dicke Theory with Logarithmic Scalar Field, *Pre-print, arXiv: gr-qc/1609.01477, (2016)"*.

Chapter 6 titled *"New Agegraphic Dark Energy in Brans-Dicke thery"*, explores the consequences of the logarithmic form of Brans–Dicke scalar field in new agegraphic dark energy in the framework of non–flat Friedmann-Robertson-Walker Universe. We have studied non–interacting and interacting models separately. The equation of s-tate parameter, deceleration parameter and energy density ratio have been obtained to discuss the evolution of the Universe. In non–interacting model, the accelerated expansion may be observed. It has been shown that the equation of state parameter mimics cosmological constant in the late time evolution. The phantom crossing of the equation of state parameter has been observed in interacting new agegraphic dark energy model. The cosmic coincidence problem may be avoided for suitable choice of the parameters of the model. This chapter comprises the result of a research paper "New Agegraphic Dark Energy in Brans-Dicke Theory with Logarithmic Scalar Field, *Astrophysics and Space Science*, DOI 10.1007/s10509-017-3032-0 (2017); [arXiv: gr-qc/1609.02751]".

Chapter 7 titled *"Conclusion and Future Scope"* presents the conclusion of the thesis work and future research plan. During this research work, we have found that the logarithmic form of Brans–Dicke scalar field may play an important roll in the study of the Universe in the framework of Brans–Dicke theory. Some important results have already been obtained in our present work. We feel that there is a good scope of this form in the study of various phenomena of the Universe. In future, we would like to extend the thesis work in Brans–Dicke theory by taking logarithmic form of Brans–Dicke scalar field. The cosmological perturbation is also an interesting area where a lot of work could be done.

Finally, the bibliography and list of author's publications have been given at the end of the thesis.

List of Figures

List of Tables

Chapter 1

Introduction

This chapter presents a brief review of the past and present developments in the field of Cosmology. It deals with four most important discoveries of the modern cosmology: general relativity, expanding Universe, Inflationary Universe and accelerating Universe. It presents a review on the available candidates in the literature to explain the accelerated expansion of the Universe. This chapter also contains a short review on modified theories of gravitation, and discusses $f(R,T)$ gravity and Brans–Dicke theory in details. The important cosmological parameters are discussed in brief. Thus, the present chapter creates a background, gives the motive of thesis work and provides the tools to achieve the goals.

Modern cosmology began after the discovery of general theory of relativity, developed by Albert Einstein in 1916 [1]. Nowadays, the modern cosmology has entered into the era of observational precision which is changing the frontiers of our knowledge about the Universe very rapidly. In the history of modern cosmology, there are four important discoveries which have changed our understanding about the Universe. These are the following.

- General Theory of Relativity
- Expanding Universe
- Inflationary Universe
- Accelerating Universe and Dark Energy

These discoveries have provided deep insight about the hidden laws of the Universe. Some times, it is required to modify the existing theory to understand the nature of the Universe. The discovery of the accelerating Universe has propelled us to think more deeply to understand our Universe. Now, we discuss these discoveries in brief.

1.1 General Theory of Relativity

Before the 20^{th} century, Newton's gravitational theory was dominated for nearly 200 years and was used to study the physical nature of the Universe. However, this theory was not able to explain the strange perihelion of mercury and bending of light near the Sun. In 1905, Albert Einstein developed special theory of relativity which only accounts for inertial systems, in the region of free space, where gravitational effects can be neglected. Soon after completing his formulation of special relativity, Einstein started working on a relativistic theory of gravitation. In 1916, he proposed a theory for non–inertial systems, known as *general theory of relativity* (GTR) which explains the phenomenon of gravitation. GTR is a classical field theory of gravitation because it does not take into account quantum effects. Besides being a theory of gravitation, GTR also provides us a new understanding of space and time. The most important thing about Einstein's GTR is that it combines space and time (called space time) where gravity is an effect of the geometry of the space time, not a force as it was supposed in Newtonian theory. GTR is formulated in a geometric framework of curved space time. It provides a complete classical description of space time structure and gravitation. GTR successfully passes the solar system tests, and explains the perihelion of mercury and bending of light near the Sun. GTR is based on two fundamental principles:-

- **The Principle of General Covariance:-** The laws of Physics retain their same form in all coordinate systems, i.e., the laws of physics remain covariant independent of the frame of reference. According to this principle, the equations governing the laws of the physics must be expressed in the tensorial form.

- **The Principle of Equivalence:-** In a local experiment, the gravitational field can be obtained by a suitable accelerating frame of reference. In other words, an inertial observer is locally equivalent to a free–falling observer in a gravitational field. Any local experiment can't distinguish these two situations. In a small enough region of space time, the laws of physics reduce to the laws of special relativity. The idea of equality of inertial mass and gravitational mass was also a main factor to lead Einstein toward the Equivalence Principle.

The study of equivalence–principle physics led Einstein to propose the identification of a gravitational field with curved space time. The geometry of space time is defined by the metric which allows the computation of path length and proper time. The motion of test particles can be determined by the metric as it gives the shortest distance between two points. The concept of space time is a combination of three dimensional space and time coordinates. Thus, the space time is a four–dimensional geometry. The four–dimensional coordinates of space time may be considered as (x^0, x^1, x^2, x^3), where $x^0 = ct$ is a time coordinate (c denotes the speed of light) and x^1, x^2, x^3 are space coordinates (x, y, z). An interval ds, which is invariant with respect to coordinate transformations, is related to the coordinates dx^μ of the the space time manifold through metric $g_{\mu\nu}$. In GTR, the most general form of the metric in tensorial form is given by

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu. \tag{1.1.1}$$

The Greek indices μ and ν run from 0 to 3 and $g_{\mu\nu}$ denotes the metric tensor of rank two obeying the transformation law

$$\bar{g_{ij}} = \frac{\partial \bar{x^i}}{\partial x^\mu}\frac{\partial \bar{x^j}}{\partial x^\nu} g_{\mu\nu} \tag{1.1.2}$$

where the quantities carrying bar correspond to the new coordinate system. The components of $g_{\mu\nu}$ are functions of the coordinates x^μ. In an orthogonal coordinate system the metric is diagonal because the coordinate lines are orthogonal to each other. In this thesis, we use only orthogonal coordinate systems. The line–element (1.1.1) represents the curved geometry of space time. We define the contravariant metric tensor $g^{\mu\nu}$ as

$$g^{\mu\nu} = \frac{\text{cofactor of } g_{\mu\nu} \text{ in } g}{g}, \tag{1.1.3}$$

where g is the determinant of $g_{\mu\nu}$, i.e., $g = |g_{\mu\nu}|$. The metric tensor $g^{\mu\nu}$ is also a symmetric tensor of rank two. This tensor is reciprocal of $g_{\mu\nu}$ and is called the conjugate metric tensor. Throughout the thesis, the summation convention is used with Greek indices running from 0 to 3 and geometrized units are used. We have used a 'time–like convention' for the metric in the thesis such that when it is diagonalized, it has the signature $(+,-,-,-)$.

Although, the metric (1.1.1) is the most general to represent curved space time and some useful results have also been obtained using this but it is very difficult to study the cosmological models in this metric. Therefore, Friedmann [2] used cosmological principle to simplify the models. According to the cosmological principle, our Universe is homogeneous and isotropic on large scales (scales more than 100 Mpc). Recent observations obtained from various probes of the cosmic microwave background radiation also confirm the homogeneity and isotropy of space time.[1] The cosmological principle gives rise to a picture of the Universe as a physical system of a "cosmic fluid". The fundamental particles of this fluid are galaxies, and a fluid element has a volume that contains many galaxies. An individual galaxy may be thought of as a particle in a smooth fluid assigned with three spatial coordinates x^μ for all instants. The coordinates x^μ are, therefore said to be comoving. In comoving coordinate system, t is the proper time of each fluid element and x^μ are the spatial coordinates carried by each fluid element.

The homogeneous and isotropic metric can be written in various forms depending on the choice of four dimensional coordinates. In spherical coordinates, where $x^0 = ct$, $x^1 = r\sin\theta\,\cos\varphi$, $x^2 = r\sin\theta\,\sin\varphi$, $x^3 = r\cos\theta$, the line element (1.1.1) can be written as [3]

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2)\right]. \tag{1.1.4}$$

[1] The homogeneity of space time means the Universe is same at all points, no position is preferable. Isotropy means the geometry of the Universe is spherically symmetric about any one point. The homogeneity and isotropy are symmetries of space only and not of space time.

We consider the unit of speed of light, $c = 1$ throughout this thesis. Here, $a(t)$ denotes the cosmic scale factor of the Universe and the term k represents the curvature parameter. Lemaitre [4–6] also have done considerable work on this metric to obtain the solutions of Einstein equations. In the 1930's, Robertson [7] and Walker [8] geometrically proved that the cases $k = -1,\ 0,\ 1$ are the only possible cases of the metric (1.1.4). Therefore, the metric is jointly known as the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, which is now commonly known as Friedmann-Robertson-Walker (FR-W) line element. The three values of k defines three possible geometries of space time, namely, open geometry for $k = -1$, flat geometry for $k = 0$ and closed geometry for $k = 1$. These three possible geometries also define the possible ultimate fate of the Universe. The closed geometry represents a Universe which collapses at the end and, open and flat geometries represent an ever expanding Universe. The FRW line element for a flat geometry of space time in the cartesian coordinates is given by

$$ds^2 = dt^2 - a^2(t)\left[dx^2 + dy^2 + dz^2\right], \tag{1.1.5}$$

where x, y and z are cartesian spatial coordinates.

1.1.1 Einstein's Field Equations

GTR explains gravity as the curvature of space time. It tells how the curved space time influences the behavior of matter, and matter determines the curved geometry of space time. Therefore, the equations governing the dynamics of the Universe should relate the matter content (energy) of the Universe and the geometry of space time. The basic equations of GTR are called Einstein field equations (EFE), which is given by

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{1.1.6}$$

where $T_{\mu\nu}$ is the energy–momentum tensor, G represents Newton's gravitational constant and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

is the Einstein tensor, with $R_{\mu\nu}$ the Ricci curvature tensor and R the Ricci scalar curvature. Taking in to account the above expression of $G_{\mu\nu}$, Eq. (1.1.6) can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}. \tag{1.1.7}$$

The left hand side of EFE (1.1.6) or (1.1.7) represents the geometry of space time as the Ricci tensor and Ricci scalar are obtained from the metric tensor directly, and the expression on the right hand side represents the matter (energy) content of the Universe. These equations together with the geodesic equation

$$\frac{d^2x^{\mu}}{dt^2}+\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}=0,$$

where

$$\Gamma^{\mu}_{\alpha\beta}=\frac{1}{2}g^{\nu\mu}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}}+\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}}-\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right]$$

is the Christoffel symbol, form the mathematical formulation of GTR. It is to be noted that depending on the form of the metric which represents the geometry of the Universe, we obtain different field equations from (1.1.7). From Bianchi identities, it is found that the Einstein tensor $G_{\mu\nu}$ has zero divergence, i.e., $G^{\mu\nu}{}_{;\nu}=\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)_{;\nu}=0$. Therefore, Eq. (1.1.7) must have $T^{\mu\nu}{}_{;\nu}=0$, which is the law of conservation of energy and momentum. This conservation law is a physical requirement.

The EFE (1.1.7) can also be derived from an action principle. This problem was solved by Hilbert soon after Einstein proposed his field equations. With the (+,-,-,-,) metric signature, the Einstein–Hilbert (EH) action (also referred to as Hilbert action) with matter is given by [3]

$$S=\int d^4x\,\sqrt{-g}\left(\frac{1}{2\kappa}R+\mathscr{L}_m\right), \tag{1.1.8}$$

where $\mathscr{L}_m$ is the matter Lagrangian density and $\kappa=8\pi G$. Varying the action (1.1.8) with respect to the metric tensor $g_{\mu\nu}$, we can obtain EFE (1.1.7). The energy–momentum tensor, $T_{\mu\nu}$ of a prefect fluid is defined as

$$T_{\mu\nu}=(\rho+p)u_{\mu}\,u_{\nu}-p\,g_{\mu\nu}, \tag{1.1.9}$$

where ρ is the energy density, p is the pressure of the perfect fluid and u^{μ} is the four–velocity vector such that $g_{\mu\nu}u^{\mu}u^{\nu}=-1$. In FRW models, the distribution of matter content of the Universe, i.e, $T_{\mu\nu}$, is only a function of t not of θ and φ due to spatial homogeneity. With one index raised $T_{\mu\nu}$ takes the convenient form $T^{\mu}_{\nu}=diag(-\rho,\ p,\ p,\ p)$ and the trace is given by $T=T^{\mu}{}_{\mu}=-\rho+3p$.

1.1.2 Friedmann Equations and Solutions

In 1922, Friedmann [2] obtained two independent field equations, known as Friedmann equations from Einstein's field equations (1.1.7) for a spatially homogeneous and isotropic Universe (1.1.4) which are given as

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.1.10}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G p - \frac{k}{a^2}, \tag{1.1.11}$$

where an over dot denotes a derivative with respect to cosmic time t.

After combining the Friedmann equations (1.1.10) and (1.1.11), we get the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{1.1.12}$$

which plays an important role to explain the evolution of the Universe. For non-exotic perfect fluid (matter and radiation), we always obtain $\frac{\ddot{a}}{a} < 0$ which implies the decelerated expansion of the Universe. Thus, using Friedmann equations we get a non–static Universe.

Friedmann was the first person who obtained the exact solution with matter of Einstein's equations and obtained an expanding evolution of the Universe. Einstein said that it was merely a mathematical curiosity and dismissed the expanding solution of Friedmann. Later on in 1927, Lemaitre [4] independently obtained the solution of Einstein's equations and suggested an expanding Universe. We note that in 1917 Einstein himself proposed a relativistic cosmological model, which he assumed to be static in conformtity with the ideas of the contemporary scientific community, however, he found a non-static solution. Einstein believed that GTR should provide a unique solution to the cosmological problem. He still believed that our Universe is static and hence, he introduced a cosmological constant term referred to as Λ-term, which would produce a repulsive effect. Einstein modified his original field equations (1.1.7) to include this cosmological constant term and proposed modified field equations as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{1.1.13}$$

Since Λ is constant, the energy conservation is unaffected. However, Einstein's effort was unsuccessful because it was pointed out by Eddington [9] that the Einstein

Universe was unstable and observations by Edwin Hubble in 1929 [10] confirmed that our Universe is expanding. So, Einstein subsequently abandoned the so–called cosmological constant term from his field equations and stated that introducing Λ was the "greatest blunder in his life". For many years the cosmological constant was almost universally considered to be zero. However, it appeared again in the later development of cosmology as a strong candidate for the "dark energy", the major constituent of the present Universe to explain the accelerating Universe. A detail discussion about cosmological constant and dark energy phenonena is given in Sect. 1.4.1.

Further, the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ leads to the equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.1.14}$$

To make progress, we can choose an equation of state (EoS), which establishes a relation between ρ and p. Throughout this thesis, we use a barotropic EoS in which the pressure is a function of density only and is given by

$$p = (\gamma - 1)\rho = w\rho, \tag{1.1.15}$$

where γ or w is known as the EoS parameter whose values depend on the type of energy content of the Universe. Now, using (1.1.15) in the conservation equation (1.1.14) we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}. \tag{1.1.16}$$

If w is a constant, Eq.(1.1.16) can be integrated to obtain

$$\rho \propto a^{-3(1+w)}. \tag{1.1.17}$$

Now, depending on the type of energy content of the Universe, i.e., depending on the value of w, the exact evolution of energy density can be obtained. A Universe in which most of the energy density is in the form of radiation is known as radiation dominated. As radiation obeys $p = \rho/3$, we get $w = 1/3$. Therefore, from the Eq. (1.1.17) the energy density evolves as $\rho \propto a^{-4}$. For matter (dust) dominated Universe, we have $p = 0$ which provides $w = 0$. The evolution of energy density in this case is given by $\rho \propto a^{-3}$. It may be observed that the density of radiation decreases faster than the density of matter.

The Friedmann equations can be solved using the evolution of density (1.1.17). Let us solve the equations for a flat FRW Universe, i.e., for $k = 0$. The cosmic scale factor a may be obtained as

$$a \propto t^{\frac{2}{3(1+w)}}. \tag{1.1.18}$$

The value of a also depends on the value of w, i.e., depends on the energy content of the Universe. For radiation ($w = 1/3$) it evolves as $a \propto t^{1/2}$ and for matter ($w = 0$) it evolves as $a \propto t^{2/3}$. In the case of the flat FRW Universe, the Friedmann equations are unchanged if we replace scale factor $a(t)$ with $\lambda a(t)$, where λ is a constant. It is an important symmetry which provides freedom to rescale a(t) according to our requirements.

1.2 Expanding Universe

Many physicists and mathematician had started working on Einstein's GTR, just after Einstein's proposal, to find the solution and many of them obtained solutions describing the expanding Universe. Friedmann in 1922 [2] and Lemaitre in 1927 [4] had developed theoretical models which had shown the expansion of the Universe, but their works did not get recognization until the discovery of expanding Universe by American astronomer Edwin Hubble. In 1929, Hubble [10] performed an observation to measure the redshift of a number of distant galaxies with their relative distances. When he analyzed the redshift against relative distance through the graph, he found that the redshift of the distant galaxies is directly proportional to a linear function of their distance. This observation confirmed that the Universe was expanding not a static as Einstein thought. The observations showed that all the galaxies are receding from us. On the basis of this observation he proposed the velocity of the galaxy is directly proportional to its distance from the observer. This is known as Hubble's law. The galaxies that are more distant from us are receding faster than galaxies that are closer to us. Mathematically, the Hubble's law can be expressed as

$$v = H_0 D, \tag{1.2.1}$$

where v represents the recession velocity of the galaxy, typically expressed in Km/s and D is the proper distance of the galaxy from the observer, measured in mega parsecs (Mpc). The proportionality constant H_0 is known as Hubble's constant which is measured in kilometers per second per megaparsec. The Hubble constant, H_0 is a Hubble parameter, H measured today. The Hubble parameter is expressed in terms of the scale factor a, which is given by

$$H = \frac{\dot{a}}{a}. \tag{1.2.2}$$

The Hubble parameter is an important quantity which tells us about the expansion rate of the Universe. The Universe contracts for negative values and expands for positive values of the Hubble parameter. Our Universe is expanding because we observe a positive value for the Hubble parameter. The present value H_0 has been obtained using various observational data. The latest data of the SDSS-III Baryon Oscillation Spectroscopic Survey gives $H_0 = 67.6^{+0.7}_{-0.6}\ km\, s^{-1} Mpc^{-1}$ [11].

The main fact behind the discovery of the expanding Universe was the red shift, z observed in the spectrum of light coming from nearby galaxies, which is basically the Doppler effect applied to light waves. The recession velocities of distant galaxies are known from the redshift, but the distances are much uncertain. The red shift is defined as

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}, \tag{1.2.3}$$

where λ_{obs} and λ_{em} are the wavelength at the time of observation and emission of the light. Actually, the spectrum shifts towards the blue end for an object moving towards the observer and it shifts towards the red end if the object is moving away from the observer. Hubble observed the red shift in the spectrum of light which was coming from distance galaxies and hence he concluded that the Universe was expanding.

As Hubble's discovery tells us that everything is moving away from us, it seems that the cosmological principle will be violated at some instant. It also seems that we are at the center of the Universe. Indeed, it is not like that. In fact, every observer sees all objects moving away from them with velocity proportional to distance. Let us consider an example of a balloon having dots on its surface. As we blow up the balloon, the dots on the surface move away. Whatever the dot we will choose as an observer, we observe that every dot is receding from the observer. It is happening due to the Hubble's law. Thus, the Universe remains almost the same in all directions and at all places. Therefore, the cosmological principle still holds.

A relation between redshift z and the scale factor a is given by

$$1+z=\frac{a(t_0)}{a(t)}=\frac{1}{a(t)}, \tag{1.2.4}$$

where $a(t_0)$ denotes the scale factor at present time and can be assumed as $a(t_0)=1$. The energy density can also be represented in terms of z as

$$\rho_m=\rho_{m,0}\,a^{-3}=\rho_{m,0}\,(1+z)^3, \tag{1.2.5}$$

where $\rho_{m,0}$ represents the energy density at present time.

The discovery of an expanding Universe propose the Big Bang theory. According to this theory, roughly 13.8 billion years ago, the Universe was originated from an infinite energy density, i.e., from the point of singularity where all physical laws break. After Big Bang, the universe came into existence and gradually began to cool from the state of higher temperature and density [3]. The ΛCDM is the current “standard model" of Big Bang cosmology that can account for various measurements and observational data.

1.3 Inflationary Universe

Despite the great phenomenological success of standard Hot Big Bang scenario in describing the evolution of the Universe, there are still some important problems with the standard cosmology, many of which are related to very early evolution of the Universe. Let us discuss some of the longstanding problems of standard model.

1. **The Singularity Problem:-** The solution of Friedmann equations gives that the scale factor of the Universe $a(t)$ vanishes at $t\to 0$, whereas the energy density becomes infinte at $t\to 0$ [12]. We can say that the Universe has singularity at $t=0$. Now, the problem is that what was before the singularity. If our Universe did not exist at $t<0$, then how could it originate from ‘nothing’? The singularity problem is certainly one of the most puzzling problems of contemporary science.

2. **The Flatness Problem :-** We observe that the present Universe is very close to flat geometry, i.e., $\Omega=1$. Here, Ω denotes density parameter which is defined as $\Omega=\frac{\rho}{\rho_c}$, where ρ_c is the critical energy density. We have defined Ω and ρ_c in

detail in Sect. 1.7. Let us consider the Universe with radiation and matter as energy content. The Friedmann equation (1.1.10) in terms of density parmeter is given by

$$\Omega - 1 = \frac{k}{a^2} \tag{1.3.1}$$

It may be observed that if Ω is exactly equal to one, then it remains one all the time. Therefore, no problem arises, we observe a flat Universe at present. But if it is not equal to one, then we face a problem. Let us discuss what the problem is? If we find the value of Ω for radiation and matter dominated Universes, then we obtain

$$|\Omega - 1| \propto t; \qquad |\Omega - 1| \propto t^{2/3} \tag{1.3.2}$$

respectively. Here, we observe that $(\Omega - 1)$ increaes with time in both cases. It shows that the flat geometry is unstable. As it deviates from the flat geometry, it becomes more and more curved during the course of time. Therefore, the Universe must be very fine tuned to flat geometry in very early times to observe flat Universe in the present time. This is called the flatness problem of the standard model.

3. **The Horizon Problem :-** The Big Bang model also suffers from the horizon problem. The problem is that why does the Universe, especially the cosmic microwave background (CMB) is very closed to isotropic, i.e., looks the same in all directions. The light in all directions possesses to very great accuracy of same temperature of $2.725k$. Therefore, the distant opposite parts of the Universe have been in thermal equilibrium in the past. As the velocity of light is finite, therefore, it can travel a finite distance within the age of the Universe. The CMB that we are observing today from opposite sides of the sky, has been traveling towards us since the time near the Big Bang itself. Therefore, there is no possibility that the distant photons on opposite sides have been in interaction with each other. Then, why are we measuring the same temperature in all directions with great accuracy.

4. **The Monopole Problem :-** The Grand Unified theory of particle physics predicts that in very early Universe magnetic monopoles were produced with high abundance. They are supposed to be extra ordinary massive, about 10^{16} GeV. Monopoles are highly stable particles and once they have been created, they are indestructible. Therefore, they would survive as relics in the present epoch.

As we don't observe magnetic monopoles at present, this imposes a problem on the standard model of cosmology called the monopole problem.

To overcome the above problems faced by Big Bang model, an inflationary Universe was proposed as a solution. The most accepted theory of inflationary Universe was first proposed in 1981 by American physicist Alan Guth [13], which was based on Weinberg's grand unified theory (GUT). According to GUT, the particle interactions possess certain symmetry. Further, this theory was discussed and modified by Linde [14, 15], and Albrecht and Steinhardt [16, 17]. It was accepted in the scientific community due to its promising and elegant way to understand the very early Universe. The basic principal of cosmic inflation is that in the first split–second after the Big Bang, the Universe underwent an exponential expansion. The cosmic inflation theory solves all problems that had been identified with Big Bang theory as discussed above except singularity problem.

The cause of inflationary phase is still not clear to cosmologists, however, the best guess is that there were some kind of a negative "vacuum energy density" generated by the separation of strong nuclear force from the other elementary forces. This separation caused a kind of symmetry breaking or phase transition which is due the existence of a certain spin–zero field ϕ, known as Higgs field. This field possesses a potential energy density function $V(\phi)$ [13]. It is now believed by some cosmologists that the inflation can't be the Higgs field [18], although the recent discovery of the Higgs boson has increased the number of works considering the Higgs field as inflation.

A new inflationary model was proposed by Russian physicist Andrei Linde [15, 19] who hypothesised a slow breaking of symmetry. The scalar field was supposed to initiate inflation, which decayed into radiation and matter to stop this scenario. The kinetic energy of the scalar field was dominant during inflation which allows de-Sitter like expansion of the Universe. Inflation happened for a very short time in the very early Universe and the Universe experienced a very rapid expansion in this duration. The Universe first experienced accelerated expansion, i.e., $\ddot{a} > 0$.

Inflation resolves all the mentioned problems associated with the Hot Big Bang model except the singularity problem. The flatness problem occurred because $(\Omega - 1)$ given by $(1.3.1)$ was an increasing function of time which forces Ω away from one. If we apply the concept of the inflation then we have

$$\ddot{a} > 0 \implies \frac{d}{dt}(\dot{a}) > 0 \implies \frac{d}{dt}(aH) > 0. \tag{1.3.3}$$

Now, it is easy to observe from (1.3.1) that inflation makes Ω closer to one which is required to observe a flat Universe at present. Thus, inflation resolves the flatness problem. Due to very rapid expansion of the Universe during the time of inflation, the small size of the Universe was increased to a much larger size. Thus, the small size of the Universe, which was able to achieve thermal equilibrium before inflation has expanded to a greatly large size and it may even be larger than our observable Universe. Thus, inflation provides the mechanism through which distant opposite sides of the Universe may have come close in very early times to establish thermal equilibrium. Therefore, the horizon problem has been solved. The monopole problem has also been resolved by inflation. The rapid expansion of the Universe during the inflationary era has diluted the density of magnetic monopoles. Therefore, we don't observe monopoles at the present time.

Inflation resolves the problems of Hot Big Bang cosmology successfully. Inflation is an active field of research. In the literature, mainly scalar field theory and supersymmetry theory are being used to get inflation.

1.4 Accelerating Universe and Dark Energy

The last decades of the 20^{th} century witnessed major developments in observational cosmology. The reasons for this dramatic development are new measurements of the geometry and the matter contents of the Universe. Most current cosmological models are based on the Big Bang theory which predicts not only the universal expansion, but also the baryonic matter content and a relic radiation from the original hot phase. The extension of Big Bang theory is the inflationary phase at very earliest times of the evolution. Therefore, the last decade of the 20^{th} century was truly exciting from the view point of progress in observational cosmology.

An attempt was made to measure the geometry through the measurement of distances. This has been measured through standard candles, that is, objects with identical absolute luminosity. There are many candidates of standard candles which have been proposed. One of such a standard candle is type Ia supernova (SN Ia). [2]

Before 1998, the cosmologists assured that the Universe was expanding with decelerated rate after a very short and rapid expansion (inflation). But, in 1998 two separate

[2]SN Ia are thought to be the result of explosion of a carbon–oxygen white dwarf in a binary system. Supernova are extremely rare, a galaxy like Milky Way may produce a SN Ia only every 400 years.

teams headed by Perlmuter [20] and Riess [21] who were working on the observations of distant SN Ia to measure the rate of expansion of the Universe, announced that our Universe is experiencing a phase of accelerated expansion. These results have been confirmed repeatedly by several other observations such as Wilkinson Microwave Anisotropy Probe (WMAP) [22, 23], Sloan Digital Sky Survey (SDSS) [24], Planck collaboration [25], etc. The observations show that the accelerating expansion of the Universe is due to an unknown component, named as "dark energy" (DE) [20,21]. DE has a strong negative pressure which is responsible for such an acceleration.

DE is the most accepted hypothesis to explain accelerated expansion of the Universe. Basically, there are two way to accommodate the observed accelerated expansion of the Universe. In one of the ways, one can consider an exotic type of energy content which could provide sufficient amount of negative pressure to observe the acceleration. This is known as the modification in the matter part of the Einstein equations and is called DE models of the Universe. In the other way, one can modify the geometric part of Einstein equations, which is known as modified theories of gravity models. Let us discuss both the concepts in brief.

1.4.1 Dark Energy Models

The observation of SN Ia about the accelerated expansion in 1998 suggested that there must be an exotic energy component called DE, which constitutes 68.3% of the total energy density of the Universe. The energy density of dark matter (DM) contributes 26.8% whereas the ordinary (baryonic) matter has 4.9% contribution in the Universe. The other components such as neutrinos and photons contribute a very small. A number of DE models have been proposed in the literature to accommodate the accelerated expansion. The most natural candidate for DE is the cosmological constant Λ which was introduced by Einstein to obtain the static Universe [26–30]. It was reconsidered to explain the accelerated expansion because it supplies sufficient negative pressure to the cosmic system. The Lambda Cold Dark Matter (ΛCDM) model of the Universe which contains cosmological constant Λ and cold dark matter (CDM) are known as the standard model of cosmology. The ΛCDM model accommodates the observations extremely well. Although, the standard model of cosmology is able to explain almost all aspects of the cosmic evolution like structure formation, black hole phenomenon, accelerated expansion etc., but it faces the theoretical problems of cosmic coincidence and fine tuning.

To avoid these problems of ΛCDM model, the dynamical dark energy models were proposed in the literature. The dynamical DE models includes scalar field models like the quintessence model, phantom model, quintom model, Chaplygin gas model, holographic dark energy (HDE) models, agegraphic dark energy (ADE) models, new agegraphic dark energy (NADE) models, etc. Let us discuss the standard ΛCDM model and dynamical DE models in brief.

The Standard ΛCDM Model

The standard ΛCDM model is the combination of two most convincing concepts, the cosmological constant and CDM. The cosmological constant Λ was reconsidered in Einstein's equations to achieve the observed acceleration. DM is required to explain the rotation curves of galaxies, stability of galaxies and galaxy clusters, and structure formation. It has almost negligible pressure. The temperature of DM is supposed to be very low, therefore, it is called CDM. The model is commonly known as ΛCDM to indicate two main components of it. The ΛCDM model is extremely predictive and observationally robust. It explains the formation of large–scale structure, early Universe cosmology and late time accelerated expansion of the Universe. It also accommodates precision observational tests, and its prediction is continually being vindicated through the discovery of polarization and gravitational lensing.

In Einstein's modified equations (1.1.13), the cosmological constant Λ was included in the left hand side, i.e., in the geometric part of the equations. In the standard model, it has been considered in the right hand side, i.e., in the matter part of the equations because it is being considered a candidate of DE. Therefore, the modified equations of (1.1.13) may be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left(T_{\mu\nu} + T^{\Lambda}_{\mu\nu}\right). \tag{1.4.1}$$

The energy denoted by the energy–momentum tensor of the cosmological constant $T^{\Lambda}_{\mu\nu}$ is supposed to be the energy of space (vacuum) itself. It remains constant during the evolution. The concept of vacuum energy and cosmological constant are equivalent and are used in the literature interchangeably. The energy–momentum tensor of the vacuum energy is defined as

$$T^{(vac)}_{\mu\nu} = -\rho_{\Lambda} g_{\mu\nu} = T^{\Lambda}_{\mu\nu}. \tag{1.4.2}$$

The vacuum energy density and cosmological constant have the relation $\rho_\Lambda = \frac{\Lambda}{8\pi G}$.

The EoS of the cosmological constant is given by $p_\Lambda = -\rho_\Lambda$, where $w = -1$ is the EoS parameter. Let us see the acceleration equation given by (1.1.12) to get a condition for acceleration. We observe accelerated expansion of the Universe if we have $\ddot{a} > 0$,i.e., $\rho + 3p < 0$. Using the EoS $p = w\rho$, we obtain the condition $w < -\frac{1}{3}$ for accelerated expansion. Thus, accelerated expansion can be easily achieved in the standard model.

Despite of many great features and observational verifications, the ΛCDM model faces theoretical challenges like fine–tuning and cosmic coincidence problems. The cosmological coincidence problem was addressed for first time by Steinhardt [31] and Zlatev et al. [32]. The Observations tell us that the ratio of DE and matter is of order one at present. In the framework of the standard model, very fine tuned initial conditions of the early Universe are required to observe this ratio of order one. This means it is like a coincidence that we are living in a time where the densities of DE and matter are of same order. The fine tuning problem of the ΛCDM model is related to its observed value and theoretical value. The observations predicts a very tiny value in comparison to the theoretical value of it which is about to 120 orders higher.

Dynamical Dark Energy Models

Fine–tuning and coincidence problems associated with the cosmological constant have led to the search for dynamical DE models [33]. A phenomenological solution of these problems is to consider a time dependent cosmological term [34, 35]. One of the simplest and probably the most common candidate of dynamical DE is *'quintessence'* [32, 36–43]. The concept of quintessence basically uses a *scalar particle field* [44, 45]. Historically, scalar fields are used as the responsible agents for inflation [46]; to seed the primordial perturbation for structure formation during an early inflationary epoch; and as a candidate for CDM, responsible for the formation of the actual cosmological structures [47].

Due to remarkable qualitative similarity between the present DE and primordial DE that derives inflation in the early Universe, the inflationary models based on scalar fields have also been applied for a description of the late–time cosmic acceleration [32, 33, 39–41, 43, 48, 49]. Therefore, the scalar field cosmological models have acquired great popularity in recent years. The outcomes from different observational data [50–54] also show a possibility of the existence of some strange kind of fields in the Universe such as the *phantom field* as proposed by Caldwell [55] having negative kinetic energy [56, 57]. Some other candidates of such dynamical DE are the quintom (a combination of quintessence and phantom scalar fields) [58], tachyonic field [59, 60, 63], k-essence [61–63], Chaplygin gas [64, 65] etc.

Apart from these DE candidates, some candidates have been proposed in the literature using significant properties of quantum mechanics like HDE, ADE, NADE etc. Nowadays, it is a common issue to make the use of such types of DE candidate as a responsible agent to describe the evolution of the Universe [33, 43, 59, 63, 65–67]. Let us discuss briefly about the theory of HDE and NADE.

Holographic Dark Energy

The origin of DE is an important issue in modern cosmology. The HDE, which possesses some significant properties of the quantum theory, has been proposed as a candidate for DE to explain the recent phase transition of the Universe. HDE is based on the holographic principle proposed by 't Hooft [68]. According to this principle, the information contained in a volume of space may be obtained from the bounding surface area. In other words, the number of degree of freedom of physical systems scales with their bounding area rather than with their volume. The holographic principle is based on black hole thermodynamics according to which the maximal entropy in any region is proportional to square of radius, and not cube of it. Holographic principle suggests that the information of all the objects that have fallen into the black hole might be entirely contained in surface fluctuations of the event horizon. This principle further was discussed by Susskind [69] in the context of string theory. The origin of HDE contains more scientific approach in comparison to other DE candidates, and presents a better way to deal with the accelerated expansion.

Cohen et al. [70] have shown that the formation of black holes imposes an upper bound on the total energy of size L and should not exceed the mass of black hole of the same size, thus $L^3\rho_h \leq LM_p^2$, where M_p stands for the reduced Planck mass and it is related with G, $M_p^{-2} = 8\pi G$. In the paper [71], the author assumed the largest infra–red (IR) cut–off to saturate the inequality imposed by black hole formation and obtained the density of HDE

$$\rho_h = 3d_1^2 M_p^2 L^{-2}, \tag{1.4.3}$$

where d_1^2 is a dimensionless constant which is taken to be of the order of unity and L denotes the IR cut–off. Taking L as the size of the current universe, for instance the Hubble scale, the resulting energy density is comparable to present date DE.

The Hubble horizon is a natural candidate for IR cut–off which is also free from causality but Hsu [72] found that it gives a wrong EoS of DE. Li [71] studied HDE with other possible IR cut–off and found that the future event horizon may be a suitable IR cut–off to observe EoS of DE. Later on, Pavón and Zimdhal [73] observed that the identification of L with the Hubble horizon, $L = H^{-1}$ may give a suitable EoS for DE for an interacting HDE model. It has also been observed that there must be a constant ratio of the energy densities of HDE and DM irrespective of type of interaction. Nojiri and Odintsov [74] have studied a generalized model of HDE and found that a unified model of the Universe may be achieved. The authors also claimed that the coincidence problem may be resolved in a generalized HDE model. Thus, the HDE models may also alleviate the cosmic coincidence problem which provides an advantage of HDE models over the other DE models. It has been shown that HDE is favoured by the latest observational data including the sample of SN Ia, the shift parameter of CMB and BAO measurement.

New Agegraphic Dark Energy

The NADE model is the new version of the ADE model. ADE model is based on the quantum fluctuations of the space time metric. Using the concept of quantum fluctuations of space time metric, Károlyházy and his collaborators [75–77] observed that the distance t in Minkowaski space time can't be known to a better accuracy than

$$\delta t = \lambda \, t_p^{2/3} \, t^{1/3}, \tag{1.4.4}$$

where t_p denotes the reduced Planck time and λ is a dimensionless constant of order unity. The units have been chosen such that $\hbar = k_b = c = 1$. Maziashvili [78, 79] argued that the quantum energy density of the metric fluctuations of Minkowaski space time can be estimated using the relation (1.4.4) with the time–energy uncertainty relation. Maziashvili [78, 79] and Sasakura [80] have independently obtained the energy density of metric fluctuations of Minkowaski space time which is given by

$$\rho_d \sim \frac{1}{t_p^2 t^2} \sim \frac{M_p^2}{t^2}, \tag{1.4.5}$$

This energy density can be viewed as the density of DE, i.e., the ADE. Following the line of the relations (1.4.4) and (1.4.5), Cai [81] proposed the original ADE model in which the energy density of ADE has the form

$$\rho_d = \frac{3d_2^2 {M_p}^2}{T^2}, \tag{1.4.6}$$

where T is age of the Universe. The numerical factor $3d_2^2$ has been introduced to parameterize some uncertainties like the species of quantum fields in the Universe, the effects of curved space time, etc. A number of papers on ADE models are available which deal with various aspects of the evolution of the Universe [82–84]. Wei and Cai [85] have proposed a new version of this model referred as "new agegraphic dark energy" model, abbreviated as NADE by replacing the cosmic age T with the cosmic conformal age η for the time scale in (1.4.6). Thus, in this model, the dark energy density is of the form

$$\rho_d = 3d_2^2 M_p^2 \, \eta^{-2}, \tag{1.4.7}$$

where the conformal time η is defined as

$$\eta = \int_0^t \frac{dt}{a} = \int_0^a \frac{da}{Ha^2}. \tag{1.4.8}$$

The most attractive merit of NADE is that it has been proven to fit the data well. The cosmological constraints and other aspects of NADE models have been studied in the literature [86–90].

Although, the DE models explain the recent accelerated expansion of the Universe very well and also accommodate the observations but the origin and evolution of DE is still mysterious and unknown. Many other problems like the fine–tuning problem, coincidence problem etc. associated with DE models compel us to think beyond the standard model and other DE models. The modified gravity models are also bright candidates to explain the evolution of the Universe. Let us discuss modified theories of gravity model in brief.

1.4.2 Modified Theories of Gravity Models

Modified theories of gravity are active research fields parallel to GTR to study the hidden realities of the Universe. Modified theories of gravity are not new in the literature and there have been a number of proposals just after Einstein's GTR. The first modification of Einstein GTR came into existence with Eddington's theory of connections [9]. Weyl [91] proposed the scale independent theory. The other theories like, Kaluza–Klein theory [92, 93] and string theory [94, 95] are examples of higher dimensional theories. The scalar–tensor theories [96, 97] are example of extra fields included in the field equations. String theory has been considered as a potential candidate of quantum gravity and considerable developments have been made in this theory in the last two decades. The scalar–tensor theories are also one of the well established and extensively studied theories of gravity. These theories often used to model Newton's constant G as a variable. The most simplest and well studied scalar–tensor theory is Brans–Dicke (BD) theory.

Utiyama and De Witt [98] used higher order terms of scalar curvature R to modify GTR. Starobinsky [99] in 1980 presented an inflationary model using the higher order corrections to GTR which has been a remarkably successful model of inflation. These modified theories of gravity are known as higher order gravity theories. Buchdahl [100] was first who considered the general function $f(R)$ in the EH action $(1.1.8)$ to present a more general model of modified gravity. $f(R)$ gravity is consistent with the observations and successfully passes the solar system tests [101–103].

The study on evolution of the Universe through modified gravity models have become a popular area of research in recent years [101,104–112]. A number of modified theories of gravity are available in the literature namely, $f(R)$ theories [103, 104, 108, 113, 114], Gauss–Bonnet $f(G)$ theory [115, 116], BD theory [117, 118], Brane world gravity [119, 120], Horava–Lifshitz gravity [121, 122], $f(T)$ theory [123, 124], $f(R,T)$ theory [125, 126], etc. However, none of these presents a complete theory of gravity [127]. But, the search for a complete theory of gravity continues. In this thesis, we discuss two different theories, namely modified $f(R,T)$ gravity and BD theory. The following are the backgrounds and field equations of these two theories.

The Modified $f(R,T)$ Gravity Theory

Bertolami et al. [128] generalized $f(R)$ gravity by assuming the maximal coupling between the curvature term R and the matter Lagrangian density. Further, Harko [129] extended this model to case of the arbitrary couplings in both geometry and matter. Harko and Lobo [130] proposed a maximal extension of EH by assuming the gravitational Lagrangian as an arbitrary function of the Ricci scalar R and of the matter Lagrangian $\mathscr{L}_m$.

In 2011, Harko at al. [125] proposed a modification of $f(R)$ theory known $f(R,T)$ gravity, where R as usual is the Ricci scalar and T is the trace of the energy–momentum tensor. Authors argued that the dependency on T may be induced by some exotic imperfect fluids or quantum effects (conformal anomaly). The modified EH action for $f(R,T)$ gravity is given by [125]

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}f(R,T)+\mathscr{L}_m\right], \tag{1.4.9}$$

where $f(R,T)$ is an arbitrary function of Ricci scalar R and trace of energy–momentum tensor T. We take $8\pi G=1$ in action (1.4.9). The energy–momentum tensor of matter is defined as [131]

$$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_m)}{\delta g^{\mu\nu}}, \tag{1.4.10}$$

Let us assume that the matter Lagrangian $\mathscr{L}_m$ depends only on the metric tensor components $g_{\mu\nu}$, and not on its derivatives. Therefore, from (1.4.10) we get

$$T_{\mu\nu}=g_{\mu\nu}\mathscr{L}_m-2\frac{\partial\mathscr{L}_m}{\partial g^{\mu\nu}}. \tag{1.4.11}$$

Varying the action (1.4.9) with respect to $g_{\mu\nu}$, we obtain the field equations of $f(R,T)$ theory as

$$f_R(R,T)R_{\mu\nu}-\frac{1}{2}f(R,T)g_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R(R,T)=T_{\mu\nu}-f_T(R,T)(T_{\mu\nu}+\Theta_{\mu\nu}), \tag{1.4.12}$$

where f_R and f_T denote the partial derivatives of $f(R,T)$ with respect to R and T, respectively, ∇_μ is the covariant derivative, $\Box\equiv\nabla_\mu\nabla^\mu$ is the d'Alembert operator, and $\Theta_{\mu\nu}$ is defined by

$$\Theta_{\mu\nu}\equiv g^{ij}\frac{\delta T_{ij}}{\delta g^{\mu\nu}}, \qquad i,j=0,1,2,3. \tag{1.4.13}$$

Using (1.4.11) into (1.4.13), we obtain

$$\Theta_{\mu\nu}=-2T_{\mu\nu}+g_{\mu\nu}\mathscr{L}_m-2g^{\alpha\beta}\frac{\partial^2\mathscr{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{1.4.14}$$

Authors suggested that the models of $f(R,T)$ gravity depend on a source term and this source term is a function of the matter Lagrangian $\mathscr{L}_m$. Therefore, the choice of $\mathscr{L}_m$ will decide the field equations of the model. $f(R,T)$ gravity due to the coupling between matter and geometry produces an extra acceleration as an additional feature of this gravity theory. The authors have obtained general constraints on the magnitude of the extra acceleration using the precession of the perihelion of the planet Mercury.

Let us take matter lagrangian $\mathscr{L}_m = -p$, then with the use of Eq. (1.4.14) we obtain

$$\ominus_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}. \tag{1.4.15}$$

As $f(R,T)$ gravity depends on the source term, various theoretical models may be obtained for different choices of matter source. In the thesis, we have assumed the form $f(R,T) = R + 2f(T)$. Therefore, we framework the field equation with this assumption only. Therefore, the field equations immediately follow from (1.4.12) as

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = T_{\mu\nu} - 2f'(T)T_{\mu\nu} - 2f'(T)\ominus_{\mu\nu} + f(T)g_{\mu\nu}, \tag{1.4.16}$$

where the prime represents a derivative with respect to T. Using (1.4.15) the field equations (1.4.16) become

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = T_{\mu\nu} + 2f'(T)T_{\mu\nu} + [2pf'(T) + f(T)]g_{\mu\nu}, \tag{1.4.17}$$

On applying covariant divergence on (1.4.12) and using the following identity

$$\nabla^{\mu}[f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f_R(R,T)] = 0, \tag{1.4.18}$$

We obtain the divergence of $T_{\mu\nu}$ as

$$T^{\mu\nu}{}_{;\nu} = \frac{f_T(R,T)}{8\pi - f_T(R,T)}[(T_{\mu\nu} + \ominus_{\mu\nu})\nabla^{\mu} ln f_T(R,T) + \nabla^{\mu}\ominus_{\mu\nu}] = 0. \tag{1.4.19}$$

A number of authors have studied various aspects of $f(R,T)$ gravity to resolve different issues of evolution of the Universe. Houndjo et al. [132] investigated $f(R,T)$ gravity models to reproduce the four known finite–time future singularities. Pasqua et al. [126] studied a particular model $f(R,T) = \mu R + \nu T$ which describes a quintessence–like behavior and exhibits a transition from a decelerated to an accelerated phase. Sharif and Zubair [133] considered two forms of the energy–momentum tensor of dark components and demonstrated that the equilibrium description of thermodynamics can't be achieved at the apparent horizon of the FRW Universe in $f(R,T)$ gravity. Azizi [134] showed in a paper that the effective stress–energy may be considered as a possibility for violation of the null energy condition in $f(R,T)$ gravity. Chakraborty [135] has assumed that the conservation equation holds for $f(R,T)$ gravity and discussed the energy conditions with a perfect fluid.

The dynamics of scalar perturbations was discussed in $f(R,T)$ gravity in the paper [136] in the metric formalism assuming a specific model that guarantees the standard continuity equation. The authors obtained the constrains on the form $f(R,T)$ such that the $f(R,T)$ gravity will satisfy the standard continuity equation for the energy–momentum tensor. Baffou et al. [137] discussed the dynamics and the stability of $f(R,T)$ gravity for de-Sitter and power–law expansion of the Universe taking conservation of the energy–momentum tensor into account. These authors concluded that $f(R,T)$ gravity is stable and dynamically consistent with the observational data for the low and high redshift regimes.

In the paper [138], the authors have studied some general models of $f(R,T)$ gravity. They found that matter and DE are of same order not only at the present time, but they have been of the same order at early times too. Further, they have concluded that $f(R,T)$ gravity models may be constructed which are consistent with solar system observations. The HDE and NADE models were discussed in $f(R,T)$ gravity in the framework of anisotropic Bianchi type metric [139]. Harko [140] generalized the conservation equation of $f(R,T)$ gravity by using the concept of irreversible matter creation in open thermodynamic systems. A number of authors [141–144] have obtained successful reconstruction of $f(R,T)$ gravity to reconstruct the known history of the Universe using various types of matter content. Many other works have been carried out to discuss the evolution of the Universe in $f(R,T)$ gravity considering different energy contents and formalisms, see refs. [145–152].

The maximum coupling between matter and geometry incorporated in $f(R,T)$ gravity theory opens up a new way to understand the hidden realities of cosmic evolution. In this theory, an effective cosmological constant model may also be constructed. $f(R,T)$ gravity theory is relatively a new theory, therefore, a lot of scope is there to study a general class of $f(R,T)$ gravity models to describe the evolution of the Universe.

The Brans–Dicke Theory

BD theory, proposed in 1961 by Brans and Dicke [117], is a scalar–tensor theory which was initially in terms of an action constructed from a metric $g_{\mu\nu}$ and a scalar field ϕ. It is solely based on dimensionless arguments and with the matter lagrangian being minimally coupled. It is considered as a viable alternative to GTR, one which respects Mach's principle and weak equivalence principle. In BD theory, Newton's gravitational

constant G is not presumed to be constant but is proportional to the inverse of the scalar field ϕ, which can vary from place to place and with time. The scalar field does not exert any direct influence on matter, its only role is that of participating in the field equations that determine the geometry of the space time.
The modified EH action for BD theory in Jordan frame is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}(\phi R - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi) + \mathscr{L}_m\right], \tag{1.4.20}$$

where R denotes the Ricci scalar curvature, $\phi = (8\pi G)^{-1}$ is a time dependent scalar field called BD scalar field which couples with gravity, ω is a coupling parameter between the scalar field and gravity called BD parameter and $\mathscr{L}_m$ represents the matter Lagrangian density.

The variation of the action (1.4.20) with respect to the metric tensor $g_{\mu\nu}$ gives the following field equations.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\sigma\phi\partial^\sigma\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right). \tag{1.4.21}$$

The wave equation for scalar field is given by

$$\Box\phi = \frac{1}{8+3\omega}T. \tag{1.4.22}$$

In the above field equations, ω is dimensionless coupling constant, $T = T^\mu_\mu$ the trace of energy–momentum tensor, $\Box$ is the d'Alembert operator or covariant wave operator and $\Box\phi = \partial_\sigma\partial^\sigma\phi$.

The field equations (1.4.21) and (1.4.22) for the line element (1.1.5) and energy–momentum tensor (1.1.9) yield

$$H^2 + H\frac{\dot{\phi}}{\phi} - \frac{\omega}{6}\frac{\dot{\phi}^{\,2}}{\phi^{\,2}} = \frac{\rho}{3\phi}, \tag{1.4.23}$$

$$2\frac{\ddot{a}}{a} + H^2 + 2H\frac{\dot{\phi}}{\phi} + \frac{\omega}{2}\frac{\dot{\phi}^{\,2}}{\phi^{\,2}} + \frac{\ddot{\phi}}{\phi} = -\frac{p}{\phi}, \tag{1.4.24}$$

$$\ddot{\phi} + 3H\dot{\phi} = \frac{\rho - 3p}{2\omega+3}, \tag{1.4.25}$$

The inflationary epoch has been studied widely in this theory [19, 118, 153]. Banerjee and Beesham [154] have investigated a second order thermodynamic viscous model in the framework of BD theory. The authors have chosen a power–law form of the BD scalar field to obtain exact solutions of the FRW model. Ram and Singh [155] have studied flat FRW model with variable equation of state parameter in this theory. Liddle et al. [156] have studied the transition from radiation to matter dominated epoch and constrained the BD parameter using microwave anisotropy and large–scale structure data in Jordan–Brans–Dicke theory. The emergent Universe model was discussed in the paper [157] in this theory. Early time cosmology with particle creation was studied by Singh [158] to analyze its thermodynamical effect in open thermodynamical systems within the framework of BD theory. In BD theory, the various aspects of black hole cosmology have been investigated in the works [159–163].

Recently, BD theory has got interest to explain the accelerated expansion due to its association with string theory and higher dimensional theories. This theory explains the recent accelerated expansion of the Universe and accommodates the observational data as well [128,164,165]. DE models like HDE model, NADE model, Ricci DE model etc. have been discussed in the frame work of BD theory to explain the problems of modern cosmology like accelerated expansion, cosmic coincidence problem etc.

1.5 Viscous Cosmology

A perfect fluid model may explain many processes in cosmology but some processes can't be explained without the investigation of dissipative phenomena arising in the cosmic fluid. Viscous cosmology has been studied to observe both early time inflation and late time acceleration of the Universe. Eckart in 1940 [166] formulated the relativistic thermodynamic theory of dissipative fluids. He considered first order deviations from thermodynamic equilibrium. However, this theory faces causality problem as dissipative perturbations propagate at infinite speed. According to Eckart theory, if p denotes the thermodynamic pressure of matter content and cosmic fluid has viscosity then the effective pressure is given by

$$\bar{p} = p - \Pi, \tag{1.5.1}$$

where $\Pi = 3\zeta H$ denotes pressure due to viscosity, known as viscous pressure. Here, ζ is the coefficient of bulk viscosity. In this thesis, we have considered only homogeneous and isotropic FRW models to study cosmological models. Therefore, the bulk viscosity is the only dissipative process possible in the models studied here. Misner [167] was the first to use the viscosity concept in cosmology.

Israel and Stewart [168] developed a second order dissipative thermodynamic theory known as the causal theory of relativistic viscosity. In this theory, the dissipative variables were included to describe non–equilibrium states due to which this theory is causal and stable. The full Israel–Stewart transport equation is given by [169]

$$\tau\dot{\Pi}+\Pi=-3\zeta H-\frac{\varepsilon\tau\Pi}{2}\left[3H+\frac{\dot{\tau}}{\tau}-\frac{\dot{\zeta}}{\zeta}-\frac{\dot{T}}{T}\right], \tag{1.5.2}$$

where τ denotes relaxation time associated with the dissipative effect. In view of the above relationship, we can study the behavior of viscosity in Truncated theory, Full Causal theory and Eckart's theory by putting $\varepsilon=0$, $\varepsilon=1$ and $\tau=0$, respectively. The truncated version of the full Israel–Stewart transport equation is given by

$$\tau\dot{\Pi}+\Pi=-3\zeta H. \tag{1.5.3}$$

To ensure that the viscous signals do not exceed the speed of light, we consider the following relation

$$\tau=\frac{\zeta}{\rho}. \tag{1.5.4}$$

In Eckart's non–causal theory, the evolution equation (1.5.2) reduces to

$$\Pi=-3\zeta H. \tag{1.5.5}$$

The energy–momentum tensor (1.1.9) with a bulk viscous term now modifies to

$$T_{\mu\nu}=(\rho+\bar{p})u_{\mu}\,u_{\nu}-\bar{p}\,g_{\mu\nu}, \tag{1.5.6}$$

where $\bar{p}=p-3\zeta H$ is the effective pressure.

Scalar dissipation processes in cosmology may be treated via the relativistic theory of bulk viscosity. The effects of bulk viscosity in an expanding Universe is to reduce the equilibrium pressure. Therefore, sufficient large bulk viscous pressure could make the effective pressure negative. Thermodynamics states with negative pressure are meta stable and cannot be excluded by any law of nature. These states are connected with phase transitions. We know that there is a problem of singularity either in GTR or modified gravity models. Many authors [169–171] have shown that the bulk viscosity removes the initial singularity. Modified gravity models with bulk viscosity have been discussed in many ways to describe the evolution of the Universe.

The phenomenon of bulk viscosity arises in the cosmological fluid when the fluid expands (contracts) too fast due to which the system is out of thermal equilibrium. Then, the effective pressure become negative to restore the thermal equilibrium [172]. Therefore, it is natural to consider bulk viscosity in an accelerating Universe. It has been found that the bulk viscous fluid can produce acceleration in the expansion of the Universe even without dark energy [176, 177, 201]. This happens because bulk viscous mechanism induces an effective negative pressure on the field. In an accelerating Universe, it may be natural to assume that the expansion process is actually a collection of state out of thermal equilibrium in a small fraction of time due to the existence of bulk viscosity.

In the framework of the homogeneous and isotropic FRW metric, the concept of bulk viscosity has been studied extensively to achieve early time inflation as well as late time acceleration [170, 172–178, 180–182]. Ren and Meng [183] have discussed a cosmological model with dark viscous fluid described by an effective equation of state and compared with SN Ia data. Feng and Li [184] showed that the age problem of Ricci dark energy can be alleviated using bulk viscosity. Further, the concept of viscous DE has also been discussed in the literature [185–187]. A phase transition in the early Universe with viscosity was discussed by Tawfik and Harko [188].

1.6 Particle Creation Cosmology

The theory of particle creation has a long history. In the framework of GTR, adiabatic irreversible particle creation was first time studied by Prigogine et al. [189,190]. This particle creation corresponds to an irreversible energy flow from the gravitational field to the created matter field. Phenomenological cosmological particle creation has been extensively discussed in the literature within the context of standard GTR [191–195]. In the context of the recent acceleration, the concept of irreversible particle creation has been reconsidered due to its capability to produce an effective negative pressure [174, 196–200]. On a phenomenological level, the particle creation model has been described in the literature in terms of a bulk viscous stress [171, 201]. However, Prigogine et al. [189, 190] pointed out that the bulk viscosity and particle creation are not only two independent processes but, in general, lead to different histories of the evolution of the Universe. Triginer and Pavón [202], and Brevik and Stokkan [203] discussed the equivalence between bulk viscosity and particle creation. Authors have concluded that however, the dynamics of both the concepts might be same, but thermodynamically both had the different aspect of the Universe. Singh and Beesham [204] have studied the observational consequence of particle creation in early Universe.

In the presence of a gravitational particle source, the number of fluid particles is not conserved, i.e., $N^{\mu}_{;\mu} \neq 0$. In this case the particle number density flow $N^{\mu} = n\, u^{\mu}$ is assumed to satisfy the balance equation [192]

$$N^{\mu}_{;\mu} \equiv \dot{n} + 3nH = n\psi, \tag{1.6.1}$$

Here n is the particle number density and ψ is the particle production rate of dimension $time^{-1}$. The positivity of the parameter ψ indicates creation of particles while there is annihilation of particles for $\psi < 0$. Defining the total number of particles in the comoving volume by $N = na^3$, the balance equation can be rewritten as $\frac{\dot{N}}{N} = \psi$, showing that ψ drives the particle creation rate in the comoving volume. Equation (1.6.1), when combined with the second law of thermodynamics for the adiabatic Universe, naturally leads to the appearance of a negative pressure directly associated with the particle creation rate ψ, which takes the following form [174, 195, 202, 205]:

$$p_c = -\left(\frac{\rho + p}{3H}\right)\psi, \tag{1.6.2}$$

where p_c denotes particle creation pressure. Therefore, p_c being negative, it can help to drive the era of accelerated cosmic expansion we are witnessing today. Once the particle number increases with volume, we find a negative pressure. Let us assume the particle creation rate evolves as [195]

$$\psi = 3\beta H, \tag{1.6.3}$$

where β is a constant lying in the range $0 \leq \beta < 1$.

The second law of thermodynamics due to presence of particle creation leads naturally to a modification of energy–momentum tensor (1.1.9). In the presence of particle creation the energy–momentum tensor of prefect fluid (1.1.9) modifies to

$$T_{\mu\nu} = (\rho + p + p_c)u_{\mu}\, u_{\nu} - (p + p_c)\, g_{\mu\nu}, \tag{1.6.4}$$

where p_c is the particle creation pressure given by Eq. (1.6.2). From (1.6.1) we find the solution for the particle number density n:

$$n(a) = n_0 \left(\frac{a_0}{a}\right)^{3(1-\beta)}, \tag{1.6.5}$$

where n_0 is a constant of integration and the subscript zero denotes the present value. The entropy flux vector takes the form $S^\mu = n\sigma u^\mu \equiv s u^\mu$, where $\sigma = S/N$ is the specific entropy per particle and $S = sa^3$ is the entropy in a comoving volume. The second law of thermodynamics imposes the relation

$$S^\mu_{;\mu} = n\dot{\sigma} + \sigma n \psi \geq 0. \tag{1.6.6}$$

If the creation process occurs in such a way that the specific entropy per particle is constant, the divergence of S^μ reduces to $S^\mu_{;\mu} = \sigma n \psi$. However, the particle creation and bulk viscosity are considered as separate processes and therefore, the specific entropy rate per particle due to the viscous term [202] is calculated as

$$\dot{\sigma} = \frac{\zeta \theta^2}{nT}, \tag{1.6.7}$$

where T is the temperature. We may observe that the production of entropy per particle is independent of the nature of ψ and depends only on the nature of bulk viscosity. If $\zeta = 0$, we find that the entropy per particle is constant as discussed above.

1.7 Some Cosmological Parameters

Rapid advances in observational cosmology are leading to the establishment of the first precision cosmological model, with many of the key cosmological parameters. The cosmological parameters play a very important role in the study of the evolution of the Universe. Let us discuss some parameters here which are used to study the cosmological phenomena in this thesis.

1.7.1 Hubble Parameter

The Hubble parameter measures the rate of expansion of the Universe and is defined by

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \tag{1.7.1}$$

where a as usual denotes the cosmic scale factor and an over dot defines a derivative with respect to time t. Note that Hubble parameter is not a constant. The Hubble constant is the Hubble parameter measured today- we denote it by H_0 (see, Sect. 1.2 for detail).

1.7.2 Critical Density

The critical density is the density of the Universe which is required for the Universe to be flat. It is denoted by ρ_c and defined as

$$\rho_{\mathrm{c}} = \frac{3H^2}{8\pi G}\,. \tag{1.7.2}$$

It is observed that the critical density of the Universe depends on H. Since the Hubble parameter is a function of time, the critical density also evolves with time. The present value of the critical density may be calculated using the value H_0.

1.7.3 Density Parameter

Sometimes it is very useful to express cosmological quantities and cosmic field equations in the terms of the density parameter Ω. It is the ratio of matter density to critical density at the same time, that is,

$$\Omega = \frac{\rho}{\rho_{\mathrm{c}}} = \frac{8\pi G\rho}{3H^2}, \tag{1.7.3}$$

where ρ may be the density of matter, DE, scalar field etc. The density parameter determines the geometry of the Universe, i.e., a closed, flat or open Universe. Observations have shown that the present Universe is very close to a spatially flat geometry ($\Omega \simeq 1$).

The total energy content of the Universe may be divided in two parts, matter (Baryonic + Dark matter) and DE. The density of both are represented by ρ_m and ρ_Λ, respectively, and the total density given by $\rho = \rho_m + \rho_\Lambda$. The density parameters for matter and DE are given as

$$\Omega_m = \frac{\rho_m}{\rho_c}; \qquad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}; \qquad \Omega = \frac{\rho}{\rho_c}, \tag{1.7.4}$$

where $\Omega = \Omega_m + \Omega_\Lambda$.

1.7.4 Deceleration Parameter

The deceleration parameter (DP) plays an important role in the study of the evolution of the Universe. It provides the information whether our Universe is decelerating or accelerating depending on its positive or negative value, respectively. It is defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2}, \tag{1.7.5}$$

where a and the overdot have their usual meanings. Our Universe has experienced two phase transitions, early time inflation to decelerated expansion and decelerated expansion to late time accelerated expansion. The DP changes its sign to observe these phase transitions and plays an important role to explain these phase transitions.

Observations of the cosmic microwave background demonstrate that the Universe is very nearly flat, therefore, DP q and EoS parameter w are related by

$$q = \frac{1}{2}(1+3w). \tag{1.7.6}$$

This relation shows that the Universe decelerates for any cosmic fluid with EoS parameter w greater than -1/3. However, the observations of distant type Ia supernovae indicate that q is negative, i.e., the expansion of the Universe is accelerating.

1.7.5 Statefinder Parameter

Sahni el al. [206] proposed statefinder parameters to discriminate among DE models. The preexisting parameters like H and q are not able to discriminate among DE models. The statefinder parameters are geometrical in nature as they are derived from the space time metric itself. The statefinder parameters $\{r,\ s\}$ are defined as

$$r = \frac{\dddot{a}}{aH^3}, \qquad s = \frac{r-1}{3(q-1/2)}. \tag{1.7.7}$$

Authors found that the values of the statefinder parameters for the ΛCDM model are $r = 1$ and $s = 0$. We may plot trajectories in the $r-s$ and $r-q$ planes to discriminate among DE models. In the present era where a number of DE model are available in the literature, the statefinder parameters $\{r,s\}$ plays an important to discriminate among the DE models.

1.8 Motivation

Even some twenty years ago, it was believed that the Universe is expending with deceleration. So when in 1998, it was found the Universe is expanding with acceleration, it comes like a bolt from the blue. Detection and further experiments reconfirmation of current cosmic acceleration pose to cosmology a fundamental task of identifying and revealing the cause of such phenomenon. This fact can be reconciled with the theory if one assumes that the Universe is mostly filled with so–called DE. This form of matter (energy) is not observable in laboratory and it also does not interact with electromagnetic radiation. Based on these properties, cosmologists have suggested a number of DE models, which are able to explain the current accelerated phase of expansion of the Universe.

In this thesis, some study on accelerating cosmological models of the Universe are carried out. The most popular models dealing with DE are discussed. We specially stretch on modified $f(R,T)$ gravity theory and BD theory within framework of FRW line element.

In previous section, we have seen that the introduction of viscosity in cosmology has been investigated from different view points. There are some recent developments like viscous DE models [175].The bulk viscosity is introduced to model the current observational cosmos and the unified dark component (DM and DE). The second chapter is devoted to the introduction of bulk viscosity into $f(R,T)$ gravity within framework of FRW line element. The bulk viscosity contributes to the cosmic pressure and plays an important role to accelerate the Universe.

Singh and Singh [208] have studied the theoretical and observational consequences of thermodynamics of open systems, which allows particle creation in modified $f(R,T)$ gravity within framework of FRW line element. It has been observed that the accelerated expansion of the Universe is deriven by particle creation without any exotic component or a cosmological constant. In chapter 3, we extend our work on viscous cosmological model with particle creation in $f(R,T)$ gravity.

We have discussed HDE model to explain the accelerated expansion of the Universe. Houndjo et al. [143] have reconstructed $f(R,T)$ gravity from HDE which is able to explain the same expansion history generated in the standard GTR, by DM and HDE. Chapter 4 deals with non–viscous and viscous HDE models in $f(R,T)$ gravity within framework of FRW line element. The cosmological parameters like DP and statefinder parameters are discussed in each model. Non–viscous HDE model does not show phase transition whereas viscous HDE model achieves a smooth phase transition of the Universe.

BD theory has been widely discussed in cosmology and very recently also in connection of HDE [215]. The cosmological application of interacting HDE in BD theory has been studied by Banerjee and Pavón [218], and Sheykhi [222]. Since HDE belongs to a dynamical cosmological constant, and need a dynamical frame work to accommodate it instead of GTR. Therefore, it is worthwhile to investigate HDE model in BD theory. In chapter 5, we consider an interacting HDE model with Hubble horizon as an IR cut–off in the framework of BD theory. We propose a logarithmic form $\phi \propto ln(l+ma)$, where l, m are constants, of BD scalar field to alleviate the problems of interacting HDE models. We present a unified model of HDE which explains the early time acceleration (inflation), medieval time deceleration and late time acceleration.

As we have seen in previous section that NADE model also explains the accelerated expansion, Liu and Zhang [207] have investigated the interacting and non–interacting NADE model in the framework of BD theory. It has been shown that NADE model realizes quintom like behaviour during evolution. In chapter 6, we study the cosmological evolution of NADE models in BD theory with logarithmic form of BD scalar field. We observe that NADE mimics cosmological constant in late time evolution. It shows phase transition from matter dominated phase in early time to accelerated phase in late time. The interacting NADE model resolves the cosmic coincidence problem. Chapter 7 is devoted to the conclusion of the thesis work and some proposals for future work plan.

Chapter 2

Viscous Cosmology in $f(R,T)$ Gravity

In this chapter[1], we introduce bulk viscosity in modified $f(R,T)$ gravity theory within the framework of a flat FRW Universe. The exact solutions of field equations are obtained by assuming a particular form $f(R,T) = R + 2\alpha T$ with constant and time–dependent bulk viscosity which has the form $\zeta = \zeta_0 + \zeta_1 H$, where ζ_0 and ζ_1 are constants. We obtain DP and analyze all possible dynamical behaviors of the Universe for whole range of the coupling parameter α. We observe deceleration, acceleration and phase transition of the Universe under certain constraints. It is observed that bulk viscosity plays an important role to observe the phase transition of the Universe. A big–rip singularity is also observed under certain constraints. The motive of this work is to analyze the effects of bulk viscosity on the evolution of the Universe in the formalism of $f(R,T)$ gravity.

[1]The result of this chapter has been published is a research paper "Friedmann model with viscous cosmology in modified $f(R,T)$ gravity theory, *The European Physical Journal C* **74**, 3070 (2014)".

2.1 Introduction

The modified theories of gravity have become one of the most popular candidates to understand the accelerated expansion of the Universe. In the literature, a number of modified theories have been discussed to explain early and late time expansion of the Universe. In this context, $f(R)$ gravity is the most successful modification of GTR. Modifying the law of gravity is a possible way to explain the acceleration mechanism of the Universe. The $f(R)$ gravity has further been modified by introducing an arbitrary coupling between matter and geometry [129]. Harko and Lobo [130] generalized this concept to an explicit coupling between arbitrary function of the Ricci scalar R and the matter Lagrangian density $\mathscr{L}_m$, known as $f(R,\mathscr{L}_m)$ gravity. In this theory, it is assumed that all the properties of the matter are encoded in the matter Lagrangian $\mathscr{L}_m$.

Harko et al. [125] have introduced another extension of GTR, the so–called modified $f(R,T)$ theory of gravity, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and the trace T of the energy–momentum tensor. This theory presents a maximal coupling between matter and geometry. The authors suggested that the coupling of matter and geometry leads to a model which depends on a source term representing the variation of the energy–momentum tensor with respect to the metric.

Sharif and Zubair [133] have discussed equilibrium and non–equilibrium thermodynamics of $f(R,T)$ gravity model. They have obtained first and second laws of thermodynamics at the apparent horizon in FRW Universe for $f(R,T)$ gravity. The authors have observed a additional entropy term in this theory as compare to GTR, Gauss–Bonnet gravity and braneworld gravity. Pasqua et al. [126] studied the particular model $f(R,T)=\mu R+\nu T$, which describes a quintessence like behavior and exhibits a transition from a decelerated to an accelerated phase. Adhav [148] have studied $f(R,T)$ gravity in Bianchi type–I cosmology and observed that the model is free from Big Bang singularity. Houndjo [142] has reconstructed the form $f(R,T)=f_1(R)+f_2(T)$ in this theory. Further, author has shown that the unification of matter dominated and accelerated phases can be obtained in $f(R,T)$ gravity without neglecting ordinary matter in contrast to $f(R)$ gravity. In the paper [136], authors have studied the evolution of matter density perturbations in $f(R,T)$ gravity. They have obtained constraints required for this theory to observe convergence of energy–momentum tensor. For the model $f(R,T)=f_1(R)+f_2(T)$, they have found that only $f(T)\propto T^{\frac{1}{2}}$ is able to follow standard continuity equation. Singh and Singh [144] have presented the reconstruction of $f(R,T)$ gravity with perfect fluid in the framework of FRW model for two well–known scale factors.

In most of the cosmological models, the Universe has been considered to be filled with perfect fluid. However, it is important to investigate more realistic models that take into account dissipative processes due to viscosity. It is known that when neutrino decoupling occurred, the matter behaved like a viscous fluid in the early stage of the Universe. The first suggestion was investigated by Misner [167] who proposed that the neutrino viscosity acting in the early era might have considerably reduced the present anisotropy of the black–body radiation during the process of evolution. Murphy [170] showed that the bulk viscosity can push the initial singularity in FRW model to the infinite past.

In the context of inflation, it has been known since long time ago that an imperfect fluid with bulk viscosity can produce an acceleration without the need of a cosmological constant or some scalar field [201]. Thus, the bulk viscosity plays a very important role in the history of the Universe. In the framework of homogeneous and isotropic Universe, for a sufficiently large bulk viscosity, the effective pressure become negative and hence it can explain the late time acceleration of the Universe. Indeed, it has been shown that for appropriate viscosity coefficient, an accelerating cosmology can be achieved without the need of a cosmological constant [175–177, 183]. Therefore, dissipative processes are thought to be present in any realistic theory of the evolution of the Universe, and it is reasonable and practical to study in the cosmological models.

In the present Chapter, our motive is to study bulk viscosity in modified $f(R,T)$ gravity theory and investigate the effects of bulk viscosity in explaining the acceleration of the Universe. The exact solutions of field equations are obtained by assuming a particular form of $f(R,T) = R + 2\alpha T$ with constant and varying coefficient of bulk viscosity. We study all possible scenarios to analyze the behavior of the scale factor, matter density and discuss the expansion history of the Universe.

2.2 Field Equations of Modified $f(R,T)$ Gravity

In modified $f(R,T)$ gravity, an arbitrary function $f(R,T)$ of the scalar curvature R and the trace T of energy–momentum tensor is introduced at the place of gravitational Lagrangian R. Therefore, the EH action for $f(R,T)$ gravity in the unit $8\pi G = 1 = c$

takes the form [125]

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}f(R,T)+\mathscr{L}_m\right]. \tag{2.2.1}$$

Variation of the action (2.2.1) with respect to metric tensor $g_{\mu\nu}$ gives field equations which are given by (1.4.12).

The field equations of $f(R,T)$ gravity are much more complicated with respect to the ones of GTR, even for the FRW metric. For this reason many possible forms of $f(R,T)$ have been proposed to solve the modified field equations, for example, $f(R,T)=R+2f(T)$, $f(R,T)=\mu f_1(R)+\nu f_2(T)$ [125, 126, 142, 143], where $f_1(R)$ and $f_2(T)$ are arbitrary functions of R and T, and μ and ν are real constants, respectively, and $f(R,T)=Rf(T)$ [135], etc.

In this work, we consider the simplest particular form [125] $f(R,T)=R+2f(T)$, i.e., the action is given by the same EH one plus a function of T. The term $2f(T)$ in the gravitational action modifies the gravitational interaction between matter and curvature. Using this form in (1.4.12), we obtain field equations given by (1.4.16) as

$$R_{\mu\nu}-\frac{1}{2}R\,g_{\mu\nu}=T_{\mu\nu}-2[T_{\mu\nu}+\ominus_{\mu\nu}]f'(T)+f(T)g_{\mu\nu}, \tag{2.2.2}$$

where the prime denotes derivative with respect to argument.

We assume a spatially flat FRW metric defined for cartesian coordinates in Eq. (1.1.5). In comoving coordinates, the components of the four–velocity u^{μ} are $u^0=1$, $u^i=0$. Here, we assume that the Universe is filled with viscous cosmological fluid. Due to isotropy and homogeneity of the Universe, the only dissipative phenomenon may exist is bulk viscosity. The energy–momentum tensor for a viscous fluid may be written with the help of projection tensor $h_{\mu\nu}=g_{\mu\nu}+u_{\mu}u_{\nu}$ as

$$T_{\mu\nu}=\rho u_{\mu}u_{\nu}-\bar{p}h_{\mu\nu}, \tag{2.2.3}$$

where $\bar{p}$ denotes the effective pressure. In the first order thermodynamic theory of Eckart [166] of dissipative processes, $\bar{p}$ is given by

$$\bar{p}=p-3\zeta H, \tag{2.2.4}$$

where ζ is the coefficient of bulk viscosity. It may be observed that for a large ζ it is possible for negative pressure term to dominate and an accelerating cosmology to ensue. The Lagrangian density may be chosen as $\mathscr{L}_m=-\bar{p}$ [125], and the tensor $\ominus_{\mu\nu}$

defined in (1.4.15) modifies to

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - \bar{p}g_{\mu\nu}. \tag{2.2.5}$$

Using (2.2.5), the field equations (2.2.2) for bulk viscous fluid become

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} + 2f'(T)T_{\mu\nu} + \left[2\bar{p}f'(T) + f(T)\right] g_{\mu\nu}. \tag{2.2.6}$$

The field equations (2.2.6) with the particular choice of the function $f(T) = \alpha T$ [see, Harko et al. [125]], where α is a constant, for the metric (1.1.5) yield

$$3H^2 = \rho + 2\alpha(\rho + \bar{p}) + \alpha T, \tag{2.2.7}$$

$$2\dot{H} + 3H^2 = -\bar{p} + \alpha T, \tag{2.2.8}$$

where $T = \rho - 3\bar{p}$. Now, we have two independent equations (2.2.7) and (2.2.8), and four unknown variables, namely H, ρ, p and ζ to be solved as functions of time. In the following section, we choose an equation of state and a coefficient of bulk viscosity, and try to solve for H.

2.3 Solution of Field Equations

From (2.2.7) and (2.2.8), we get a single evolution equation for H:

$$2\dot{H} + (1+2\alpha)(\rho + p) - 3(1+2\alpha)\zeta H = 0. \tag{2.3.1}$$

If an EoS connecting p and ρ is chosen in the form

$$p = (\gamma - 1)\rho, \tag{2.3.2}$$

where γ is a constant known as the EoS parameter lying in the range $0 \leq \gamma \leq 2$, then equation (2.3.1) can be solved for any particular choice of ζ.

Let us assume the bulk viscosity ζ of the form [177, 183]

$$\zeta = \zeta_0 + \zeta_1 H, \tag{2.3.3}$$

where ζ_0 and ζ_1 are positive constants. The motivation of considering this bulk viscosity is to be found in fluid mechanics. We know that the transport/viscosity phenomenon is involved with the velocity $\dot{a}$, which is related to the scalar expansion $\theta = 3\frac{\dot{a}}{a}$. Both $\zeta = \zeta_0$ (constant) and $\zeta \propto \theta$ are separately considered by many authors. Therefore, a linear combination of the two is more general.

Using (2.2.4), (2.3.2) and (2.3.3) in (2.2.7), we obtain

$$\rho = \frac{3H[(1-\alpha\zeta_1)H - \alpha\zeta_0]}{1+4\alpha-\alpha\gamma}. \tag{2.3.4}$$

Now, let us discuss the cosmology of $f(R,T)$ gravity for different choices of ζ_0 and ζ_1.

2.3.1 Cosmology with Non-viscous Fluid

First, let us discuss non–viscous case to observe the effects of bulk viscosity in next subsections. In this case, where $\zeta_0 = \zeta_1 = 0$, equation (2.3.1) with the help of (2.3.2), (2.3.3) and (2.3.4) reduces to

$$\dot{H} + \frac{3\gamma(1+2\alpha)H^2}{2(1+4\alpha-\alpha\gamma)} = 0. \tag{2.3.5}$$

On solving (2.3.5) for $\gamma \neq 0$, we get

$$H = \frac{1}{\left[c_1 + \frac{3(1+2\alpha)\gamma}{2(1+4\alpha-\alpha\gamma)}\, t\right]}, \tag{2.3.6}$$

where c_1 is a constant of integration. Using $H = \dot{a}/a$, equation (2.3.6) gives the power–law expansion for the scale factor of the form

$$a = c_2 \left[c_1 + \frac{3\gamma(1+2\alpha)}{2(1+4\alpha-\alpha\gamma)}\, t\right]^{\frac{2(1+4\alpha-\alpha\gamma)}{3\gamma(1+2\alpha)}}, \quad (\alpha \neq -1/2), \tag{2.3.7}$$

where c_2 is another constant of integration. This scale factor can be rewritten as

$$a = a_0 \left[1 + \frac{3\gamma(1+2\alpha)H_0}{2(1+4\alpha-\alpha\gamma)}\, t\right]^{\frac{2(1+4\alpha-\alpha\gamma)}{3\gamma(1+2\alpha)}}, \tag{2.3.8}$$

where $H = H_0 > 0$ at $t = t_0$. The cosmic time t_0 corresponds to the time where dark component begins to become dominant. From (2.3.4), the energy density in terms of t is given by

$$\rho = \rho_0 \left[1 + \frac{3\gamma(1+2\alpha)H_0}{2(1+4\alpha-\alpha\gamma)}\, t\right]^{-2}, \tag{2.3.9}$$

where $\rho_0 = 3H_0^2/(1+4\alpha-\alpha\gamma)$. For $\gamma < 0$, we get a big-rip singularity at finite time $t_{br} = -2(1+4\alpha-\alpha\gamma)/3\gamma(1+2\alpha)H_0 > t_0$ as the scale factor and energy density tend to infinite at this time.

The cosmological inflation or the accelerated expansion of the Universe is characterized by the DP defined by Eq. (1.7.5). In this case, we get

$$q = \frac{3\gamma(1+2\alpha)}{2(1+4\alpha-\alpha\gamma)} - 1, \tag{2.3.10}$$

which is constant through out the evolution of the Universe. As we know that $q > 0$ determines expansion of Universe with decelerated rate, $q < 0$ describes the accelerated expansion of the Universe and $q = 0$ gives the coasting or marginal inflation. Thus, for suitable values of α and γ we can obtain decelerated and accelerated expansion of the Universe. In this case, the model does not exhibit phase transition due to constant value of q.

For $\gamma = 0$, i.e., $p = -\rho$, Eq. (2.3.5) gives $H = H_0$, which corresponds to $a = a_0 e^{H_0 t}$. Thus, we observe a de-Sitter type expansion of the Universe. Both ρ and p are constant and $q = -1$ through out the evolution of the Universe.

2.3.2 Cosmology with Viscous Fluid

On the thermodynamical grounds, ζ in (2.3.3) is conventionally chosen to be a positive quantity and may depend on the cosmic time t, or the scale factor a, or the energy density ρ. Therefore, different forms of viscosity can be used to make Eq. (2.3.1) solvable numerically or exactly. We investigate in the following some different choices for ζ.

Case I : Solution with Constant Bulk Viscosity

From bulk viscosity point of view, the simplest case is thought to be a constant bulk viscosity. Therefore, assuming $\zeta_1 = 0$ we get $\zeta = \zeta_0$. In this case, Eq. (2.3.4) reduces to

$$\rho = \frac{3H^2 - 3\alpha\zeta_0 H}{(1+4\alpha-\alpha\gamma)}. \tag{2.3.11}$$

Substituting (2.3.2) and (2.3.11) into (2.3.1), we get

$$\dot{H} + \frac{3}{2}\frac{\gamma(1+2\alpha)H}{(1+4\alpha-\alpha\gamma)}\left[H - \frac{\zeta_0(1+4\alpha)}{\gamma}\right] = 0. \tag{2.3.12}$$

In what follows we solve (2.3.12) for $\gamma \neq 0$ and $\gamma = 0$ separately.

(i) Solution for $\gamma \neq 0$:

Solving (2.3.12) for $\gamma \neq 0$, we find

$$H = \frac{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}}{c_3 + \frac{\gamma}{(1+4\alpha)\zeta_0}\, e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}}, \tag{2.3.13}$$

where c_3 is a constant of integration. Using $H = \dot{a}/a$, the scale factor in terms of t is given by

$$a = c_4 \left[c_3 + \frac{\gamma}{(1+4\alpha)\zeta_0}\, e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t} \right]^{\frac{2}{3}\frac{(1+4\alpha-\alpha\gamma)}{\gamma(1+2\alpha)}}, \tag{2.3.14}$$

where $c_4 > 0$ is another integration constant. This scale factor may be rewritten as

$$a(t) = a_0 \left[1 + \frac{\gamma H_0}{(1+4\alpha)\zeta_0} \left(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t} - 1 \right) \right]^{\frac{2}{3}\frac{(1+4\alpha-\alpha\gamma)}{\gamma(1+2\alpha)}}. \tag{2.3.15}$$

The energy density can be calculated as

$$\begin{aligned} \rho &= \frac{3H_0}{(1+4\alpha-\alpha\gamma)} \left[\frac{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}}{1 + \frac{\gamma H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t} - 1 \right)} \right] \\ &\times \left[\frac{H_0\, e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}}{1 + \frac{\gamma H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t} - 1 \right)} - \alpha\, \zeta_0 \right]. \end{aligned} \tag{2.3.16}$$

For $0 \leq \gamma \leq 2$, viscous solution satisfies the dominant energy condition (DEC), i.e., $\rho + p \geq 0$. If $\gamma < 0$ we have a big–rip singularity at a finite value of cosmic time

$$t_{br} = \frac{2(1+4\alpha-\alpha\gamma)}{3(1+2\alpha)(1+4\alpha)\zeta_0}\, ln\left(1 - \frac{(1+4\alpha)\zeta_0}{\gamma H_0} \right) > t_0. \tag{2.3.17}$$

One may observe that there is a violation of DEC. The energy density grows up to infinity at a finite time $t_{br} > t_0$, which leads to a big–rip singularity characterized by the scale factor and Hubble parameter blowing up to infinity at this finite time. Therefore, there are cosmological models with viscous fluid which may develop a sudden future singularity.

Let us discuss the dynamical behaviour of the Universe using DP. The DP for this

Table 2.1: Variation of q for $\gamma = \frac{2}{3}$.

Range of α	Constraints on ζ_0	q	Evolution of Universe
$\alpha > 0$	for all $\zeta_0 > 0$	negative	accelerated expansion
$-0.25 < \alpha < 0$	$0 < \zeta_0 < \frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	+ve to -ve	transition from dec. to acc.
	$\zeta_0 \geq \frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	negative	accelerated expansion
$-0.30 < \alpha \leq -0.25$	for all $\zeta_0 > 0$	positive	decelerated expansion
$-0.50 \leq \alpha < -0.30$	for all $\zeta_0 > 0$	negative	accelerated expansion
$\alpha < -0.50$	$0 < \zeta_0 < \frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	-ve to +ve	transition from acc. to dec.
	$\zeta_0 \geq \frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	positive	decelerated expansion

Table 2.2: Variation of q for $\gamma = \frac{4}{3}$.

Range of α	Constraints on ζ_0	q	Evolution of Universe
$\alpha > -0.25$ ($\alpha \neq 0$)	$0 < \zeta_0 < \frac{2(3+4\alpha)H_0}{9(1+4\alpha)(1+2\alpha)}$	+ve to -ve	transition from dec. to acc.
	$\zeta_0 \geq \frac{2(3+4\alpha)H_0}{9(1+4\alpha)(1+2\alpha)}$	negative	accelerated expantion
$-0.375 < \alpha \leq -0.25$	for all $\zeta_0 > 0$	positive	decelerated expansion
$-.50 \leq \alpha < -0.375$	for all $\zeta_0 > 0$	negative	accelerated expansion
$\alpha < -0.50$	$0 < \zeta_0 < \frac{2(3+4\alpha)H_0}{9(1+4\alpha)(1+2\alpha)}$	-ve to +ve	transition from acc. to dec.
	$\zeta_0 \geq \frac{2(3+4\alpha)H_0}{9(1+4\alpha)(1+2\alpha)}$	positive	decelerated expansion

case is calculated as

$$q = \frac{\frac{3}{2}\frac{(1+2\alpha)}{(1+4\alpha-\alpha\gamma)}[\gamma - \frac{(1+4\alpha)\zeta_0}{H_0}]}{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}} - 1, \tag{2.3.18}$$

which is time dependent in contrast to the perfect fluid. Thus, the constant bulk viscous coefficient generates time dependent q which may describe the transition phases of the Universe along with deceleration or acceleration of the Universe. Let us observe the variation of q with bulk viscous coefficient ζ_0 for various ranges of α in different phases of evolution of the Universe for $\gamma > 0$, which are presented in the following Tables (2.1)–(2.3).

We observe from the Tables (2.1), (2.2) and (2.3) that the Universe accelerates through out the evolution when $\alpha > 0$ for any $\zeta_0 > 0$ in inflationary phase where as it shows transition from decelerated phase to accelerated phase when $\alpha > -0.25$ for s-

Table 2.3: Variation of q for $\gamma = 1$.

Range of α	Constraints on ζ_0	q	Evolution of Universe
$\alpha > -0.25$ ($\alpha \neq 0$)	$0 < \zeta_0 < \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	+ve to -ve	transition from dec. to acc.
	$\zeta_0 \geq \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	negative	accelerated expantion
$-0.33 < \alpha \leq -0.25$	for all $\zeta_0 > 0$	positive	decelerated expansion
$-0.50 \leq \alpha < -0.33$	for all $\zeta_0 > 0$	negative	accelerated expansion
$\alpha < -0.50$	$0 < \zeta_0 < \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	-ve to +ve	transition from acc. to dec.
	$\zeta_0 \geq \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	positive	decelerated expansion

maller values of ζ_0 and acceleration for larger values of ζ_0 in radiation and matter dominated phases. It is to be noted here that the larger values of ζ_0 makes the effective pressure more negative to accelerate the Universe through out the evolution. We find that the Universe decelerates in $-0.30 < \alpha \leq -0.25$ for $\gamma = 2/3$, in $-0.375 < \alpha \leq -0.25$ for $\gamma = 4/3$ and in $-0.33 < \alpha \leq -0.25$ for $\gamma = 1$ for all $\zeta_0 > 0$. Further, we find that the Universe accelerates in $-0.50 \leq \alpha < -0.30$ for $\gamma = 2/3$, in $-0.50 \leq \alpha < -0.375$ for $\gamma = 4/3$ and in $-0.50 \leq \alpha < -0.33$ for $\gamma = 1$ for all $\zeta_0 > 0$. When $\alpha < -0.50$, then model shows transition from acceleration to deceleration phase for small values of ζ_0 and decelerates for large values of ζ_0. In conclusion, we can say that the Universe accelerates or shows transition from decelerated phase to accelerated phase for $\alpha > -0.25$ with constant viscous term in all phases of its evolution.

For $\gamma < 0$, where the solution has a big–rip singularity, q is always negative for $\alpha > 0$ during any cosmic time.

(ii) Solution for $\gamma = 0$:

In this case, the evolution Eq. (2.3.12) of H reduces to

$$\dot{H} - \frac{3}{2}(1+2\alpha)\zeta_0 H = 0, \tag{2.3.19}$$

which gives the solution for H in terms of t as

$$H = c_5 e^{\frac{3}{2}(1+2\alpha)\zeta_0 t}, \tag{2.3.20}$$

where $c_5 > 0$ is a constant of integration. The scale factor in terms of t is given by

$$a = c_6\, e^{\frac{2c_5}{3(1+2\alpha)\zeta_0}\, e^{\frac{3}{2}(1+2\alpha)\zeta_0 t}}, \tag{2.3.21}$$

where $c_6 > 0$ is another constant of integration. This scale factor may be rewritten as

$$a = a_0\, e^{\frac{2H_0}{3(1+2\alpha)\zeta_0}\,(e^{\frac{3}{2}(1+2\alpha)\zeta_0 t} - 1)}. \tag{2.3.22}$$

We find that the scale factor shows the superinflation in the presence of constant bulk viscous coefficient where as it has de-Sitter expansion in non–viscous case. The

Table 2.4: Variation of q for $\gamma = 0$.

Range of α	Constraints on ζ_0	q	Evolution of Universe
$\alpha \geq -0.50$ ($\alpha \neq 0$)	for all $\zeta_0 > 0$	negative	accelerated expansion
$\alpha < -0.50$	$0 < \zeta_0 < \frac{-2H_0}{3(1+2\alpha)}$	-ve to +ve	transition from acc. to dec.
	$\zeta_0 \geq \frac{2H_0}{3(1+2\alpha)}$	positive	decelerated expansion

energy density is given by

$$\rho = \frac{3H_0\, e^{\frac{3}{2}(1+2\alpha)\zeta_0 t}}{(1+4\alpha)} \left[H_0 e^{\frac{3}{2}(1+2\alpha)\zeta_0 t} - \alpha\zeta_0\right]. \tag{2.3.23}$$

We observe that ρ varies with time in contrast to perfect fluid solution where it is constant. Both $a(t)$ and ρ tend to constant at $t = 0$ and tend to infinity at $t \to \infty$. In this case the deceleration parameter is given by

$$q = -\frac{3\zeta_0(1+2\alpha)}{2H_0}\, e^{-\frac{3}{2}(1+2\alpha)\zeta_0 t} - 1, \tag{2.3.24}$$

which is time–dependent. We study the variation of q with bulk viscous coefficient ζ_0 for various ranges of α, which are summarized in Table (2.4). From the table we observe that q is always negative for $\alpha \geq -0.50$ and the universe accelerates for all values of $\zeta_0 > 0$. For $\alpha < -0.50$, the value of q varies from negative to positive, that is, the universe shows transition from accelerated phase to decelerated phase for small values of ζ_0 and always decelerates for large values of ζ_0.

Case II : Solution with Variable Bulk Viscosity

In this section, we present the solutions for $\zeta_0 = 0$ and $\zeta_0 \neq 0$.

(i) When $\zeta_0 = 0$, i.e., $\zeta = \zeta_1 H$:

For $\zeta_0 = 0$, the form of bulk viscosity assumed in (2.3.3) reduces to $\zeta = \zeta_1 H$ [209]. As ζ is assumed to be positive, for this the constant ζ_1 must be positive.

Using (2.3.2) and (2.3.4) into (2.3.1) we get

$$\dot{H} + \frac{3}{2}\frac{(1+2\alpha)(\gamma - \zeta_1 - 4\alpha\zeta_1)}{(1+4\alpha-\alpha\gamma)} H^2 = 0. \tag{2.3.25}$$

Solving (2.3.25), the Hubble parameter can be obtained in terms of t for any value of γ

in the range $0 \leq \gamma \leq 2$, except $\gamma \neq (1+4\alpha)\zeta_1$ and $\alpha \neq -1/2$ as

$$H = \frac{1}{[\frac{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma)}\, t + c_7]}, \tag{2.3.26}$$

where c_7 is a constant of integration. Correspondingly, the scale factor in terms of t is given by

$$a = c_8 \left[c_7 + \frac{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma)}\, t \right]^{\frac{2(1+4\alpha-\alpha\gamma)}{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)}}, \tag{2.3.27}$$

where $c_8 > 0$ is another constant of integration. We find that the form $\zeta = \zeta_1 H$ yields a power–law expansion for the scale factor. This scale factor may be rewritten as

$$a = a_0 \left[1 + \frac{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)H_0}{2(1+4\alpha-\alpha\gamma)}\, (t-t_0) \right]^{\frac{2(1+4\alpha-\alpha\gamma)}{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)}}, \tag{2.3.28}$$

where $H = H_0$ at $t = t_0$ and t_0 corresponds to the time where dark component begins to dominant, i.e., describes the present value. The energy density and bulk viscous coefficient are respectively given by

$$\rho = \frac{3(1-\alpha\zeta_1)H_0^2}{(1+4\alpha-\alpha\gamma)} \left[1 + \frac{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)H_0}{2(1+4\alpha-\alpha\gamma)}\, (t-t_0) \right]^{-2}, \tag{2.3.29}$$

$$\zeta = \zeta_1 H_0 \left[1 + \frac{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)H_0}{2(1+4\alpha-\alpha\gamma)}\, (t-t_0) \right]^{-1}. \tag{2.3.30}$$

In this case, we get deceleration parameter as

$$q = \frac{3(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma)} - 1, \tag{2.3.31}$$

which is constant. The sign of q depends on the values of parameters γ, α and ζ_1. It is always negative for $\gamma \leq 0$, $\alpha > 0$ and $\zeta_1 > 0$.

If we demand to have the occurrence of a big–rip singularity in the future then we have following constraint on the parameters α, ζ_1 and γ

$$\zeta_1(1+4\alpha) > \gamma, \tag{2.3.32}$$

which leads the scale factor and energy density tending to infinity at a finite time

$$t_{br} = \frac{2(1+4\alpha-\alpha\gamma)}{3(1+2\alpha)(\zeta_1+4\alpha\zeta_1-\gamma)} H_0^{-1}. \tag{2.3.33}$$

We can observe from (2.3.28) and (2.3.29) that the energy density of the dark component increases with scale factor for $\zeta_1(1+4\alpha) > \gamma$.

(ii) When $\zeta_0 \neq 0$, i.e., $\zeta = \zeta_0 + \zeta_1 H$:

Using (2.3.2), (2.3.3) and (2.3.4) into (2.3.1) we get

$$\dot{H} + \frac{3}{2}\frac{(1+2\alpha)(\gamma-\zeta_1-4\alpha\zeta_1)}{1+4\alpha-\alpha\gamma} H\left[H - \frac{(1+4\alpha)\zeta_0}{\gamma-\zeta_1-4\alpha\zeta_1}\right] = 0. \tag{2.3.34}$$

Solving (2.3.34), we find the following solution for any value of γ in the range $0 \leq \gamma \leq 2$, except $\gamma \neq (\zeta_1 + 4\alpha\zeta_1)$ and $\alpha \neq -1/2$ as

$$H = \frac{e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma)}t}}{c_9 + \frac{(\gamma-\zeta_1-4\alpha\zeta_1)}{(1+4\alpha)\zeta_0} e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma)}t}}, \tag{2.3.35}$$

where c_9 is constant of integration. The scale factor in terms of t is given by

$$a = c_{10}\left[c_9 + \frac{(\gamma-\zeta_1-4\alpha\zeta_1)}{(1+4\alpha)\zeta_0}\, e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}\right]^{\frac{2(1+4\alpha-\alpha\gamma)}{3(\gamma-\zeta_1-4\alpha\zeta_1)(1+2\alpha)}}, \tag{2.3.36}$$

where $c_{10} > 0$ is a constant of integration. This scale factor may be rewritten as

$$a = a_0\left[1 + \frac{H_0(\gamma-\zeta_1-4\alpha\zeta_1)}{(1+4\alpha)\zeta_0}\left(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t} - 1\right)\right]^{\frac{2(1+4\alpha-\alpha\gamma)}{3(\gamma-\zeta_1-4\alpha\zeta_1)(1+2\alpha)}}. \tag{2.3.37}$$

The energy density ρ and bulk viscosity ζ can be calculated as

$$\begin{aligned}\rho &= \frac{3H_0}{(1+4\alpha-\alpha\gamma)} \frac{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}}{\left[1+\frac{(\gamma-\zeta_1-4\alpha\zeta_1)H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}-1\right)\right]} \\ &\times\left[\frac{(1-\alpha\zeta_0)H_0\, e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}}{1+\frac{(\gamma-\zeta_1-4\alpha\zeta_1)H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}-1\right)} - \alpha\,\zeta_0\right],\end{aligned} \tag{2.3.38}$$

$$\zeta = \zeta_0 + \zeta_1\left[\frac{H_0\, e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma)}t}}{1+\frac{H_0(\gamma-\zeta_1-4\alpha\zeta_1)}{(1+4\alpha)\zeta_0}\left(e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma)}t}-1\right)}\right]. \tag{2.3.39}$$

Table 2.5: Variation of q for $\gamma=\frac{2}{3}$.

Range of α	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$\alpha>0$	for all $\zeta_0>0$ and $\zeta_1>0$	negative	accelerated expansion
$-0.25<\alpha<0$	$0<(\zeta_0+H_0\zeta_1)<\frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	+ve to -ve	transition from dec. to acc.
	$(\zeta_0+H_0\zeta_1)\geq\frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	negative	accelerated expansion
$-0.30<\alpha\leq-0.25$	for all $\zeta_0>0$ and $\zeta_1>0$	positive	decelerated expansion
$-0.50\leq\alpha<-0.30$	for all $\zeta_0>0$ and $\zeta_1>0$	negative	accelerated expansion
$\alpha<-0.50$	$0<(\zeta_0+H_0\zeta_1)<\frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	-ve to +ve	transition from acc. to dec.
	$(\zeta_0+H_0\zeta_1)\geq\frac{-8\alpha H_0}{9(1+2\alpha)(1+4\alpha)}$	positive	decelerated expansion

Table 2.6: Variation of q for $\gamma=\frac{4}{3}$.

Range of α	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$\alpha>-0.25$ $(\alpha\neq0)$	$0<(\zeta_0+H_0\zeta_1)<\frac{2(3+4\alpha)H_0}{9(1+2\alpha)(1+4\alpha)}$	+ve to -ve	transition from dec. to acc.
	$(\zeta_0+H_0\zeta_1)\geq\frac{2(3+4\alpha)H_0}{9(1+2\alpha)(1+4\alpha)}$	negative	accelerated expantion
$-0.375<\alpha\leq-0.25$	for all $0<(\zeta_0+H_0\zeta_1)$	positive	decelerated expansion
$-0.50\leq\alpha<-0.375$	for all $0<(\zeta_0+H_0\zeta_1)$	negative	accelerated expansion
$\alpha<-0.50$	$0<(\zeta_0+H_0\zeta_1)<\frac{2(3+4\alpha)H_0}{9(1+2\alpha)(1+4\alpha)}$	-ve to +ve	transition from acc. to dec.
	$(\zeta_0+H_0\zeta_1)\geq\frac{2(3+4\alpha)H_0}{9(1+2\alpha)(1+4\alpha)}$	positive	decelerated expansion

With the constraint given in (2.3.32), we get a big–rip singularity at

$$t_{br}=\frac{2(1+4\alpha-\alpha\gamma)}{3(1+2\alpha)(1+4\alpha)\zeta_0}\,ln\left(1+\frac{(1+4\alpha)\zeta_0}{H_0(\zeta_1+4\alpha\zeta_1-\gamma)}\right). \tag{2.3.40}$$

In this case the deceleration parameter is given by

$$q=\frac{\frac{3}{2}\frac{(1+2\alpha)}{(1+4\alpha-\alpha\gamma)}[(\gamma-\zeta_1-4\alpha\zeta_1)-\frac{(1+4\alpha)\zeta_0}{H_0}]}{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma)}t}}-1, \tag{2.3.41}$$

which shows that q is time–dependent. Therefore, we study the variation of q with bulk viscous coefficient $\zeta_0+\zeta_1 H$ for various ranges of α in different phases of evolution of the Universe, which are summarized in Tables (2.5)–(2.8).

We observe from the Tables (2.5)-(2.8) that the Universe accelerates through out the evolution when $\alpha>0$ for any positive values of ζ_0 and ζ_1 in inflationary phase where as it shows transition from decelerated phase to accelerated phase when $\alpha>-0.25$ for smaller values of $(\zeta_0+H_0\zeta_1)$ and shows acceleration for larger values of $(\zeta_0+H_0\zeta_1)$ in radiation and matter dominated phases. It is due to the fact that the larger values of ζ_0 or ζ_1 or both make the effective pressure more negative to accelerate the Universe through out the evolution. We find that the Universe decelerates in $-0.30<\alpha\leq-0.25$ for $\gamma=2/3$, in $-0.375<\alpha\leq-0.25$ for $\gamma=4/3$ and in $-0.33<\alpha\leq-0.25$ for $\gamma=1$ for all

Table 2.7: Variation of q for $\gamma = 1$.

Range of α	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$\alpha > -0.25$ ($\alpha \neq 0$)	$0 < (\zeta_0 + H_0\zeta_1) < \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	+ve to -ve	transition from dec. to acc.
	$(\zeta_0 + H_0\zeta_1) \geq \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	negative	accelerated expantion
$-0.33 < \alpha \leq -0.25$	for all $\zeta_0 > 0$ and $\zeta_1 > 0$	positive	decelerated expansion
$-0.50 \leq \alpha < -0.33$	for all $\zeta_0 > 0$ and $\zeta_1 > 0$	negative	accelerated expansion
$\alpha < -0.50$	$0 < (\zeta_0 + H_0\zeta_1) < \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	-ve to +ve	transition from acc. to dec.
	$(\zeta_0 + H_0\zeta_1) \geq \frac{H_0}{3(1+2\alpha)(1+4\alpha)}$	positive	decelerated expansion

Table 2.8: Variation of q for $\gamma = 0$.

Range of α	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$\alpha \geq -0.50$ ($\alpha \neq 0$)	for all $\zeta_0 > 0$ and $\zeta_1 > 0$	negative	accelerated expansion
$\alpha < -0.50$	$0 < (\zeta_0 + H_0\zeta_1) < -\frac{2H_0}{3(1+2\alpha)}$	-ve to +ve	transition from acc. to dec.
	$(\zeta_0 + H_0\zeta_1) \geq -\frac{2H_0}{3(1+2\alpha)}$	positive	decelerated expansion

$\zeta_0 > 0$ and $\zeta_1 > 0$. Further, we find that the Universe accelerates in $-0.50 \leq \alpha < -0.30$ for $\gamma = 2/3$, in $-0.50 \leq \alpha < -0.375$ for $\gamma = 4/3$ and in $-0.50 \leq \alpha < -0.33$ for $\gamma = 1$ for all $\zeta_0 > 0$ and $\zeta_1 > 0$. For $\gamma = 0$ and $\alpha \geq -0.50$, we observe that Universe shows accelerated expansion for positive values of ζ_0 and ζ_1. When $\alpha < -0.50$, the model shows transition from acceleration to deceleration for small values of $\zeta_0 + H_0\zeta_1$ and decelerates for large values of $\zeta_0 + H_0\zeta_1$ for all phases.

For $\zeta_1(1+4\alpha) > \gamma$, where the model has a big–rip singularity, q is always negative for $\alpha > 0$ during any cosmic time. We also see from (2.3.25) and (2.3.34) that $\dot{H} = 0$ for $\zeta_1(1+4\alpha) = \gamma$ or $\alpha = -1/2$, which shows de-Sitter expansion of scale factor in both cases. The solutions for $\gamma = 0$ can be obtained directly from (2.3.28) and (2.3.37), and the big–rip singularity can be observed at $t_{br} = 2/[3(1+2\alpha)\zeta_1 H_0] > t_0$ in for the case $\zeta_0 = 0$ and $t_{br} = \frac{2}{3(1+2\alpha)\zeta_0}\, ln\left(1 + \frac{\zeta_0}{H_0\zeta_1}\right) > t_0$ for the case $\zeta_0 \neq 0$, respectively.

2.4 Conclusion

In this chapter, we have extensively studied the effects of viscous fluid in $f(R,T)$ gravity within the framework of a flat FRW model. We have investigated first order relativistic thermodynamic Eckart theory of dissipative processes. We have discussed the expansion history of the Universe with and without bulk viscosity. We have found cosmological solutions which exhibit a big–rip singularity under certain constraints. Therefore, the negative pressure generated by the bulk viscosity cannot avoid the dark energy of the universe to be phantom. It is to be noted that we have discussed the various phases and their possible transitions for all possible ranges of α with ζ_0

and ζ_1, which have not been studied in the past in the cited works. It contains many new solutions like power–law, exponential, and superinflationary scale factors by assuming the same form of $f(R,T)$. We have obtained both constant and time dependent DP which describe the decelerated/accelerated phases and the transition from the decelerated to the accelerated phase. As we have considered all possible positive and negative ranges of α with the inclusion of a viscous term in $f(R,T)$ theory, which is of physical interest, one may see that our results are relatively more generalized than earlier work. We summarize the results as follows.

In case of perfect fluid distribution as presented in Sect. 2.3.1, a power–law expansion for the scale factor has been obtained for $\gamma \neq 0$. It has found that the energy density decreases with time and the DP is constant throughout the evolution. For $\gamma = 0$ i.e. $p = -\rho$, we have de-Sitter solution of scale factor and both p and ρ are constant. For $\gamma < 0$, there is a big–rip singularity at a finite value of cosmic time describing the phantom cosmology.

In case of viscous cosmology, we have explored bulk viscous model composed by perfect fluid with bulk viscosity of the form $\zeta = \zeta_0 + \zeta_1 H$. We have discussed three different cases depending on the composition of the bulk viscosity. In case of constant coefficient of bulk viscosity, the scale factor varies exponentially for $\gamma \neq 0$, which avoids the Big Bang singularity. We have observed that the constant viscous term generates the time dependent DP, which describes different phases of the Universe and transition to accelerated phase. We have shown the variation of q and the corresponding evolution of the Universe in Tables (2.1)-(2.4) for various ranges of α. In this case, we also have the big–rip singularity at finite value of cosmic time for $\gamma < 0$. When $\gamma = 0$, a superinflation for the scale factor has been found whereas it has de-Sitter solution in non–viscous case.

In case where the bulk viscosity is proportional to Hubble parameter, we have obtained a power–law expansion for the scale factor similar to the perfect fluid model. The energy density varies inversely as the square of the cosmic time whereas the bulk viscosity decreases linearly with time. In this case the value of q is constant and the sign of q depends on γ, α and ζ_1. This form of bulk viscosity generates a big–rip singularity when the constraint given in Eq. (2.3.32) holds.

In case of $\zeta = \zeta_0 + \zeta_1 H$, we again have obtained exponential form of expansion for the scale factor which is similar to the form with constant bulk viscous term, and time dependent DP which can explain the expansion history of the Universe and the transition to accelerated or decelerated phase, see Tables (2.5)-(2.8).

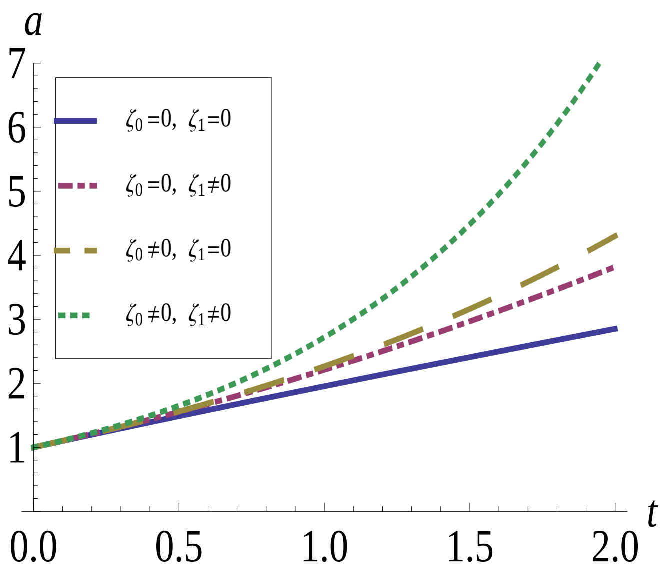

Figure 2.1: Scale factor in terms of cosmic time for $\alpha > 0$. Here $\gamma = \alpha = a_0 = H_0 = 1$ and $\zeta_0 = \zeta_1 = 0.1$.

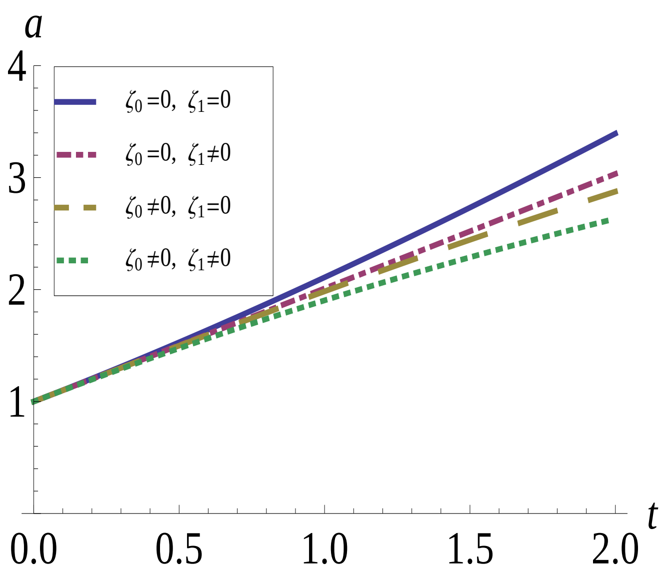

Figure 2.2: Scale factor in terms of cosmic time for $\alpha < 0$. Here $\alpha = -1$, $\gamma = a_0 = H_0 = 1$ and $\zeta_0 = \zeta_1 = 0.1$

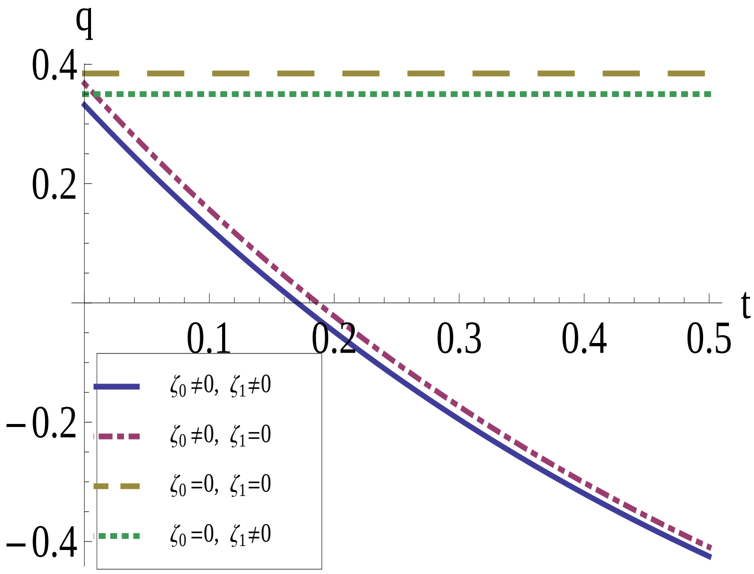

Figure 2.3: Deceleration parameter in terms of cosmic time for $\alpha > 0$. Here $\gamma = H_0 = 1, \alpha = 0.1, \zeta_0 = 0.01$ and $\zeta_1 = 0.02$.

The behavior of scale factor with cosmic time has been shown in Figs. 2.1 and 2.2 for different models with the particular choices of bulk viscosity. It has been observed that the scale factor increases rapidly in all viscous models as compared to the non–viscous model (perfect fluid model) for any positive values of α as shown in Fig. 2.1. The rate of expansion depends on the bulk viscous coefficient. The scale factor varies very close to non–viscous model when bulk viscous coefficient is assumed to be small. Fig. 2.1 shows that the expansion of scale factors deviate more rapidly from the perfect fluid expansion rate for larger bulk viscous coefficient and the rate of expansion is very fast when the bulk viscosity is considered in the form $\zeta = \zeta_0 + \zeta_1 H$. The behavior of scale factor for different models is reverse for $\alpha < 0$ as shown in Fig. 2.2. The variation of deceleration parameter with cosmic time has been shown in Fig. 2.3 for different models. The models with $\zeta = 0$ (perfect fluid) and $\zeta = \zeta_1 H$, where the scale factor vary as power–law of cosmic time, the values of q are constant. On the other hand, in models $\zeta = \zeta_0$ and $\zeta = \zeta_0 + \zeta_1 H$, where the scale factor varies exponentially, q is time dependent and transitions from positive to negative values are observed.

The main conclusion, in the context of the bulk viscous, is that the form of $\zeta = \zeta_0 + \zeta_1 H$ is suitable to describe the power–law and exponential expansion for the scale factor. It is also possible to obtain big–rip type solutions under certain constraints.

Therefore, we can explore the possibility that the present acceleration of the Universe is driven by a kind of viscous fluid by assuming this form of bulk viscous term. Though the viscous dark energy form is simple, the results are better. We emphasize that perfect fluid is just a limiting case of a general viscous media that is more practical in the astrophysical sense. Therefore, it is worthy to study the early and late time evolution of the Universe with viscous fluid.

Chapter 3

Viscous Cosmology with Matter Creation in $f(R,T)$ Gravity

In this chapter[1], we extend the study of previous chapter considering viscous fluid and matter creation together in the modified $f(R,T)$ theory of gravitation. We assume the bulk viscosity and the matter creation as two independent processes as discussed by Prigogine et al. [189, 190]. We obtain various forms of the scale factor for constant and time–dependent bulk viscous coefficient using equation of state $p = (\gamma - 1)\rho$. All possible (deceleration, acceleration and their transitions) evolution of the Universe are discussed by constraining the model on β, ζ_0 and ζ_1 in case of time–dependent DP for whole range of α. It is also noted that the finite time big–rip singularity can be removed for a specific range of α in phantom region. The role of bulk viscosity and matter creation are discussed in detail through the variation of DP.

[1]The result of this chapter is based on a research paper "Viscous cosmology with matter creation in modified $f(R,T)$ gravity, *Astrophysics and Space Science* **357**, 120 (2015)".

3.1 Introduction

It has been observed in the $f(R,T)$ theory that the motion of the particles does not take place along a geodesic path by using energy–momentum tensor as a source because there is an extra force perpendicular to the four–velocity unless we add the constraint of conservation of the energy–momentum tensor. In a paper, Harko [140] has presented the thermodynamical interpretation of the modified theories of gravity with the non–minimal matter–geometry coupling. The author has interpreted a generalized conservation equations from a thermodynamical point of view by using the formalism of open thermodynamics systems describing irreversible matter creation.

In the framework of GTR, the adiabatic irreversible particle creation was first time studied by Prigogine et al. [189, 190]. This particle creation corresponds to an irreversible energy flow from the gravitational field to the created matter field. The phenomenological cosmological particle creation has been extensively discussed in literature within the context of standard GTR [194, 210–213]. In the context of the recent acceleration, the concept of irreversible particle creation has been reconsidered due to its capability to produce an effective negative pressure [174, 196–200].

In chapter 2, we have investigated the effects of bulk viscosity in modified $f(R,T)$ gravity. In the present chapter, we extend our study with irreversible particle creation thermodynamics in open Universe and discuss the separate and combined effects of bulk viscosity and particle creation. On a phenomenological level, particle creation model has been described in the literature in terms of a bulk viscous stress [171, 201]. However, Prigogine et al. [189, 190] pointed out that the bulk viscosity and particle creation are not only two independent processes but, in general, lead to different histories of the evolution of the Universe. Triginer and Pavón [202], and Brevik and Stokkan [203] discussed about the equivalence between bulk viscosity and particle creation, and concluded that however, the dynamics of both the concepts might be same, but thermodynamically both had the different aspect of the Universe.

In the literature, the bulk viscosity with the particle creation have been studied to discuss the behaviour of the Universe (see Ref. [205] and therein). Motivated by these works, it will be interesting for us to discuss the bulk viscous cosmology and thermodynamics of the particle creation in the framework of the modified $f(R,T)$ gravity with the concept that these two fluids have different histories of the evolution of the Universe as suggested by Prigogine et al. [189].

This chapter explores the effects of the bulk viscosity and adiabatic irreversible thermodynamics of open systems with particle creation in the framework of $f(R,T)$ gravity with reference to the question whether these may cause inflation in the early and the late time acceleration of the Universe. Considering bulk viscosity and particle creation as separate irreversible processes, the field equations are obtained in $f(R,T)$ gravity theory. Here we restrict ourselves to adiabatic particle production processes. It is found that, by choosing the appropriate form of the bulk viscous coefficient and the particle creation rate, various forms of the non–singular solutions exhibit for all possible ranges of parameters. It is also noted that a finite time big–rip singularity is observed for $\gamma < 0$ but it may be avoided in some specific ranges of the parameters.

3.2 Metric and Field Equations

We consider a homogeneous and isotropic flat FRW line element given by Eq. (1.1.5). The concept of particle creation in adiabatic irreversible open thermodynamic systems and bulk viscous stress to cosmology lead to modify the energy–momentum tensor of perfect fluid (1.1.9) as [202]

$$T_{\mu\nu} = (\rho + P_{eff}) u_\mu \, u_\nu - g_{\mu\nu} \, P_{eff}. \tag{3.2.1}$$

Here ρ is the energy density, and P_{eff} stands for effective pressure which includes the pressure of perfect fluid, bulk viscous and particle creation, that is

$$P_{eff} = p + \Pi + p_c, \tag{3.2.2}$$

where p is the thermodynamical pressure and p_c denotes the pressure associated with the matter creation out of the gravitational field [189, 190]. Here, Π represents the viscous pressure which conventionally in the first order thermodynamics theory defined by Eckart (1940) [166] is assumed as $\Pi = -3\zeta H$.

Now, the matter Lagrangian may be chosen as $\mathscr{L}_m = -P_{eff}$ and the trace of the effective energy–momentum tensor is given by $T = \rho - 3P_{eff}$. Therefore, the expression of tensor $\ominus_{\mu\nu}$ given in (1.4.15) modifies to

$$\ominus_{\mu\nu} = -2T_{\mu\nu} - g_{\mu\nu} \, p_{eff}. \tag{3.2.3}$$

Using (3.2.3) into (2.2.2), the field equations reduce

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = T_{\mu\nu} + 2(T_{\mu\nu} + P_{eff})f'(T) + f(T)g_{\mu\nu}. \tag{3.2.4}$$

Using $f(T) = \alpha T$, the field equation (3.2.4) for the line element (1.1.5) and energy–momentum tensor (3.2.1) yield

$$3H^2 = \rho + 2\alpha(\rho + P_{eff}) + \alpha T, \tag{3.2.5}$$

$$2\dot{H} + 3H^2 = -P_{eff} + \alpha T. \tag{3.2.6}$$

Here, a dot denotes derivative with respect to cosmic time t. We have two independent equations (3.2.5) and (3.2.6), and five unknown variables, namely H, ρ, p, p_c and ζ to be solved as functions of time. In the next section, we choose EoS, bulk viscosity and particle creation rate, and try to solve the equations for H.

3.3 Solution of Field Equations

From (3.2.5) and (3.2.6), we obtain a single evolution equation of H as

$$2\dot{H} + (1+2\alpha)(\rho + p + p_c) - 3(1+2\alpha)\zeta H = 0. \tag{3.3.1}$$

we use the EoS for prefect fluid given in Eq. (2.3.2) and the coefficient of bulk viscosity defined in (2.3.3). The thermodynamics of particle creation has already been discussed in Sect. 1.6 of introductory chapter 1.

Using (2.3.2) and (1.6.3) into (1.6.2), we get particle creation pressure as

$$p_c = -\beta\gamma\rho. \tag{3.3.2}$$

By the use of (2.3.2) and (3.3.2), Eq. (3.3.1) reduces to

$$2\dot{H} + (1+2\alpha)(1-\beta)\gamma\rho - 3(1+2\alpha)\zeta H = 0. \tag{3.3.3}$$

Also, from (3.2.5) we obtain the value of ρ as

$$\rho = \frac{3H(H - \alpha\zeta)}{1 + 4\alpha - \alpha\gamma + \alpha\beta\gamma}. \tag{3.3.4}$$

3.3.1 Cosmology with Constant Bulk Viscosity

The simplest choice of bulk viscosity is the constant bulk viscous coefficient. Therefore, assuming $\zeta_1 = 0$ in (2.3.3), Eq. (3.3.4) becomes

$$\rho = \frac{3H(H-\alpha\zeta_0)}{1+4\alpha-\alpha\gamma+\alpha\beta\gamma}\,. \tag{3.3.5}$$

Using (3.3.5) into (3.3.3), we obtain

$$\dot{H} + \frac{3}{2}\frac{\gamma\,(1+2\alpha)(1-\beta)H}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\left[H - \frac{(1+4\alpha)\zeta_0}{\gamma(1-\beta)}\right] = 0. \tag{3.3.6}$$

In what follows we solve this equation for $\gamma = 0$ and $\gamma \neq 0$, respectively.

Case I : Solution for $\gamma \neq 0$

Equation (3.3.6) may be solved for $\gamma \neq \frac{1+4\alpha}{\alpha(1-\beta)}$ and $0 \leq \beta < 1$ to obtain

$$H = \frac{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\,t}}{c_{11} + \frac{\gamma(1-\beta)}{(1+4\alpha)\zeta_0}\,e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\,t}}\,, \tag{3.3.7}$$

where c_{11} is a constant of integration. We obtain the cosmic scale factor which is given by

$$a = c_{12}\left[c_{11} + \frac{\gamma(1-\beta)}{(1+4\alpha)\zeta_0}\,e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\,t}\right]^{\frac{2}{3}\frac{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{\gamma(1+2\alpha)(1-\beta)}}\,, \tag{3.3.8}$$

where $c_{12} > 0$ is another integration constant. Here, we get an exponential scale factor which avoids the Big Bang singularity. We can rewrite (3.3.8) as

$$a(t) = a_0\left[1 + \frac{\gamma(1-\beta)H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\,t} - 1\right)\right]^{\frac{2}{3}\frac{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{\gamma(1+2\alpha)(1-\beta)}}\,, \tag{3.3.9}$$

where $H = H_0$ denotes the present value of the Hubble parameter at $t = t_0$ and t_0 denotes the present value of the cosmic time, where the dark component begins to dominant. By use of (3.3.9) energy density in (3.3.5) is given by

$$\rho = \frac{3H_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\left[\frac{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}}{1+\frac{\gamma(1-\beta)H_0}{(1+4\alpha)\zeta_0}(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}-1)}\right] \times\left[\frac{H_0\, e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}}{1+\frac{\gamma(1-\beta)H_0}{(1+4\alpha)\zeta_0}\,(e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}-1)}-\alpha\zeta_0\right]. \tag{3.3.10}$$

The above results show that the non–singular solution can be obtained with and without particle creation and this is due to the constant bulk viscous term. In the absence of viscous term, one may get power–law solution. Using (3.3.10) into (2.3.2) and (3.3.2), one can find respective thermodynamical pressure and particle creation pressure. One can easily find the particle creation rate and entropy production by the use of scale factor (3.3.9). For $0 \leq \gamma \leq 2$, solution satisfies the dominant energy condition (DEC), i.e., $\rho + p \geq 0$.

If $\gamma < 0$, we observe a big–rip singularity at a finite value of cosmic time

$$t_{br} = \frac{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{3(1+2\alpha)(1+4\alpha)\zeta_0}\, ln\left(1-\frac{(1+4\alpha)\zeta_0}{\gamma(1-\beta)H_0}\right) > t_0. \tag{3.3.11}$$

We find that energy density and the scale factor tend to infinity at this finite cosmic time t_{br}. It can be observed that this big–rip singularity may be avoided for some specific values of the parameters α, β and γ. Let us take the case $\gamma = -1$, then for $\alpha \in (-0.50, -0.25)$ and for any β, big–rip singularity will not appear. Further, for $\gamma > 0$ we observe a finite time big–rip singularity for specific values of parameters α and β. Let us take $\gamma = 1$, then in the interval $-0.50 < \alpha < -0.33$, we always have a big–rip singularity for any value of β lying in $0 \leq \beta < 1$. However, as we increase the upper value of α to -0.25, the model shows big–rip singularity for some selected values of β only in [0, 1). Thus, it is possible that the big–rip singularity may be avoided for some values of $0 \leq \beta < 1$.

To discuss the acceleration or deceleration of the Universe and its transition, we calculate the DP for this model which is given by

$$q = \frac{\frac{3(1+2\alpha)(1-\beta)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\left[\gamma-\frac{(1+4\alpha)\zeta_0}{(1-\beta)H_0}\right]}{e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}} - 1. \tag{3.3.12}$$

Here, we obtain a time–dependent DP. Thus, for suitable values of various parameters we can present the evolution of the Universe. If we solve (3.3.6) for $\zeta = 0$, i.e., $\zeta_0 = 0$, we obtain a power–law solution for scale factor which naturally leads to a constant DP. Therefore, due to the presence of constant bulk viscous coefficient we obtain time–dependent DP which is able to describe the phase transition of the Universe. The variation of q with respect to ζ_0 and β for various ranges of α are presented in Tables (3.1)–(3.3) to observe the role of viscosity and particle creation in different phases of evolution of the Universe for $\gamma > 0$.

From the tables it is observed that the Universe accelerates throughout the evolution for $\alpha > 0$, and all values of β and ζ_0 during inflation. In the range $\alpha > -0.25$ $(\alpha \neq 0)$ for matter and radiation phases, and in $0 < \alpha < -0.25$ for inflationary phase, we observe an accelerated expansion of the Universe for large values of ζ_0 and β. It is worthy to note that both the bulk viscosity and the particle creation are capable to accelerate the Universe without the need of either as these parameters make the effective pressure more negative.

The model shows the transition from a decelerated phase to an accelerated phase for small values of ζ_0 and β. We observe that in $-0.30 < \alpha \leq -0.25$ for $\gamma = 2/3$, in $-0.375 < \alpha \leq -0.25$ for $\gamma = 4/3$ and in $-0.33 < \alpha \leq -0.25$ for $\gamma = 1$, accelerated or decelerated expansion of the Universe depends only on β. In such case the Universe accelerates for the large values of β and decelerates for the small values of β, where $0 \leq \beta < 1$. Further, we observe an accelerated expansion in $-0.50 \leq \alpha \leq -0.30$ for $\gamma = 2/3$, in $-0.50 \leq \alpha \leq -0.375$ for $\gamma = 4/3$ and in $-0.50 \leq \alpha \leq -0.33$ for $\gamma = 1$ for all values of ζ_0 and β. It is found that the Universe shows a phase transition from acceleration to deceleration for small values of ζ_0, where the value of ζ_0 depends on β also, in all phases for $\alpha < -0.50$. One may observe that q is always negative for $\gamma < 0$ and $\alpha > 0$ irrespective of bulk viscous coefficient and particle creation.

Case II : Solution for $\gamma = 0$

For $\gamma = 0$, we get a single equation of motion from (3.3.6) as

$$\dot{H} - \frac{3}{2}(1+2\alpha)\zeta_0 H = 0, \tag{3.3.13}$$

On solving (3.3.13), we obtain H in terms of t as

$$H = c_{13} e^{\frac{3}{2}(1+2\alpha)\zeta_0 t}, \tag{3.3.14}$$

Table 3.1: Variation of q for $\gamma = \frac{2}{3}$.

Range of α	Constraints on β	Constraints on ζ_0	q	Evolution of Universe
$\alpha > 0$	For all β	For all $\zeta_0 > 0$	Negative	Accelerated expansion
$-0.25 < \alpha < 0$	For all β	$\zeta_0 > \frac{2(1-\beta)H_0}{3(1+4\alpha)}$	Negatine	Accelerated expansion
	$\beta \geq \frac{-4\alpha}{3+8\alpha}$	For all $\zeta_0 > 0$	Negative	Accelerated expansion
	$\beta < \frac{-4\alpha}{3+8\alpha}$	$\zeta_0 < \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	+ve to -ve	Trans. from dec. to acc.
	$\beta < \frac{-4\alpha}{3+8\alpha}$	$\zeta_0 \geq \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Negative	Accelerated expansion
$-0.30 < \alpha \leq -0.25$	$\beta > -\frac{3+10\alpha}{2\alpha}$	For all $\zeta_0 > 0$	Negative	Accelerated expansion
	$\beta < -\frac{3+10\alpha}{2\alpha}$		Positive	Decelerated expansion
$-0.50 \leq \alpha < -0.30$	For all β	For all $\zeta_0 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	For all β	$\zeta_0 < \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	-ve to +ve	Trans. from acc. to dec.
		$\zeta_0 \geq \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Positive	Decelerated expansion

Table 3.2: Variation of q for $\gamma = \frac{4}{3}$.

Range of α	Constraints on β	Constraints on ζ_0	q	Evolution of Universe
$-0.25 < \alpha$ $(\alpha \neq 0)$	For all β	$\zeta_0 > \frac{4(1-\beta)H_0}{3(1+4\alpha)}$	Negative	Accelerated expansion
	$\beta \geq \frac{3+4\alpha}{2(3+8\alpha)}$	For all $\zeta_0 > 0$	Negative	Accelerated expansion
	$\beta < \frac{3+4\alpha}{2(3+8\alpha)}$	$\zeta_0 < \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	+ve to -ve	Trans. from dec. to acc.
	$\beta < \frac{3+4\alpha}{2(3+8\alpha)}$	$\zeta_0 \geq \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Negative	Accelerated expansion
$-0.375 < \alpha \leq -0.25$	$\beta > -\frac{3+8\alpha}{4\alpha}$	For all $\zeta_0 > 0$	Negative	Accelerated expansion
	$\beta < -\frac{3+8\alpha}{4\alpha}$		Positive	Decelerated expansion
$-0.50 \leq \alpha < -0.375$	For all β	For all $\zeta_0 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	$\beta \leq \frac{3+4\alpha}{2(3+8\alpha)}$	For all $\zeta > 0$	Positive	Decelerated expansion
	$\beta > \frac{3+4\alpha}{2(3+8\alpha)}$	$\zeta_0 < \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	-ve to +ve	Trans. from acc. to dec.
		$\zeta_0 \geq \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Positive	Decelerated expansion

Table 3.3: Variation of q for $\gamma = 1$.

Range of α	Constraints on β	Constraints on ζ_0	q	Evolution of Universe
$-0.25 < \alpha$ $(\alpha \neq 0)$	For all β	$\zeta_0 > \frac{(1-\beta)H_0}{(1+4\alpha)}$	Negative	Accelerated expansion
	$\beta \geq \frac{1}{(3+8\alpha)}$	For all $\zeta_0 > 0$	Negative	Accelerated expansion
	$\beta < \frac{1}{(3+8\alpha)}$	$\zeta_0 < \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	+ve to -ve	Trans. from dec. to acc.
	$\beta < \frac{1}{(3+8\alpha)}$	$\zeta_0 \geq \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	Negative	Accelerated expansion
$-0.33 < \alpha \leq -0.25$	$\beta > -\frac{1+3\alpha}{\alpha}$	For all $\zeta_0 > 0$	Negative	Accelerated expansion
	$\beta < -\frac{1+3\alpha}{\alpha}$		Positive	Decelerated expansion
$-0.50 \leq \alpha < -0.33$	For all β	For all $\zeta_0 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	For all β	$\zeta_0 < \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	-ve to +ve	Trans. from acc. to dec.
		$\zeta_0 \geq \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	Positive	Decelerated expansion

Table 3.4: Variation of q for $\gamma = 0$.

Range of α	Constraints on β	Constraints on ζ_0	q	Evolution of Universe
$\alpha \geq -0.50$ $(\alpha \neq 0)$	For all β	For all $\zeta_0 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	For all β	$0 < \zeta_0 < \frac{-2H_0}{3(1+2\alpha)}$	-ve to +ve	Trans. from acc. to dec.
		$\zeta_0 \geq \frac{-2H_0}{3(1+2\alpha)}$	Positive	Decelerated expansion

where $c_{13} > 0$ is an integration constant. Further, we can obtain scale factor as

$$a = c_{14} e^{\frac{2c_7}{3(1+2\alpha)\zeta_0} e^{\frac{3}{2}(1+2\alpha)\zeta_0 t}}, \tag{3.3.15}$$

where $c_{14} > 0$ is another integration constant. We can rewrite the scale factor as

$$a = a_0 e^{\frac{2H_0}{3(1+2\alpha)\zeta_0}(e^{\frac{3}{2}(1+2\alpha)\zeta_0 t} - 1)}. \tag{3.3.16}$$

Here, we observe “superinflationary" expansion of the Universe and this is due to the presence of viscous term only. The effect of particle creation is negligible. The energy density is given by

$$\rho = \frac{3H_0 e^{\frac{3}{2}(1+2\alpha)\zeta_0 t}}{(1+4\alpha)} \left[H_0 e^{\frac{3}{2}(1+2\alpha)\zeta_0 t} - \alpha\zeta_0 \right]. \tag{3.3.17}$$

The energy density is finite at $t = 0$ and hence the model is free from singularity. This type of solution helps to solve several cosmological problems like flatness, horizon, etc. The DP for obtained scale factor is given by

$$q = -\frac{3(1+2\alpha)\zeta_0}{2H_0} e^{-\frac{3}{2}(1+2\alpha)\zeta_0 t} - 1, \tag{3.3.18}$$

which is time–dependent. Table (3.4) lists the variation of q for the whole range of α with constraint of ζ_0 for all β.

From Table (3.4), it is observed that q is always negative and hence the Universe accelerates through out the evolution for $\alpha \geq -0.50$ and all values of ζ_0 irrespective of β. When $\alpha < -0.50$, we observe a phase transition from acceleration to deceleration for small values of ζ_0 and only deceleration for large values of ζ_0 as q is positive.

3.3.2 Cosmology with Variable Bulk Viscosity

In this section we consider two cases for $\zeta_0 = 0$ and $\zeta_0 \neq 0$, respectively to find the exact solution of (3.3.3).

Case I : When $\zeta_0 = 0$

In this case, the form of bulk viscous coefficient defined in (2.3.3) reduces to $\zeta = \zeta_1 H$, where ζ_1 is a positive constant. Using this form of ζ in (3.3.4), the energy density becomes

$$\rho = \frac{3(1-\alpha\zeta_1)H^2}{1+4\alpha-\alpha\gamma+\alpha\beta\gamma}. \tag{3.3.19}$$

Using (3.3.19) into (3.3.3), we obtain

$$\dot{H} + \frac{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\, H^2 = 0. \tag{3.3.20}$$

The solution of this is given by

$$H = \frac{1}{\left[\frac{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\, t + c_{15}\right]}, \tag{3.3.21}$$

where c_{15} is an integration constant. For $\alpha = 1/2$ or $\gamma = \frac{(1+4\alpha)\zeta_1}{1-\beta}$, we obtain de-Sitter solution. The scale factor is given by

$$a = c_{16}\left[c_{15} + \frac{3(1+2\alpha)}{2}\frac{(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)t}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\right]^{\frac{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}}, \tag{3.3.22}$$

where $c_{16} > 0$ is another integration constant. We can rewrite this scale factor as

$$a = a_0\left[1 + \frac{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}H_0(t-t_0)\right]^{\frac{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}}, \tag{3.3.23}$$

which represents power–law expansion of the scale factor. The respective energy density and the bulk viscous coefficient have the following expressions.

$$\rho = \frac{3(1-\alpha\zeta_1)H_0^2}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\left[1 + \frac{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}H_0(t-t_0)\right]^{-2}, \tag{3.3.24}$$

$$\zeta = \zeta_1 H_0\left[1 + \frac{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}H_0\,(t-t_0)\right]^{-1}. \tag{3.3.25}$$

The particle creation pressure, particle creation density, rate of particle creation, entropy production may be obtained by using the above results and discuss the behavior accordingly. In this case the DP is given by

$$q = \frac{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)} - 1. \tag{3.3.26}$$

Here, we get a constant DP. There is no phase transition due to constant DP. For the suitable values of the parameters α, β, γ and ζ_1, we can obtain accelerated or decelerated expansion of the Universe. However, we always get accelerated expansion for $\gamma \leq 0$, $\alpha > 0$ and $\zeta_1 > 0$.

The above results is valid for $0 \leq \gamma \leq 2$. For $\gamma < 0$ we get a big–rip singularity in the future under the following constraint.

$$\frac{\zeta_1(1+4\alpha)}{(1-\beta)} > \gamma, \tag{3.3.27}$$

which shows that this type of singularity depends both on ζ_1 and β. This constraint naturally tends the scale factor and the energy density to infinity at a finite time

$$t_{br} = \frac{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{3(1+2\alpha)(\zeta_1+\beta\gamma+4\alpha\zeta_1-\gamma)} H_0^{-1}. \tag{3.3.28}$$

It may be noted that the finite time big–rip singularity exists without satisfying the constraint mentioned in (3.3.27) if $\alpha < 0$. The energy density of the dark component increases with scale factor for $\frac{\zeta_1(1+4\alpha)}{1-\beta} > \gamma$. It is also observed that the big–rip singularity may be avoided for suitable range of various parameters as discussed above.

Case II : When $\zeta_0 \neq 0$

In this case, we consider the general form of the bulk viscous coefficient (2.3.3), i.e, $\zeta = \zeta_0 + \zeta_1 H$ for which (3.3.4) becomes

$$\rho = \frac{3H[H-\alpha(\zeta_0+\zeta_1 H)]}{1+4\alpha-\alpha\gamma+\alpha\beta\gamma}. \tag{3.3.29}$$

Using (3.3.29) in (3.3.3), we obtain

$$\dot{H} + \frac{3}{2}\frac{(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{1+4\alpha-\alpha\gamma+\alpha\beta\gamma} H \left[H - \frac{(1+4\alpha)\zeta_0}{\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1}\right] = 0. \tag{3.3.30}$$

On solving this evolution equation of H for $0 \leq \gamma \leq 2$, we get

$$H = \frac{e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}}{c_{17} + \frac{(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{(1+4\alpha)\zeta_0} e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}}, \tag{3.3.31}$$

where c_{17} is an integration constant. From (3.3.31), we obtain the scale factor as

$$a = c_{18}\left[c_{17} + \frac{(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}{(1+4\alpha)\zeta_0} e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}\right]^{\frac{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{3(1+2\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}}, \tag{3.3.32}$$

where $c_{18} > 0$ is another integration constant. This solution in not valid for $\alpha = -1/2$. For $\alpha = -1/2$ we again get the de-Sitter solution. We can rewrite the scale factor as

$$a=a_0\Big[1+\frac{(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0 t}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}}-1\right)\Big]^{\frac{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{(3+6\alpha)(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)}}. \qquad (3.3.33)$$

The energy density and the bulk viscous coefficient are respectively given by

$$\rho = \frac{3H_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\left[\frac{e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0 t}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}}}{1+\frac{(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0 t}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}}-1\right)}\right]$$
$$\times\left[\frac{(1-\alpha\zeta_1)H_0\, e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0 t}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}}}{1+\frac{(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0 t}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}}-1\right)}-\alpha\zeta_0\right] \qquad (3.3.34)$$

$$\zeta=\zeta_0+\zeta_1\left[\frac{H_0\, e^{\frac{3(1+2\alpha)(1+4\alpha\zeta_0}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}}{1+\frac{(\gamma-\zeta_1-\beta\gamma-4\alpha\zeta_1)H_0}{(1+4\alpha)\zeta_0}\left(e^{\frac{3(1+2\alpha)(1+4\alpha)\zeta_0}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}-1\right)}\right]. \qquad (3.3.35)$$

With the constraint defined in (3.3.27) we get a big–rip singularity at

$$t_{br}=\frac{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}{3(1+2\alpha)(1+4\alpha)\zeta_0}\, ln\left(1+\frac{(1+4\alpha)\zeta_0}{(\zeta_1+\beta\gamma+4\alpha\zeta_1-\gamma)H_0}\right). \qquad (3.3.36)$$

The DP for this case is given by

$$q=\frac{\frac{3(1+2\alpha)(1-\beta)}{2(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}\left[\gamma-\frac{(1+4\alpha)(\zeta_0+\zeta_1 H_0)}{(1-\beta)H_0}\right]}{e^{\frac{3}{2}\frac{(1+2\alpha)(1+4\alpha)\zeta_0}{(1+4\alpha-\alpha\gamma+\alpha\beta\gamma)}t}}-1, \qquad (3.3.37)$$

which is time–dependent as in the case of constant bulk viscosity. The nature of q is listed in Tables (3.5)-(3.8) for whole ranges of α.

We find that the Universe accelerates throughout the evolution for $\alpha>0$ and all values of β, ζ_0 and ζ_1 in inflationary phase. In the range $\alpha>-0.25(\alpha\neq 0)$ for a matter and radiation phase, and in $0<\alpha<-0.25$ for an inflationary phase, we observe an accelerated expansion of the Universe for large values of any of β, ζ_0 and ζ_1. The model shows transition from a decelerated phase to an accelerated phase for small values of ζ_0, ζ_1 and β. In $-0.30<\alpha\leq-0.25$ for $\gamma=2/3$, $-0.375<\alpha\leq-0.25$ for $\gamma=4/3$ and $-0.33<\alpha\leq-0.25$ for $\gamma=1$, an accelerated or a decelerated expansion of the Universe depends only on β. Here, the Universe accelerates for the large values of β and decelerates for the small values of β.

Further, we observe an accelerated expansion in $-0.50\leq\alpha\leq-0.30$ for $\gamma=2/3$, in $-0.50\leq\alpha\leq-0.375$ for $\gamma=4/3$ and in $-0.50\leq\alpha\leq-0.33$ for $\gamma=1$ for all values of ζ_0,

Table 3.5: Variation of q for $\gamma = \frac{2}{3}$.

Range of α	Constraints on β	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$\alpha > 0$	For all β	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
$-0.25 < \alpha < 0$	$\beta < \frac{-4\alpha}{3+8\alpha}$	$(\zeta_0 + \zeta_1 H_0) < \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	+ve to -ve	Trans. from Dec. to Acc.
	$\beta < \frac{-4\alpha}{3+8\alpha}$	$(\zeta_0 + \zeta_1 H_0) \geq \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Negative	Accelerated expansion
	$\beta \geq \frac{-4\alpha}{3+8\alpha}$	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
$-0.30 < \alpha \leq -0.25$	$\beta > -\frac{3+10\alpha}{2\alpha}$	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
	$\beta < -\frac{3+10\alpha}{2\alpha}$		Positive	Decelerated expansion
$-0.50 \leq \alpha < -0.30$	For all β	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	For allβ	$(\zeta_0 + \zeta_1 H_0) < \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	-ve to +ve	Trans. from Acc. to Dec.
		$(\zeta_0 + \zeta_1 H_0) \geq \frac{-2H_0(4\alpha+3\beta+8\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Positive	Decelerated expansion

Table 3.6: Variation of q for $\gamma = \frac{4}{3}$.

Range of α	Constraints on β	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$-0.25 < \alpha \ (\alpha \neq 0)$	For all β	$(\zeta_0 + \zeta_1 H_0) > \frac{4(1-\beta)H_0}{3(1+4\alpha)}$	Negative	Accelerated expansion
	$\beta \geq \frac{3+4\alpha}{2(3+8\alpha)}$	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
	$\beta < \frac{3+4\alpha}{2(3+8\alpha)}$	$(\zeta_0 + \zeta_1 H_0) < \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	+ve to -ve	Trans. from Dec. to Acc.
	$\beta < \frac{3+4\alpha}{2(3+8\alpha)}$	$(\zeta_0 + \zeta_1 H_0) \geq \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Negative	Accelerated expansion
$-0.375 < \alpha \leq -0.25$	$\beta > -\frac{3+8\alpha}{4\alpha}$	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
	$\beta < -\frac{3+8\alpha}{4\alpha}$		Positive	Decelerated expansion
$-0.50 \leq \alpha < -0.375$	For all β	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	$\beta \leq \frac{3+4\alpha}{2(3+8\alpha)}$	for all $\zeta, \zeta_0 > 0$	Positive	Decelerated expansion
	$\beta > \frac{3+4\alpha}{2(3+8\alpha)}$	$(\zeta_0 + \zeta_1 H_0) < \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	-ve to +ve	Trans. from Acc. to Dec.
		$(\zeta_0 + \zeta_1 H_0) < \frac{2H_0(3+4\alpha-6\beta-16\alpha\beta)}{9(1+2\alpha)(1+4\alpha)}$	Positive	Decelerated expansion

Table 3.7: Variation of q for $\gamma = 1$.

Range of α	Constraints on β	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$-0.25 < \alpha \ (\alpha \neq 0)$	For all β	$(\zeta_0 + \zeta_1 H_0) > \frac{(1-\beta)H_0}{(1+4\alpha)}$	Negative	Accelerated expansion
	$\beta \geq \frac{1}{(3+8\alpha)}$	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
	$\beta < \frac{1}{(3+8\alpha)}$	$(\zeta_0 + \zeta_1 H_0) < \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	+ve to -ve	Trans. from Dec. to Acc.
	$\beta < \frac{1}{(3+8\alpha)}$	$(\zeta_0 + \zeta_1 H_0) \geq \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	Negative	Accelerated expansion
$-0.33 < \alpha \leq -0.25$	$\beta > -\frac{1+3\alpha}{\alpha}$	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
	$\beta < -\frac{1+3\alpha}{\alpha}$		Positive	Decelerated expansion
$-0.50 \leq \alpha < -0.33$	For all β	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	for all β	$(\zeta_0 + \zeta_1 H_0) < \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	-ve to +ve	Trans. from acc. to dec.
		$(\zeta_0 + \zeta_1 H_0) \geq \frac{H_0(1-3\beta-8\alpha\beta)}{3(1+2\alpha)(1+4\alpha)}$	Positive	Decelerated expansion

Table 3.8: Variation of q for $\gamma = 0$.

Range of α	Constraints on β	Constraints on ζ_0 and ζ_1	q	Evolution of Universe
$\alpha \geq -0.50 \ (\alpha \neq 0)$	For all β	For all $\zeta_0, \zeta_1 > 0$	Negative	Accelerated expansion
$\alpha < -0.50$	For all β	$(\zeta_0 + \zeta_1 H_0) < \frac{-2H_0}{3(1+2\alpha)}$	-ve to +ve	Trans. from Acc. to Dec.
		$(\zeta_0 + \zeta_1 H_0) \geq \frac{-2H_0}{3(1+2\alpha)}$	Positive	Decelerated expansion

ζ_1 and β. In case of $\gamma = 0$, the model shows acceleration for all values of ζ_0, ζ_1 and β when $\alpha \geq -0.50$. We observe phase transition from acceleration to deceleration for small values of ζ_0 and ζ_1 in all phases when $\alpha < -0.50$. In this case also, the finite time big–rip singularity may be avoided for suitable range of the parameters in phantom region ($\gamma < 0$) as discussed in previous section. One can obtain the solutions for $\gamma = 0$ after putting directly $\gamma = 0$ in the values obtained above. The evolution of the Universe depends on the bulk viscous coefficient only. The variation of q for $\gamma = 0$ is summarized in Table (3.8).

3.4 Conclusion

In Chapter 2, the effect of the bulk viscosity in $f(R,T)$ gravity within the framework of a flat FRW model has been studied for different forms of bulk viscous coefficient. In the present chapter, we have extended the study by including the concept of particle creation with an approach similar to Triginer and Pav$\acute{o}$n [202], and Brevik and Stokkan [203] where they have studied particle creation and bulk viscosity together. It was observed that both the phenomenon might be dynamically similar, depending on their particular form, but thermodynamically both have different phenomenon. Therefore, it is interesting to observe the effects of bulk viscosity and particle creation together in this modified gravity theory as both the cosmic fluids play an important role in early and late–time evolution of the Universe. We have made assumptions on bulk viscous coefficient $\zeta = \zeta_0 + \zeta_1 H$ and particle creation rate $\psi = 3\beta H$ to find the exact solution of the modified field equations. We have observed that both the cosmic fluids explain the early and late–time evolution of the Universe together or independently. The effective cosmic fluid (bulk viscous plus particle creation) expands the scale factor in a very fast rate in comparison to the cosmic fluid of bulk viscosity or particle creation for $\alpha > 0$ whereas only particle creation is responsible to expands the Universe in a very fast rate for $\alpha < 0$. Thus, the evolution of the Universe depends on the ranges of α which we have discussed through tables and figures of variation of q and a with respect to cosmic time t, respectively. We summarize the results obtained in each section one by one as fellows:

In Sect. 3.3.1, we have discussed the constant bulk viscosity ($\zeta = \zeta_0$) with particle creation for the cases $\gamma = 0$ and $\gamma \neq 0$, separately. In case of $\gamma \neq 0$, we have found exponential expansion of the Universe which explain the whole history of the evolution of the Universe, beginning from inflation to the recent accelerated expansion of the Universe. One may observe that the presence of particle creation increases the expansion rate very fast in each phase. It has been found that the DP is time–dependent which describes the deceleration, acceleration and their transition for different ranges of the parameters. Tables (3.1)-(3.3) present the variation of q with respect to t for the suitable constraints on the parameters ζ_0 and β for all ranges of α. It has been observed that both bulk viscosity and particle creation can explain recent acceleration of the Universe. We have a big–rip singularity in the phantom region($\gamma < 0$). However, it may be avoided for a suitable range of the parameters as discussed in Sect. 3.3.1. The value of q is always negative for $\gamma < 0$ and $\alpha > 0$. We have found a superinflationary expansion of the Universe and time–dependent q for $\gamma = 0$. The variation of q with t has been listed in Table (3.4), which shows the different types of expansion and their transition of the evolution of the Universe for all possible ranges of α with the constraint on ζ_0 irrespective of β. The role of particle creation is negligible in this case which may be assumed that result is true for all β.

In Sect. 3.3.2, we have discussed two cases of variable bulk viscosity as $\zeta = \zeta_1 H$ and $\zeta = \zeta_0 + \zeta_1 H$ with particle creation. In case of $\zeta = \zeta_1 H$, we have obtained power–law expansion of the Universe for $0 \leq \gamma \leq 2$. This model also shows big–rip singularity under the constraint defined in Eq. (3.3.27). In this case we have found a constant DP where the sign of q determines the deceleration or acceleration of the evolution of the Universe depending on the suitable values of the parameters. In case of $\zeta = \zeta_0 + \zeta_1 H$, we have obtained an exponential evolution of the Universe as in the case of constant viscous coefficient. Indeed, this exponential expansion is induced by the constant term of the bulk viscous coefficient. We have found inflation and late–time acceleration, respectively for different ranges of α under suitable constraints of other parameters. For $\alpha < -0.50$, the Universe passes through a phase transition from acceleration to deceleration for small values of the parameters ζ_0 and ζ_1 for all β. The evolution of the Universe has been shown in Tables (3.5)-(3.8) for different phases. A big–rip singularity has also been observed at a finite cosmic time given in the phantom region($\gamma < 0$). It is worthy to notice that the occurrence of big–rip singularity may be avoided for a specific range of parameters, specially on α.

The graphs of the scale factor with cosmic time have been presented in Figs. 3.1 and 3.2 for all possible combinations of ζ_0, ζ_1 and β with some selected values of the other parameters for $\alpha > 0$ and $\alpha < 0$, respectively from where the effects of bulk viscosity and particle creation may be observed. In case of $\alpha > 0$ as shown in Fig. 3.1, it

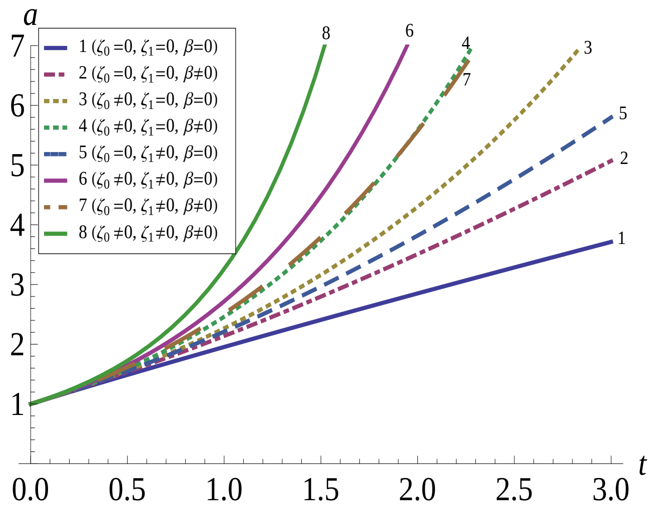

Figure 3.1: The scale factor in terms of cosmic time for $\alpha > 0$. Here $\gamma = \alpha = a_0 = H_0 = 1$, $\beta = 1/3$ and $\zeta_0 = \zeta_1 = 0.1$

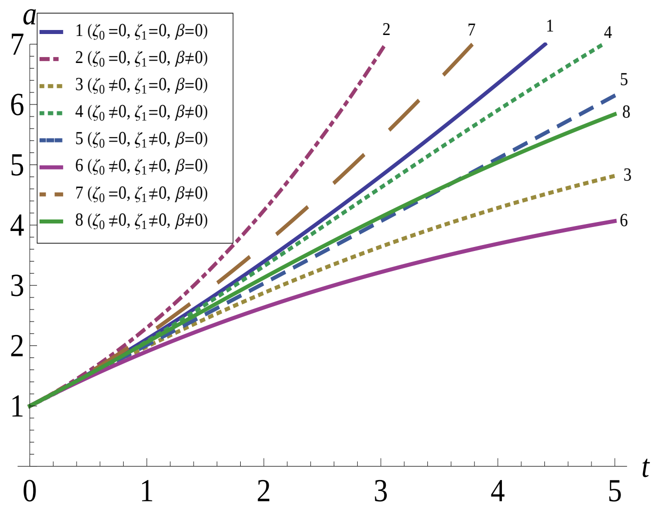

Figure 3.2: The scale factor in terms of cosmic time for $\alpha < 0$. Here $\alpha = -1$, $\gamma = a_0 = H_0 = 1$, $\beta = 1/3$ and $\zeta_0 = \zeta_1 = 0.1$

has been observed that the presence of the viscous term and particle creation together or independently in each case has enhanced the rate of expansion as compared to the perfect fluid ($\zeta_0 = 0,\ \zeta_1 = 0,\ \beta = 0$). The rate of expansion is very fast in case of $\zeta = \zeta_0 + \zeta_1 H$ and $\beta \neq 0$ as the scale factor varies exponentially. It is comparatively slow in the absence of β for viscous fluid models. For example, the expansion goes to the fastest rate in the case when $\zeta_0 \neq 0,\ \zeta_1 \neq 0$ and $\beta \neq 0$, and the slowest rate for $\zeta_0 = 0,\ \zeta_1 = 0$ and $\beta = 0$ for $\alpha = 1$. For $\alpha < 0$ we have observed that the particle creation plays an important role in the expansion history of the Universe as shown in Fig. 3.2. For example, the scale factor goes to the fastest rate in case of $\zeta_0 = 0 = \zeta_1$ and $\beta \neq 0$, and the slowest rate when $\zeta_0 \neq 0,\ \zeta_1 \neq 0$ and $\beta = 0$ for $\alpha = -1$. The effects of other combinations may also be analyzed from Figs. 3.1 and 3.2, respectively.

Figs. 3.3 and 3.4 plot the deceleration parameter as a function of time for $\gamma = 1$

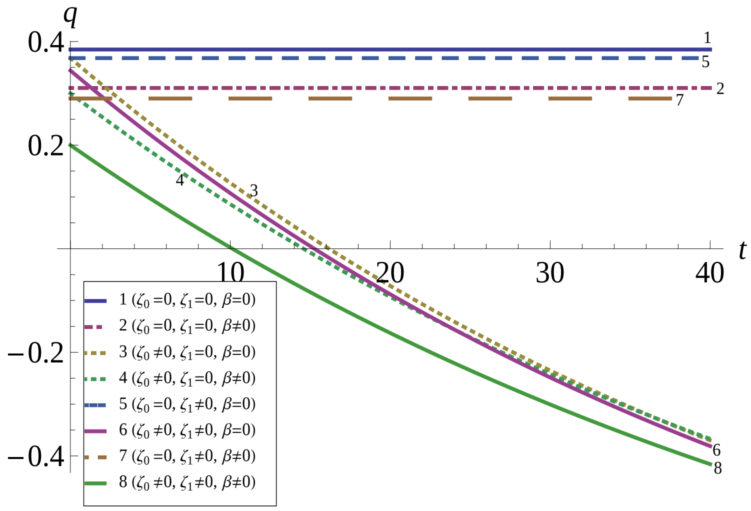

Figure 3.3: The deceleration parameter in terms of cosmic time for $\alpha > 0$. Here $\gamma = a_0 = H_0 = 1$, $\alpha = 0.1$, $\beta = 0.05$ and $\zeta_0 = \zeta_1 = 0.01$.

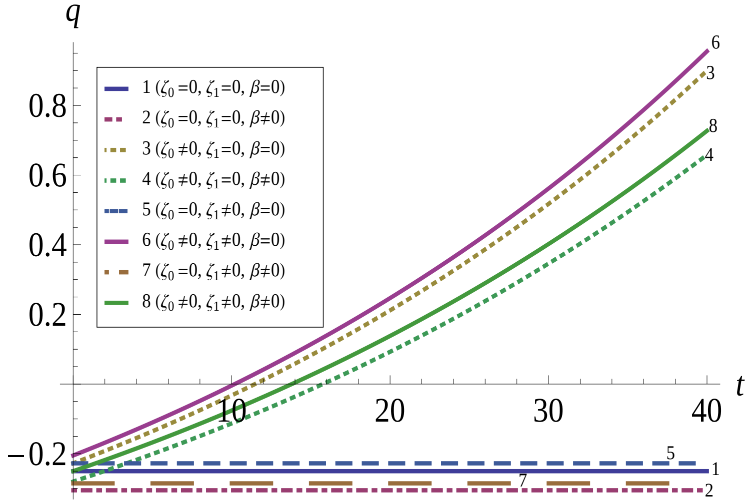

Figure 3.4: The deceleration parameter in terms of cosmic time for $\alpha < 0$. Here $\alpha = -1$, $\gamma = a_0 = H_0 = 1$, $\beta = 0.05$ and $\zeta_0 = \zeta_1 = 0.01$

and all possible combinations of ζ_0, ζ_1 and β with some selected values of other parameters for $\alpha > 0$ and $\alpha < 0$, respectively. In the cases (i) $\zeta_0 = 0$, $\zeta_1 = 0$, $\beta = 0$, (ii) $\zeta_0 = 0$, $\zeta_1 \neq 0$, $\beta \neq 0$, (iii) $\zeta_0 = 0 = \zeta_1$, $\beta \neq 0$, and (iv) $\zeta_0 = 0$, $\zeta_1 \neq 0$, $\beta = 0$, the deceleration parameter is always constant. In these cases the positive or negative value of q depends on α, β, γ and ζ_1. However, it is always negative for $\gamma \leq 0$, $\alpha > 0$ and $\zeta > 0$. With time–dependent q, we observe phase transitions from decelerated phase to accelerated phase for $\alpha > 0$ (see, Fig. 3.3), and accelerated to decelerated in the case

$\alpha < 0$ (see, Fig. 3.4) for other combinations with $\zeta_0 \neq 0$, respectively. It is to be noted that the constant α acts as a coupling parameter between geometry and matter in such form of $f(R,T)$. The purpose of taking the various ranges of α is to observe the evolution of the Universe by changing the coupling between geometry and matter in different phases of the Universe. Therefore, in this paper we have presented the both decelerating and accelerating and the transition between them for different phases of the evolution of the Universe depending on the ranges of α and constraints on other parameters.

In concluding remark we can say that the bulk viscosity and matter creation either as a combined cosmic fluid or in separate form may play an important role in the expansion history of evolution of the Universe as both have the negative pressure.

Chapter 4

Holographic Dark Energy in $f(R,T)$ Gravity

In this chapter[1], we study non–viscous and viscous HDE models with the Hubble horizon as an IR cut-off in the formalism of modified $f(R,T)$ gravity to check whether it is suitable to explain the recent accelerated expansion or not. It has been observed that this IR cut–off is suitable to explain accelerated expansion in $f(R,T)$ gravity. It is also observed that the viscous HDE model explains the phase transition from decelerated phase to accelerated phase. The cosmological parameters like DP and statefinder parameters are discussed to analyze the dynamics of evolution of the Universe for both non–viscous and viscous models. The trajectories for viscous model are plotted in $r-s$ and $r-q$ planes to discriminate our model with the existing DE models which show the quintessence like behavior.

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[1]The content of this chapter is based on research paper "Statefinder diagnosis for holographic dark energy models in modified f(R, T) gravity, *Astrophysics and Space Science* **361**, 157 (2016)".

4.1 Introduction

The prediction of accelerated expansion made by Riess et al. [21] and Perlmutter et al. [20] has been confirmed by the recent observations. The ΛCDM model presents the simplest and most successful description of the recent accelerated expansion scenario, but it has some theoretical problems like fine–tuning and cosmic coincidence problems [28–30, 39]. To overcome from these problems, a number of dynamical DE models such as scalar field models [33, 55, 59, 61], chaplygin gas models [64, 65] etc. have been proposed in the literature. Some dynamical DE models have been proposed based on the principle of quantum gravity one of which is HDE model [71, 73, 217]. The concept of HDE is based on the holographic principle proposed by 't Hooft [68] and found it's roots in the quantum field theory. In the recent years, HDE models have been emerged as a viable candidates to explain the problems of modern cosmology. The HDE models explain the recent accelerated expansion as well as the coincidence problem of the Universe [71, 73, 74].

In the formalism of HDE, the Hubble horizon is a most natural choice for IR cut–off, but Hsu [72] in GTR and Xu et al. [217] in BD theory have shown that the Hubble horizon as an IR cut–off is not a suitable candidate to explain the recent accelerated expansion. However, Zimdhal and Pavón [73] in GTR, and Banerjee and Pavón [218] in BD theory have shown that the interaction between HDE and DM can change the scenario and Hubble horizon as an IR cut–off may explain the recent accelerated expansion.

The $f(R,T)$ gravity, where R as usual stands for the Ricci scalar and T denotes the trace of energy-momentum tensor, presents a maximal coupling between geometry and matter. The HDE models have not been yet discussed in detail in the framework of $f(R,T)$ gravity. In some papers [139, 143] reconstruction of $f(R,T)$ gravity from HDE and anisotropic model of HDE have been discussed. Our interest is to study HDE model with Hubble horizon as an IR cut–off in $f(R,T)$ gravity without considering the interaction between HDE and DM and to check wether it is able to explain accelerated expansion or not. We show that Hubble horizon as an IR cut–off is suitable to explain the accelerated expansion in this theory without interaction between HDE and DM.

The Hubble parameter H and the DP q are well known cosmological parameters which explain the evolution of the Universe. However, these two parameters can not discriminate among various DE models. Therefore, we use new geometrical diagnostic pair $\{r,s\}$, known as statefinder parameters, which was proposed by Sahni et al. [48] to discriminate among various DE models. The statefinder pair $\{r,s\}$ is geometrical in the nature as it is constructed from the space time metric directly. Therefore, the statefinder parameters are more universal parameters to study the DE models than any other physical parameters. One can plot the trajectories in $r-s$ and $r-q$ planes to discriminate various DE models. We discuss the statefinder diagnostic and obtain the fixed point values of statefinder pair $\{r,s\} = \{1,0\}$ as in the case of ΛCDM model.

To be more realistic, the perfect fluid Universe is just an approximation of the viscous Universe. It has been shown that inflation and recent acceleration can be explained using the viscous behavior of the Universe, and it plays an important role in the phase transition of the Universe [170, 175, 177, 178, 201]. The concept of viscous DE has been discussed extensively in the literature [185–187]. It has been shown that the bulk viscosity alleviated the age problem of the Ricci dark energy [184] (for more details about bulk viscosity, see Section 1.5). Motivated by the above works, we extend our analysis to viscous HDE with the same IR cut–off. We observe that viscous HDE may explian the recent phase transition of the Universe. We obtain the statefinder parameters for viscous HDE which achieve the value of ΛCDM model and show the quintessence like behavior.

The present chapter explores the possibility whether HDE model with Hubble horizon as IR cut–off in $f(R,T)$ gravity may explain accelerated expansion or not. First, we discuss non–viscous HDE model without considering interaction between HDE and DM. Further, we extend your study to viscous HDE model which explains phase transition of the Universe. We also find statefinder parameters and discuss their behaviors for both the models.

4.2 Non-viscous HDE in $f(R,T)$ Gravity

Let us consider the Universe filled with HDE and pressureless DM (excluding baryonic matter) for which energy–momentum tensor is given by

$$T_{\mu\nu} = (\rho_m + \rho_h + p_h)u_\mu\, u_\nu - p_h\, g_{\mu\nu} \tag{4.2.1}$$

where ρ_m, ρ_h and p_h denote the energy density of DM, the energy density of HDE and the pressure of HDE, respectively.

The general form of EH action for modified $f(R,T)$ gravity is given by (1.4.9). Varying action (1.4.9) with respect to the metric tensor $g_{\mu\nu}$ for $f(R,T) = R + f(T)$, we get field equation of the modified $f(R,T)$ gravity which is given by

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = T_{\mu\nu} - (T_{\mu\nu} + \ominus_{\mu\nu})f'(T) + \frac{1}{2}f(T)g_{\mu\nu}, \tag{4.2.2}$$

where a prime denotes derivative with respect to the argument.

Many authors have described the recent accelerated expansion by assuming the interaction between HDE and DM in the different theories of gravity. Here, we consider a non–interacting matter field in $f(R,T)$ gravity. The matter Lagrangian $\mathscr{L}_m$ may be chosen as $\mathscr{L}_m = -p_h$ for which we get $\ominus_{\mu\nu} = -2T_{\mu\nu} - p_h\, g_{\mu\nu}$ from Eq. (1.4.15). For the line element (1.1.5) and energy–momentum tensor (4.2.1), the $f(R,T)$ gravity (4.2.2) yield the following field equations for $f(T) = \alpha T$ [125]

$$3H^2 = \rho_m + \rho_h + \alpha(\rho_m + \rho_h + p_h) + \frac{1}{2}\,\alpha\, T, \tag{4.2.3}$$

$$2\dot{H} + 3H^2 = -p_h + \frac{1}{2}\,\alpha\, T. \tag{4.2.4}$$

An over dot denotes the derivative with respect to cosmic time t. The EoS of HDE is given by $p_h = w_h\rho_h$ and the trace of energy–momentum tensor is given by $T = \rho_m + \rho_h - 3p_h$. Now, from (4.2.3) and (4.2.4), a combined evolution equation for H can be written as

$$2\dot{H} + (1+\alpha)[(1+w_h)\rho_h + \rho_m] = 0. \tag{4.2.5}$$

In the literature, various forms of HDE (the general form is $\rho_h = 3d_1^2 M_p^2\, L^{-2}$) have been discussed depending on the choices of IR cut–off (L)such as Hubble horizon, future event horizon, apparent horizon, Granda–Oliveros cut–off, etc. In this work, we consider the Hubble horizon ($L = H^{-1}$) as an IR cut–off to describe the recent acceleration for which the energy density ρ_h is given by

$$\rho_h = 3\, d_1^2 H^2, \tag{4.2.6}$$

where $M_p^2 = (8\pi G)^{-1}$.
Using(4.2.6) into (4.2.3), the energy density of DM is given by

$$\rho_m = \frac{3(\alpha d_1^2 w_h - 3\alpha d_1^2 - 2d_1^2 + 2)}{(3\alpha + 2)}\, H^2, \quad (\alpha \neq -\frac{2}{3}). \tag{4.2.7}$$

Using (4.2.6) and (4.2.7) into (4.2.5), we finally get

$$\dot{H}+\frac{3(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}{3\alpha+2}H^2=0, \tag{4.2.8}$$

which, on solving gives (where $\alpha\neq-1$, $\alpha\neq-2/3$)

$$H=\frac{1}{c_{19}+\frac{3(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}{3\alpha+2}t}, \tag{4.2.9}$$

where c_{19} is a positive constant of integration. The value of H can be rewritten as

$$H=\frac{H_0}{1+\frac{3H_0(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}{3\alpha+2}(t-t_0)}, \tag{4.2.10}$$

where H_0 is the present value of the Hubble parameter at the cosmic time $t=t_0$, the time where the HDE starts to dominate. Using the relation $H=\frac{\dot{a}}{a}$ in Eq. (4.2.10), the cosmic scale factor a can be obtained as

$$a=c_{20}\left[1+\frac{3H_0(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}{3\alpha+2}(t-t_0)\right]^{\frac{3\alpha+2}{3(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}}, \tag{4.2.11}$$

where c_{20} is an another positive constant of integration. One can rewrite a as

$$a=a_0\left[1+\frac{3H_0(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}{3\alpha+2}(t-t_0)\right]^{\frac{3\alpha+2}{3(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}}, \tag{4.2.12}$$

where a_0 is the present value of the scale factor at the cosmic time $t=t_0$. Thus, we obtain the power–law evolution of the scale factor of the Universe which avoids the Big Bang singularity.

The value of q is given by

$$q=\frac{3(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}{3\alpha+2}-1. \tag{4.2.13}$$

Here, we obtain a constant DP as expected due to the power–law scale factor. The accelerated expansion of the Universe may be obtained if the Eq. (4.2.13) satisfies the constraint $\frac{3(\alpha+1)(2\alpha d_1^2 w_h+d_1^2 w_h+1)}{3\alpha+2}<1$. In a paper, Li et al. [219] have studied the Planck constraints on HDE and obtained the tightest and self–consistent value of constant d_1 from Planck+WP+BAO+HST+lensing as $d_1=0.495\pm0.039$. Therefore, let us consider here and thereafter $d_1=0.5$ for further discussion which is lying in this observed range. Now, for example, let us take $\alpha=1$ and $w_h=-0.5$, we obtain $q=-0.25$ which shows the accelerated expansion. Moreover, one may obtain the accelerated expansion even when the HDE does not violate the strong energy condition $\rho_h + 3p_h > 0$ for suitable values of coupling parameter α.

Thus, HDE with Hubble horizon as an IR cut–off may successfully explain the accelerated expansion in the framework of $f(R,T)$ gravity without assuming the interaction between HDE and DM in contrast to the works done in GTR [72] and in BD theory [217]. It is to be noted that this model does not show the phase transition as the DP is constant.

In order to get a robust analysis to discriminate among DE models, we use statefinder parameters $\{r,s\}$ proposed by Sahni et al. [48] and Alam et al. [52] which is constructed from the scale factor and its derivatives up to the third order. The statefinder pair $\{r,s\}$ provides a very comprehensive description of the dynamics of the Universe and consequently the nature of the DE. In this model, the statefinder parameters r and s as defined in Eq. $(1.7.7)$ are given by

$$r = \frac{18(\alpha+1)^2(2\alpha d_1^2 w_h + d_1^2 w_h + 1)^2}{(3\alpha+2)^2} - \frac{9(\alpha+1)(2\alpha d_1^2 w_h + d_1^2 w_h + 1)}{3\alpha+2} + 1, \tag{4.2.14}$$

$$s = \frac{(\alpha+1)(2\alpha d_1^2 w_h + d_1^2 w_h + 1)[2(\alpha+1)(2\alpha d_1^2 w_h + d_1^2 w_h + 1) - (3\alpha+2)]}{(3\alpha+2)[(\alpha+1)(2\alpha d_1^2 w_h + d_1^2 w_h + 1) - \frac{1}{2}]}. \tag{4.2.15}$$

We observe that the statefinder parameters $\{r,s\}$ are constant and the values of these parameters depend on the coupling parameter α, constant d_1 and EoS parameter w_h of HDE. In the papers [48, 52], it has been observed that SCDM model and ΛCDM model have fixed point values of statefinder pair $\{r,s\} = \{1,1\}$ and $\{r,s\} = \{1,0\}$, respectively. In our work, it is observed that fixed point of ΛCDM model can be achieved for $\alpha = -\frac{1+d_1^2 w_h}{2d_1^2 w_h}$ as a particular case of this model. Thus, for a suitable value of w_h, which may be obtained by observations, we can find the coupling parameter α for which we have $\{r,s\} = \{1,0\}$ and viceversa.

4.3 Viscous HDE in $f(R,T)$ Gravity

It has been shown in previous section that non–viscous HDE could not explain the phase transition of the Universe. It has been found in the literature that the viscosity may play an important role in the explanation of phase transition. Therefore, it would be of interest to investigate whether a viscous HDE with Hubble horizon as an IR cut-off could be helpful to explain the phase transition of the Universe. The energy–momentum tensor (4.2.1) with the inclusion of bulk viscous fluid modifies to

$$T_{\mu\nu} = (\rho_m + \rho_h + P_{eff})g_{\mu\nu} - P_{eff}\, g_{\mu\nu}. \tag{4.3.1}$$

It has been found that only the bulk viscous fluid remains compatible with the assumption of large scale homogeneity and isotropy. The other processes, like shear and heat conduction, are directional mechanisms and they decay as the Universe expands. We have discussed the effect of viscous fluid in $f(R,T)$ gravity and discussed the recent phase transition of the Universe in previous chapters.

Using the Eckart formalism for dissipative fluids, we can assume that the effective pressure of HDE is a sum of the thermodynamical pressure (p_h) and the bulk viscous pressure (Π), i.e.,

$$P_{eff} = p_h + \Pi = p_h - 3\zeta H, \tag{4.3.2}$$

where ζ is the positive coefficient of the bulk viscosity. The matter Lagrangian is taken as $\mathscr{L}_m = -P_{eff}$ for which Eq. (1.4.15) gives $\ominus_{\mu\nu} = 2T_{\mu\nu} - P_{eff}\, g_{\mu\nu}$. In this model, we follow the same concept as discussed in Sect. 4.2 to analyze the behavior of the Universe. Using $f(T) = \alpha T$, the $f(R,T)$ field equations (4.2.2) for viscous HDE with energy–momentum tensor (4.3.1) yield

$$3H^2 = \rho_m + \rho_h + \alpha(\rho_m + \rho_h + p_h - 3\zeta H) + \frac{1}{2}\alpha T, \tag{4.3.3}$$

$$2\dot{H} + 3H^2 = -p_h + 3\zeta H + \frac{1}{2}\alpha T. \tag{4.3.4}$$

In this case, the trace of energy-momentum tensor is $T = \rho_m + \rho_h - 3(p_h - 3\zeta H)$. Using the value of T in (4.3.3) and (4.3.4), a single evolution equation of H is given by

$$2\dot{H} + (\alpha+1)(\rho_m + \rho_h + p_h) - 3(\alpha+1)\zeta H = 0.. \tag{4.3.5}$$

Using (4.2.6) with EoS $p_h = w_h \rho_h$ into (4.3.3), the energy density for matter is given by

$$\rho_m = \frac{3}{(3\alpha+2)}\, H\, [(\alpha d_1^2 w_h - 3\alpha d_1^2 - 2d_1^2 + 2)H - \alpha\zeta]. \tag{4.3.6}$$

Now, Using (4.2.6) and (4.3.6) into (4.3.5), we get

$$\dot{H} + \frac{3(\alpha+1)}{3\alpha+2}(2\alpha d_1^2 w_h + d_1^2 w_h + 1)H^2 - \frac{3(\alpha+1)(2\alpha+1)\zeta}{(3\alpha+2)} H = 0. \tag{4.3.7}$$

Equation (4.3.7) is solvable for H if the coefficient of bulk viscosity ζ is known. Many authors have studied the cosmological models by assuming the various forms of the bulk viscous coefficient (for review, see Maartens [169]). Here, we assume the bulk viscous coefficient of the form $\zeta = \zeta_0 + \zeta_1 H$ as taken in chapters 2 and 3. Using this form of bulk viscous coefficient, Eq. (4.3.7) reduces to

$$\dot{H} + \frac{3(\alpha+1)}{3\alpha+2}(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)H^2 - \frac{3(\alpha+1)(2\alpha+1)\zeta_0}{(3\alpha+2)}H = 0. \quad (4.3.8)$$

Now, this evolution equation in H is solvable for all values of α except $\alpha = -\frac{2}{3}$. The solution is given by

$$H = \frac{e^{\frac{3(\alpha+1)(2\alpha+1)\zeta_0}{(3\alpha+2)}t}}{c_{21} + \frac{(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)}{(2\alpha+1)\zeta_0} e^{\frac{3(\alpha+1)(2\alpha+1)\zeta_0}{(3\alpha+2)}t}}, \quad (4.3.9)$$

where c_{21} is a constant of integration. The scale factor a in the terms of cosmic time t is

$$a = c_{22}\left[c_{21} + \frac{(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)}{(2\alpha+1)\zeta_0} e^{\frac{3(\alpha+1)(2\alpha+1)\zeta_0}{(3\alpha+2)}t}\right]^{\frac{3\alpha+2}{3(\alpha+1)(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)}}, \quad (4.3.10)$$

where $c_{22} > 0$ is another constant of integration. The Eq. (4.3.10) can be rewritten as

$$a = a_0\left[1 + \frac{(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)H_0}{(2\alpha+1)\zeta_0}\left(e^{\frac{3(\alpha+1)(2\alpha+1)\zeta_0(t-t_0)}{(3\alpha+2)}} - 1\right)\right]^{\frac{(3\alpha+2)/(\alpha+1)}{3(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)}}. \quad (4.3.11)$$

One can observe that the model avoids the Big Bang singularity. In this case, the DP is obtained as

$$q = \frac{3(\alpha+1)}{(3\alpha+2)H_0}\left[(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)H_0 - (2\alpha+1)\zeta_0\right] e^{\frac{3(\alpha+1)(2\alpha+1)\zeta_0}{(3\alpha+2)}(t_0 - t)} - 1. \quad (4.3.12)$$

It is observed that the value of q is time dependent which comes due to the introduction of bulk viscous term in HDE. The phase transition of the Universe can be explained using this value of q. The DP must change it's sign from positive to negative to explain the recent phase transition (deceleration to acceleration) of the Universe. In fact, q must change the sign at the time $t = t_0$ because we have assumed t_0 is the time where the viscous HDE begins to dominate. In other words, the Universe must decelerate for $t < t_0$ (matter dominated epoch) and accelerate for $t > t_0$ (HDE dominated epoch). We observe that the Universe shows the transition from decelerated to accelerated phase at cosmic time t_0 if $3(\alpha+1)[(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)H_0 - (2\alpha+1)\zeta_0] = (3\alpha+2)H_0$. Therefore, the value of coupling parameter α can be obtained for a given value of w_h, which may be obtained from the observations, or vice–versa to observe the phase transition. Thus, we have shown that the bulk viscous HDE with Hubble horizon as an IR cut–off can explain the recent phase transition of the Universe in the framework of $f(R,T)$ gravity.

Let us discuss statefinder parameters to compare our viscous HDE model with existing DE models. In this case, the statefinder parameter r is obtained as

$$\begin{aligned} r &= \frac{9(\alpha+1)^2}{(3\alpha+2)^2 H_0}[(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)H_0 - (2\alpha+1)\zeta_0] \\ &\Big[\frac{(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1)H_0 - (2\alpha+1)\zeta_0}{H_0 e^{\frac{6(\alpha+1)(2\alpha+1)\zeta_0}{(3\alpha+2)}(t-t_0)}} \\ &+ \frac{(2\alpha d_1^2 w_h + d_1^2 w_h - 2\alpha\zeta_1 - \zeta_1 + 1) - \frac{3\alpha+2}{\alpha+1}}{e^{\frac{3(\alpha+1)(2\alpha+1)\zeta_0}{(3\alpha+2)}(t-t_0)}}\Big] + 1. \end{aligned} \tag{4.3.13}$$

The second statefinder parameter s is not given here due to complexity but one can find it by using the values of q and r in $s = \frac{r-1}{3(q-1/2)}$. Our model reproduces the fixed point value $\{r,s\} = \{1,0\}$ of ΛCDM model when the parameter α satisfies the condition $\alpha = -\frac{1}{2}(1 + \frac{H_0}{d_1^2 w_h H_0 - \zeta_1 H_0 - \zeta_0})$. For this value of α, the statefinder pair is independent of time and remains fixed throughout the evolution as in ΛCDM model. Indeed, we have obtained a time dependent statefinder pair which means that a general study of the behaviour of this pair is needed.

We plot the trajectories in $r-s$ plane for some particular values of the parameters α and w_h to discriminate our model with existing models of DE. Here, we have taken $d_1 = 0.5$, $H_0 = 1$, $\zeta_0 = \zeta_1 = 0.05$ and $t_0 = 1$. In the Figs. 4.1 and 4.2, fixed points $\{r,s\} = \{1,1\}$ and $\{r,s\} = \{1,0\}$ are corresponding to SCDM model and ΛCDM model, respectively. It is obvious from both the figures that for any values of α and w_h, which are consistent with the model, the viscous HDE model always approaches to the ΛCDM model, i.e., $\{r,s\} = \{1,0\}$ in the late time evolution. In the early time, our model may approach in the vicinity of SCDM model for some values of α as can be seen in Figs. 4.1 and 4.2. It is interesting to note that for larger negative values of α, the trajectories may start from ΛCDM in the early time and approach to the same ΛCDM model in the late time.

In the quiessence model with constant EoS (Q_1–model) [206, 220] and in the Ricci dark energy (RDE) model [221], it has been shown that the trajectories in $r-s$ plane

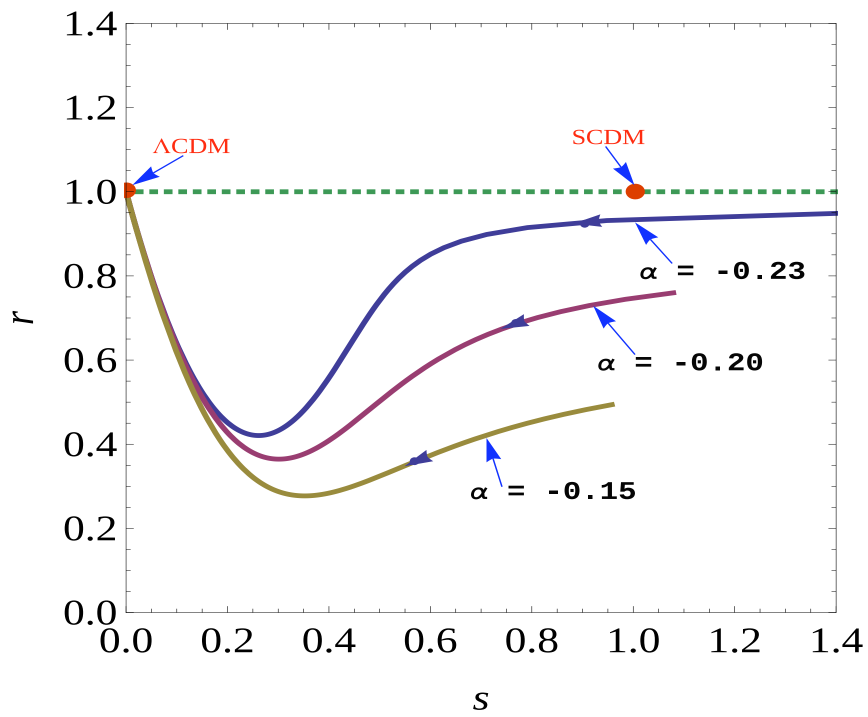

Figure 4.1: The trajectories in $r-s$ plane are plotted for $w_h = -0.5$ and different values of α.

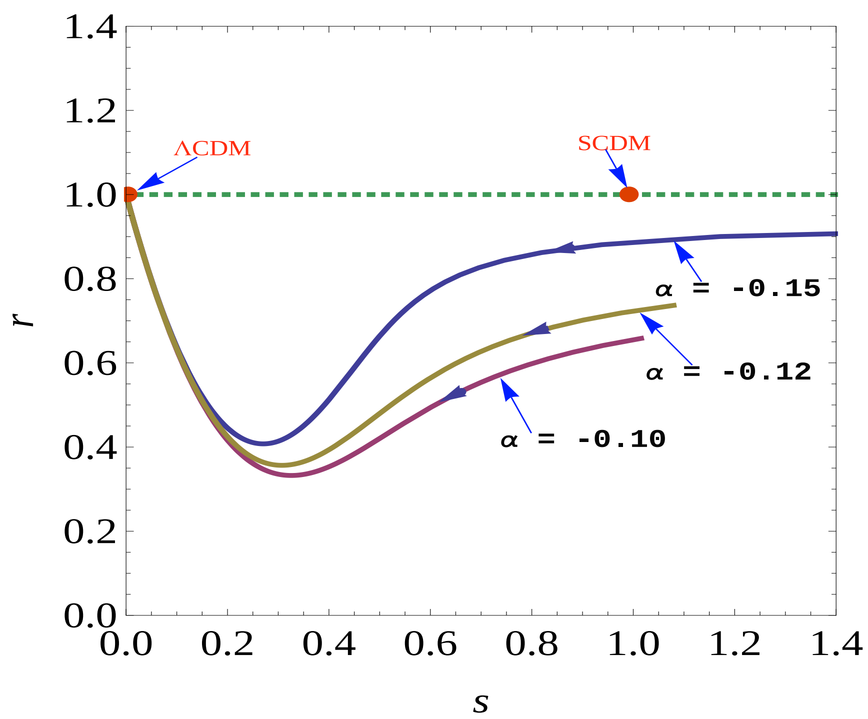

Figure 4.2: The trajectories in $r-s$ plane are plotted for $w_h = -1$ and different values of α.

are vertically straight lines. In the both models, s is constant throughout the evolution, while r increases in RDE model and decreases in Q_1–model starting from the initial point $r = 1$. It has been shown in the papers [206, 220] that the trajectories for the quintessence scalar field model (Q_2–model), where the scalar potential $V(\phi)$ varies as $V(\phi) \propto \phi^{-\varepsilon}, \varepsilon \geq 1$, and chaplygin gas model approach asymptotically to the ΛCDM model in the late time. Comparing our viscous HDE model with Q_1–model and RDE model, we find that our model produces the curved trajectories which approach to ΛCDM model in the late time. Further, we observe that our model almost shows the similar trajectories like Q_2–model for some values of α and w_h in $r-s$ plane. For $\alpha = -0.15$, $w_h = -0.5$ and $\alpha = -0.23$, $w_h = -1$ as shown in Figs. 4.1 and 4.2, respectively, the trajectories show almost similar behavior as Q_2–model for $\varepsilon = 2$.

It may be observed from Figs. 4.1 and 4.2 that for a fixed value of α, say $\alpha = -0.15$, the trajectories show more deviation from the Q_2–model as we increase the negative value of w_h. Further, for small negative values or positive values of α, the trajectories deviate more and more from the Q_2–model for fixed value of w_h [206,220]. We may fix α, and vary ζ_0 and ζ_1, the same behavior of trajectories can be observed for suitable values of ζ_0 and ζ_1.

In Figs. 4.3 and 4.4, we plot the trajectories in $r-q$ plane. Here, we have taken $c = 0.5$, $H_0 = 1$, $\zeta_0 = \zeta_1 = 0.05$, and $t_0 = 1$. The SCDM model and SS model (steady–state cosmology) have been shown by the fixed points $\{r,q\} = \{1,0.5\}$ and $\{r,q\} = \{1,-1\}$, respectively. We observe the signature flip in the values of q from positive to negative which explain the recent phase transition successfully. The ΛCDM model starts from the fixed point of SCDM model, evolves along the doted line and converges to the fixed point of SS model in the late time. It can be observed from both the figures that for any values of α and w_h, the viscous HDE model always approaches to the SS model, i.e., $\{r,q\} = \{1,-1\}$ as ΛCDM, Q_2 and chaplygin gas models approach in the late time evolution. However, in the early time of evolution, the model stats close to SCDM model for some values of α as shown in Figs. 4.3 and 4.4. Moreover, it may start exactly from the fixed point $\{r,q\} = \{1,0.5\}$ of SCDM for suitable value of α. In $r-q$ plane also, the trajectories corresponding to our model show the Q_2-model like behaviour. Again, comparing our viscous HDE model with Q_1–model, RDE model, Q_2–model and chaplygin gas model, we find that viscous HDE model is compatible with Q_2–model.

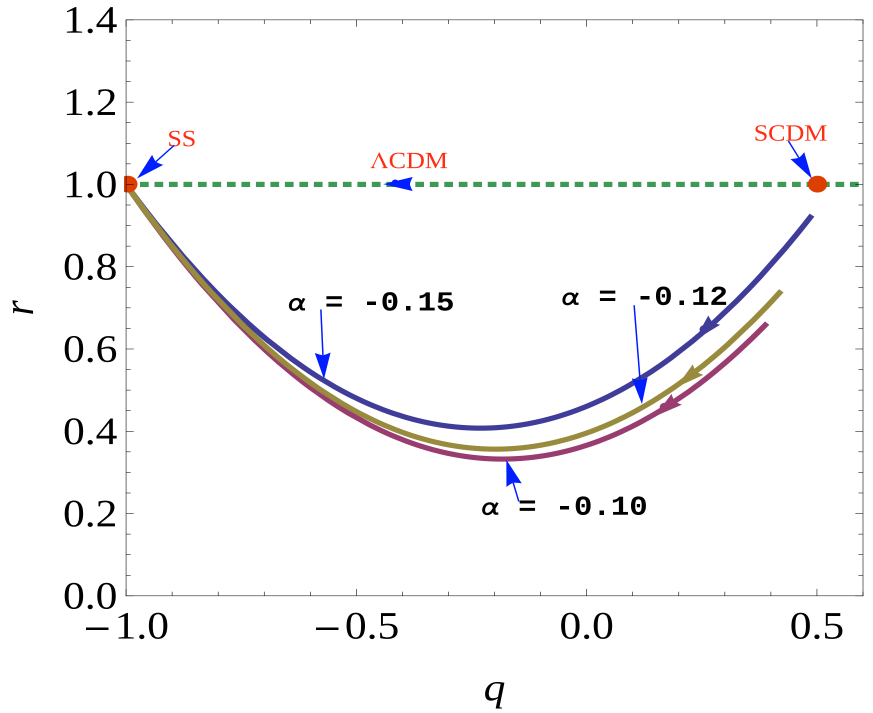

Figure 4.3: The trajectories in $r-q$ plane are plotted for $w_h = -0.5$ and different values of α.

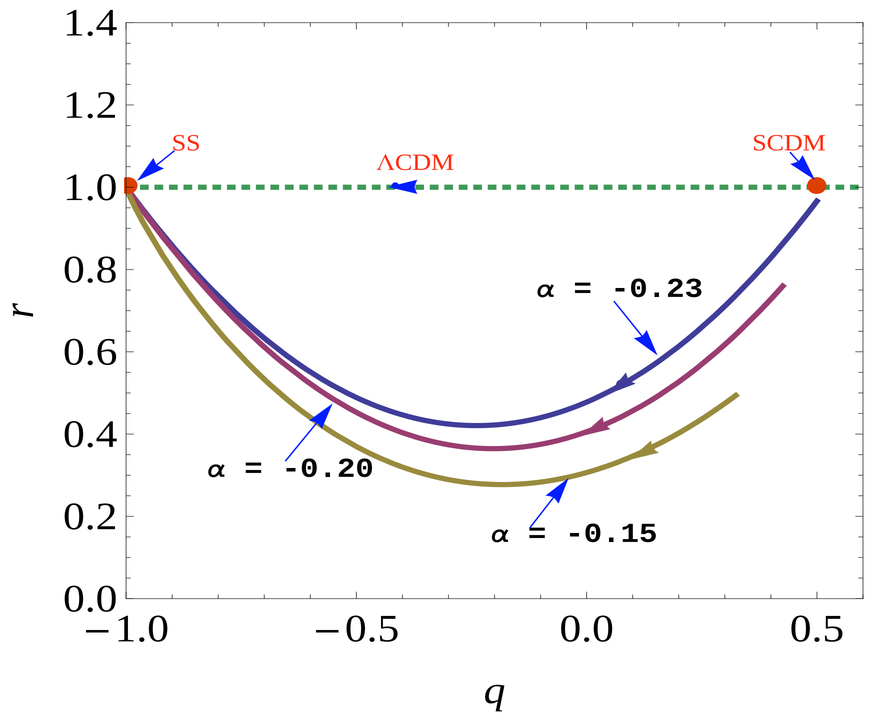

Figure 4.4: The trajectories in $r-q$ plane are plotted for $w_h = -1$ and different values of α.

4.4 Conclusion

In GTR and BD theory, the authors [72, 217] have found that the Hubble horizon as an IR cut–off is not a viable candidate for HDE model to explain the accelerated expansion of the Universe. However, Pavón and Zimdahl [73], and Banerjee and Pavón [218] have shown that the interaction between HDE and DM can describe the accelerated expansion. Therefore, it is clear that one can observe accelerated expansion if the interaction between dark components is considered. The $f(R,T)$ gravity theory presents a maximal coupling between geometry and matter. In this work, we have studied HDE models with Hubble horizon as an IR cut–off in the frame work of modified $f(R,T)$ gravity. Therefore, we have explored the consequences of the coupling of matter with the geometry of the Universe instead of taking the interaction between HDE and DM as many authors have studied. We have investigated the possibility whether the Hubble horizon as an IR cut–off could explain an accelerated expansion in $f(R,T)$ gravity. We have studied non–viscous and viscous HDE models in $f(R,T)$ gravity. It has been shown that the non–viscous and viscous HDE models with Hubble horizon as an IR cut–off may explain the accelerated expansion in the frame work of this modified theory. Further, we have investigated statefinder pair $\{r,s\}$ to discriminate our non–viscous and viscous HDE models with other existing DE models. We summarize the results of these two models as follows.

In non–viscous HDE model, we have found an accelerated expansion under the constraint of parameters. In this case, we have obtained constant deceleration and statefinder parameters. Due to constant q, it is not possible to analyze the phase transition of the Universe. We have found the fixed point $\{r,s\} = \{1,0\}$ of ΛCDM model as a particular case of this model. Thus, non–viscous HDE model is consistent with ΛCDM model.

In viscous HDE model, we have obtained the recent phase transition of the Universe as the DP comes out to be time–dependent and shows signature flip from positive to negative. In this model, the statefinder parameters also are the function of cosmic time t. These time–dependent parameters are possible due to the bulk viscous HDE which could explain the recent phase transition in a better way. It is interesting to note that the viscous HDE model gives the ΛCDM model fixed point $\{r,s\} = \{1,0\}$ and remains fixed in ΛCDM model throughout the evolution for a specific value of α as discussed in Sect. 4.3. The statefinder diagnostic have been discussed through the trajectories of $r-s$ and $r-q$ planes as shown in Figs. 4.1–4.4 to discriminate our model with the existing DE models. In Figs. 4.1 and 4.2, it has been observed that some of the trajectories pass through the vicinity of SCDM during early time but ultimately all approach to ΛCDM model in the late time. It can be seen in the Figs. 4.3 and 4.4 that the trajectories may start from SCDM model for a suitable value of α in early time but all the trajectories approach to SS model in the late time evolution. We observe that for some values of α, the trajectories of the viscous HDE model are similar to the trajectories of Q_2–model [206, 220]. Therefore, the viscous HDE in the framework of $f(R,T)$ gravity gives more general results in comparison to ΛCDM and Q_2–model at least at the level of statefinder diagnostic as we are able to achieve the behavior of both the models.

Chapter 5

Holographic Dark Energy in Brans-Dicke Theory

In this chapter[1], an interacting HDE model with Hubble horizon as an IR cut–off is considered in the framework of BD theory. We propose a logarithmic form $\phi \propto ln(l + m\,a)$ of the BD scalar field to alleviate the problems of interacting DE models in BD theory. The EoS parameter w_h of HDE and DP q for the model have been obtained. We presents a unified model of the Universe which explains the early time acceleration (inflation), medieval time deceleration and late time acceleration. It is also observed that w_h may cross the phantom divide line in the late time evolution. A time–varying density ratio of HDE to DM is obtained which is a constant of order one ($r \sim \mathcal{O}(1)$) during early and late time evolution. Therefore, the model successfully resolves the cosmic coincidence problem.

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[1]The work presented in this chapter comprises the results of a research paper entitled "Holographic dark energy in Brans-Dicke theory with logarithmic form of scalar field", Pre-print, arXiv:1609.01477 [gr-qc], (2016).

5.1 Introduction

The dynamical DE candidates were proposed to over come the shortcomings of the standard ΛCDM model like fine–tuning and cosmic coincidence problem [28, 39]. HDE models, which possess some significant properties of the quantum theory, have been proposed as a candidate of dynamical DE to explain the problems of modern cosmology. The origin of HDE contains the more scientific approach in comparison of other DE candidates and presents a better way to deal with the accelerated expansion. HDE models may also alleviate the cosmic coincidence problem which provides an advantage to HDE models over the other DE models.

As HDE models belong to dynamical DE models, therefore, a dynamical framework may be more suitable to study these models rather than GTR. BD theory is a natural extension of GTR. Recently, the BD theory has got interest to explain the accelerated expansion due to its association with the string theory and extra dimensional theories. This theory explains the recent accelerated expansion of the Universe and accommodates the observational data as well [128, 164, 165]. BD theory provides a dynamical framework which is more suitable to study HDE models as HDE belongs to the family of dynamical DE candidates. Therefore, it is quite natural to study HDE models in the framework of BD theory. HDE models have been studied in the framework of BD theory to explain the recent accelerated expansion and to alleviate the problems associated with DE models like cosmic coincidence problem [215–218, 222].

In the literature, most of the models have been discussed in BD theory by assuming the power–law form of BD scalar field $\phi \propto a^n$, where a is the scale factor and n is a constant. It has been shown in the papers [224–226] that the assumption $\phi \propto a^n$ naturally leads to a constant DP in BD theory irrespective of the matter content of the Universe. In the recent papers [217, 218, 222], authors have used the same power–law form of BD scalar field in HDE model and found a time dependent DP. Thus, it has been observed that a constant as well as a time dependent DP may be obtained with this power–law form of BD scalar field in HDE model. In our point of view there should not be two different values of DP with in a same model. Therefore, it is natural to explore more options for BD scalar field to over come from the shortcoming of this power–law form of BD scalar field.

In this context, we propose a logarithmic form $\phi \propto ln(l + m\,a)$ of BD scalar field, where a as usual denotes the scale factor, and l and m are positive constants such that $l > 1$ and $m > 0$. This form of BD scalar field is free from the constant value of DP which naturally arises in the power–law form. We successfully obtain the EoS of DE and the time dependent DP. We observe unification of early time acceleration (inflation) and late time acceleration including matter dominated era. In the early and late time evolution we find that the density ratio of HDE to DM is a constant of order one ($r \sim \mathscr{O}(1)$) which successfully solve the cosmic coincidence problem.

5.2 HDE in BD Theory

Let us consider the modified EH action for BD theory in Jordan frame given by Eq. (1.4.20). The variation of action (1.4.20) with respect to metric tensor $g_{\mu\nu}$ gives field equations of BD theory (1.4.21) and (1.4.22).

We consider a homogeneous and isotropic flat FRW Universe given by (1.1.5) and assume that the Universe is filled with perfect fluid containing pressureless DM (excluding baryonic matter) and HDE. Therefore, we consider the energy–momentum tensor given in Eq. (4.2.1). The field equations (1.4.21) and (1.4.22) for FRW line element (1.1.5) and energy–momentum tensor (4.2.1) yield the following equations.

$$H^2 + H\frac{\dot{\phi}}{\phi} - \frac{\omega}{6}\frac{\dot{\phi}^{\,2}}{\phi^{\,2}} = \frac{\rho_m + \rho_h}{3\phi}, \tag{5.2.1}$$

$$2\frac{\ddot{a}}{a} + H^2 + 2H\frac{\dot{\phi}}{\phi} + \frac{\omega}{2}\frac{\dot{\phi}^{\,2}}{\phi^{\,2}} + \frac{\ddot{\phi}}{\phi} = -\frac{p_h}{\phi}, \tag{5.2.2}$$

$$\ddot{\phi} + 3H\dot{\phi} = \frac{\rho_m + \rho_h - 3p_h}{2\omega + 3}, \tag{5.2.3}$$

where ρ_m and ρ_h are, respectively the energy density of DM and HDE, and p_h denotes the pressure of HDE. Here, an over dot denotes the derivative with respect to the cosmic time t. The equation (5.2.3) is dynamical equation of the BD scalar field ϕ.

Many cosmologists [227–231] have considered the interaction between DM and DE. The recent cosmological observations [232–235] also support this interaction. Considering the interacting factor Q between DM and HDE, the conservation equations of DM and HDE are respectively given by [218, 223]:

$$\dot{\rho}_m + 3H\rho_m = Q, \tag{5.2.4}$$

$$\dot{\rho}_h + 3H(\rho_h + p_h) = -Q, \tag{5.2.5}$$

where $Q = \Gamma\rho_h$, Γ stands for interaction rate. The sign of interaction rate Γ is crucial and defines the direction of the energy transfer. For $\Gamma > 0$, there is an energy transfer from HDE to DM, and for $\Gamma < 0$, there is an energy transfer from DM to HDE. In the literature [216,223], Γ has been assumed to be proportional to the Hubble parameter, i.e., $\Gamma \propto H$ to maintain the interaction term Q as a function of a quantity with units of inverse of time multiplied with the energy density. Therefore, let us consider $\Gamma = 3b^2H$ so that the interaction term becomes $Q = 3b^2H\rho_h$, where b^2 is a coupling constant. This assumption relies purely on dimension basis in the absence of a suitable theory.

As we have discussed in introduction, based on holographic principle [68] Li [71] obtained the density of HDE, $\rho_h = 3d_1^2M_p^2L^{-2}$ defined in (1.4.3). In the framework of BD theory, the HDE density has the form $\rho_h = 3d_1^2\phi L^{-2}$, where $\phi = M_p^2 = (8\pi G)^{-1}$ is a time dependent scalar field which couples to gravity with a coupling parameter ω. In the literature, there are various forms of HDE depending on the different choices of IR cut–off (L) like particle horizon, future event horizon, Hubble horizon, Granda–Oliveros cut–off etc. The Hubble horizon is a most natural candidate of IR cut–off and it is free from causality problem. Therefore, we choose Hubble horizon as an IR cut–off which gives HDE density as:

$$\rho_h = 3\, d_1^2\phi H^2, \tag{5.2.6}$$

where d_1^2 is a dimensionless constant.

5.3 Logarithmic Form of BD Scalar Field and Cosmological Consequences

Many authors [205, 224–226] have assumed that BD scalar field ϕ evolves as a power–law of the scale factor a, i.e., $\phi \propto a^n$. They have observed that this assumption leads to a constant value of DP. The constant value can be obtained irrespective of matter content of the Universe. However, some authors [217,218,222] have studied HDE models in BD theory with the same form of BD scalar field and have obtained a time–dependent DP. Now, the question is that why does the same form of BD scalar field lead to two different values of DP, constant and time–dependent in a same model?

In spite of several advantages of this form of BD scalar field, it seems from the above mentioned works that this may not be a suitable assumption to discuss the evolution of the Universe in HDE models. In other words, this form may not be suitable for those models where we want to study the phase transition of the Universe. Taking into consideration to this problem, we hereby propose that the BD scalar field evolves as a logarithmic function of the scale factor which is given by

$$\phi \propto ln(l+m\,a), \tag{5.3.1}$$

where $l>1$ and $m>0$ are constants.

In principle, BD scalar field should evolve slowly to observe a slow variation of G. This logarithmic form of ϕ fulfills this requirement. In this process, the value of m also plays an important role. It is worth noting that GTR can be recovered for $m=0$. One can observe that this form does not give a constant value of DP when we combine Eqs. (5.2.1) and (5.2.2) with (5.2.3). Thus, the constant value problem of the power–law form can be resolved. It provides an initial advantage to the logarithmic form over the power–law form of BD scalar field. Therefore, it will be interesting to investigate the role of this form in the evolution of the Universe within the formalism of interacting HDE model.

Using the form $Q=3b^2H\rho_h$ in (5.2.5), we have

$$\dot{\rho}_h+3H(1+w_h)\rho_h=-3b^2H\rho_h, \tag{5.3.2}$$

where $w_h=p_h/\rho_h$ is the EoS parameter of HDE. Using (5.2.6) and (5.3.1) into (5.3.2), we get

$$\frac{m;a}{(l+m\,a)ln(l+m\,a)}+2\frac{\dot{H}}{H^2}+3(1+w_h)=-3b^2. \tag{5.3.3}$$

From (5.2.1) and (5.2.2), we obtain

$$\begin{aligned}\frac{\dot{H}}{H^2} &= \frac{1}{2+\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}}\Big[-3-\frac{\omega\,m^2a^2}{2(l+m\,a)^2\,[ln(l+m\,a)]^2}-\frac{3\,m\,a}{(l+m\,a)\,ln(l+m\,a)}\\ &+ \frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)}-\frac{3w_h}{1+r}\Big\{1+\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}\\ &- \frac{\omega\,m^2a^2}{6(l+m\,a)^2\,[ln(l+m\,a)]^2}\Big\}\Big],\end{aligned} \tag{5.3.4}$$

where $r=\rho_m/\rho_h$ represents the energy density ratio. Using (5.3.4) into (5.3.3), we get

$$\begin{aligned} w_h &= \frac{(r+1)}{3\left[2r+\frac{(r-1)\,m\,a}{(l+m\,a)\,ln(l+m\,a)}+\frac{\omega\,m^2a^2}{3(l+m\,a)^2\,[ln(l+m\,a)]^2}\right]} \\ &\times\ \Big[-3b^2\left(2+\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}\right)-\frac{2\,m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)} \\ &+\ \frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}+\frac{(\omega-1)\,m^2a^2}{(l+m\,a)^2\,[ln(l+m\,a)]^2}\Big]. \end{aligned} \tag{5.3.5}$$

Let us first discuss the terms present in (5.3.5) which have significant contribution in the behavior of w_h. We observe that the terms $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ and $\frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)}$ are zero at $a=0$ and both the terms converse to zero as $a\to\infty$. We find that both the terms start from zero, achieve maximum value during the evolution and then converge to zero in the late time evolution. Further, we observe that both the terms attain the maximum value asymptotically for sufficiently small values of l and sufficiently large values of m. The maximum value of the term $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ lies in the interval]0, 1[whereas the maximum value of the term $\frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)}$ lies in the interval]0, 0.41[depending on the value of l only. It is to be noted that the parameter m does not has any effect on the maximum values of both the terms.

The value of w_h given by (5.3.5) contains two more parameters ω and b^2 which also play an important role in the value of w_h. It has been observed that the value of b^2 is a small positive constant. The value of BD parameter ω has been constrained by various astronomical and cosmological observations. The solar system experiment Cassini gave a very stringent high bound $\omega>40000$ [236, 237] for spherically symmetric solution in the parameterized post Newtonian formalism. The solar system constraints on ω may not be consistent at the cosmological scales, therefore, the cosmological constraints are required to study the large scale properties of the Universe. In this context, Acquaviva et al. [238] have found $\omega>120$ at 95% confidence level, Wu and Chen [239] have found $\omega<-120$ or $\omega>97.8$ at 95.5% confidence level whereas Li et al. [240] have obtained $\omega>181.65$ at 95% confidence level. However, all these constraints on ω depend on the choice of a model. In the present paper, we will consider only the values $\omega>0$.

Now, let us discuss the behavior of EoS parameter w_h of HDE as obtained in Eq. (5.3.5). The sign of w_h depends on the sign of numerator as the denominator has only positive values. In the beginning of the evolution ($a=0$), all the terms of the numerator are zero except the first term which has negative sign. Therefore, we observe a negative value of w_h as

$$w_h = -b^2 \left(1 + \frac{1}{r}\right). \tag{5.3.6}$$

Thus, HDE has EoS of DE in the beginning which is required to explain the inflation supposed to happen in very early time of the evolution. As the Universe evolves, the terms $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ and $\frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)}$ attain their maximum value, therefore, the last term of numerator containing ω starts to dominate as a large value of ω has been expected from the observations. Thus, we may observe a change in the sign of EoS parameter w_h from negative to positive and HDE may start behaving like radiation ($w_h > 0$). It is evident from (5.3.5) that the first term of the numerator has a constant part, and the terms $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ and $\frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)}$ converse to zero in the late time evolution. Therefore, ultimately the sign of w_h changes from positive to negative. Thus, HDE achieves the EoS of DE again during the evolution which is required to explain the late time acceleration of the Universe. It is also obvious that HDE achieves dust like behavior ($w_h = 0$) whenever w_h changes sign from positive to negative or viceversa. Zimdahl and Pavón [241] for HDE, and Zlatev et al. [32] for tracker quintessence scalar field have also observed radiation and dust like behavior in the early time evolution.

We also emphasise that for a small value of ω or a large value of b^2 or a suitable combination of their values, the EoS parameter w_h is negative throughout the evolution. Thus, HDE behaves like DE throughout the evolution. As it is mentioned that the maximum value of the terms $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ and $\frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)}$ depend on the value of l, the change in sign of w_h also depends on the value of l. The possible behaviors of w_h are shown in Figs. 5.1 and 5.2 for different values of parameters which show the similar behaviors as discussed above. We have used value of r from (5.4.2) and have assumed the present value of scale factor $a_0 = 1$ here and thereafter for our convenience.

It is worthy to note that if we take constant value of Γ as taken in [73], the Eq. (5.3.5) in the late time evolution ($a \to \infty$) reduces to its respective expression in GTR for HDE model, that is,

$$w_h \approx -\left(1 + \frac{1}{r}\right)\frac{\Gamma}{3H}. \tag{5.3.7}$$

Now, for time–dependent $\Gamma = 3b^2H$, we get the expression of w_h for late time evolution as

$$w_h \approx -b^2 \left(1 + \frac{1}{r}\right). \tag{5.3.8}$$

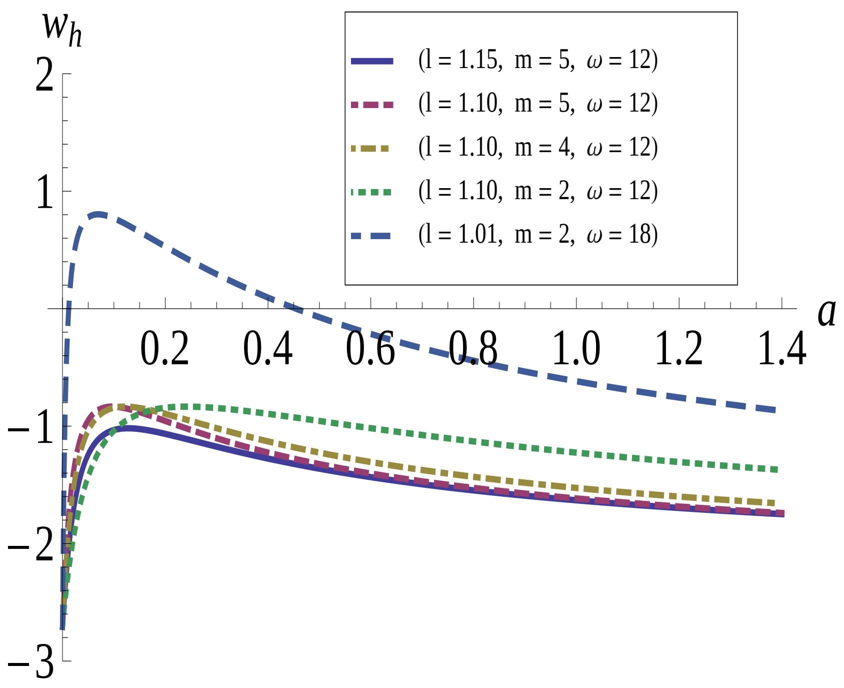

Figure 5.1: The evolution of w_h verses a for different values of l, m and ω with $b = 1.05$ and $d_1 = 0.77$.

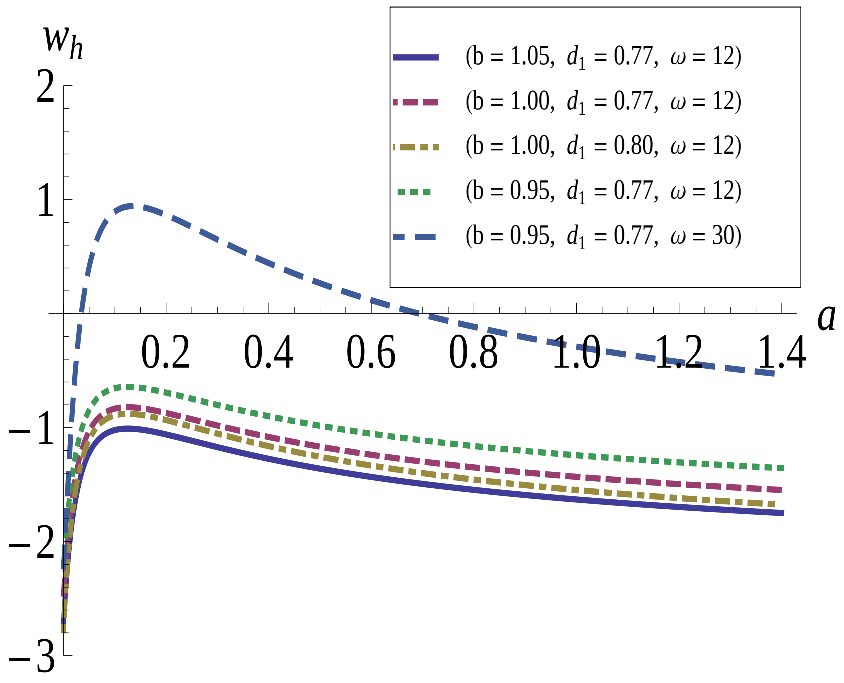

Figure 5.2: The evolution of w_h verses a for different values of b, d_1 and ω with $l = 1.15$ and $m = 5$.

One can observe that the EoS parameter (5.3.8) converses to a negative constant value in the late time evolution. It means that as the Universe enters in the late time accelerating phase, the HDE behave like DE forever. In the absence of interaction ($b^2 = 0$), we obtain a positive EoS parameter of HDE ($w_h > 0$) from (5.3.5) through out the evolution. It means accelerated expansion is not possible in this case as observed in the papers [73, 217]. It is also observed that w_h may cross the phantom divide ($w_h = -1$) for a suitable value of b^2 in the late time evolution.

In the present model, the dynamics of the Universe depends not only on DM and HDE, but also on the BD scalar field. It will be early to conclude about the evolution of the Universe only on the basis of EoS of HDE. Therefore, it is important to discuss the behavior of DP to make a precise conclusion. We obtain the value of q after dividing Eq. (5.2.2) by H^2, and using (5.2.6) and (5.3.1) which is given as

$$q = \frac{3\,d_1^2\,w_h + 1 - \frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)} + \frac{2\,m\,a}{(l+m\,a)\,ln(l+m\,a)} + \frac{\omega\,m^2a^2}{2(l+m\,a)^2\,[ln(l+m\,a)]^2}}{2 + \frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}}. \tag{5.3.9}$$

Let us examine the sign of DP to discuss the early and late time evolution. In the beginning of evolution at $a = 0$, we obtain $q = \frac{3\,d_1^2\,w_h+1}{2}$. Using the initial value of w_h given by Eq. (5.3.6), we get $q = \frac{1}{2}\left[1 - 3b^2d_1^2\left(1+\frac{1}{r}\right)\right]$, which produces accelerated expansion for $3b^2d_1^2\left(1+\frac{1}{r}\right) > 1$. Thus, we observe inflationary era which has been expected to happen in the beginning of the evolution to resolve the problems of Big Bang cosmology. As w_h may show sign change from negative to positive during the evolution and the last term containing ω in numerator of Eq. (5.3.9) starts to dominate due to large value of ω, a change in the sign of DP from negative to positive may also be observed. If w_h is negative throughout the evolution then also we observe sign change in q as shown in Figs. 5.3 and 5.4. Thus, we observe a decelerated expansion of the Universe after inflation. Since the terms $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ and $\frac{m^2a^2}{(l+m\,a)^2\,ln(l+m\,a)}$ converse to zero in the late time evolution, the late time value of q is obtained as

$$q \approx \frac{3d_1^2w_h + 1}{2}. \tag{5.3.10}$$

Using the late time value of w_h given by (5.3.8), we get

$$q \approx \frac{1}{2}\left[1 - 3b^2d_1^2\left(1+\frac{1}{r}\right)\right]. \tag{5.3.11}$$

Here, q will be negative if $3b^2d_1^2\left(1+\frac{1}{r}\right) > 1$, the same as for early time. Therefore,

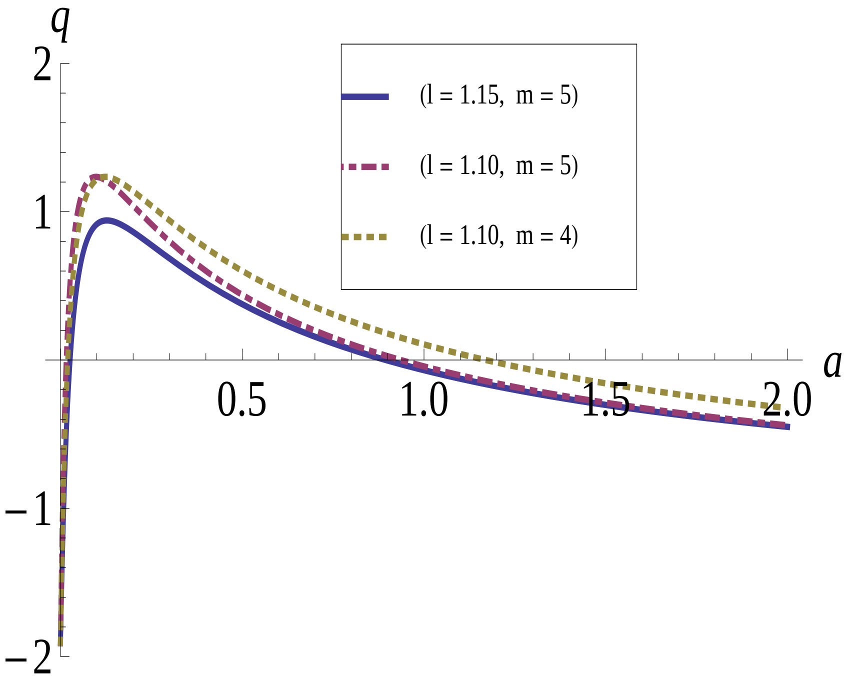

Figure 5.3: The evolution of q verses a for different values of l and m with $b = 1.05$ and $d_1 = 0.77$.

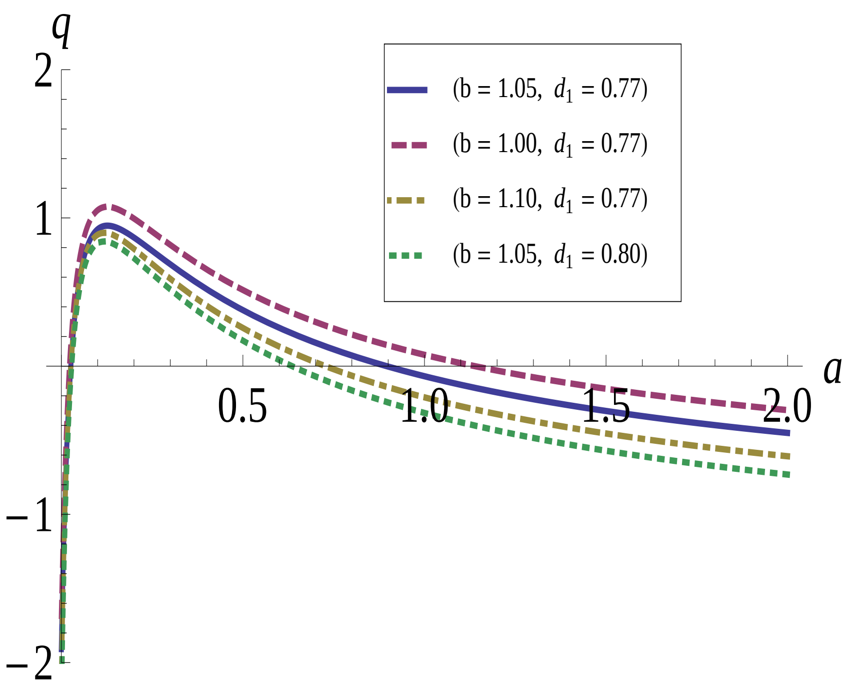

Figure 5.4: The evolution of q verses a for different values of b and d_1 with $l = 1.15$ and $m = 5$.

a sign change of q from positive to negative may be observed which successfully explains the late time phase transition (deceleration to acceleration) of the Universe. Indeed, HDE shows almost same behavior in the early and late time evolution of the Universe. If there is no interaction ($b^2 = 0$), we observe $q > 0$ through out the evolution. Thus, using the value of q we reconfirm that the accelerated expansion is not possible in the case of no interaction. It can be seen that it is not only for $b^2 = 0$, but also for a small value of $b^2 << 1$, a decelerating Universe may be observed throughout the evolution. The other parameters of the model will not change this scenario. Further, we observe that a sufficiently large value of b^2 is able to accelerate the expansion through out the evolution. Therefore, the value of coupling parameter b^2 plays an important role to observe the evolution of the Universe which is consistent with the observations. It means that a suitable interaction between the dark components of the Universe, i.e., a suitable value of b^2 is required to be consistent with the observations.

After having a suitable interaction, the most important parameters are l and m, which are due to the logarithmic form of BD scalar field. The terms containing l and m in Eq. (5.3.9) start from zero, evolve up to maximum value and converse to zero in the late time which lead to observe two phase transitions of the Universe, i.e., the early time inflation and the late time acceleration. It is worthy to mention here that for the power–law form of BD scalar field, only a late time acceleration is possible. The values of l and m are important to decide the time when the phase transitions, inflation and late time acceleration, will happen which can be seen in Figs. 5.3 and 5.4. Thus, to accommodate the observed history of the Universe the value of l and m are significant. It is also evident from (5.3.9) that for a large value of ω the inflation ends early and the late time acceleration occurs late, and viceversa. So far we have observed that a number of combination of the parameters are possible to explain the evolution of the Universe but let observations to decide the best combination of the parameters.

We have shown that the present model of HDE is able to unify the inflation and late time acceleration including the matter dominated era (radiation+dust) of the Universe. Thus, this model presents a unified model of interacting HDE in the framework of BD theory with logarithmic form of BD scalar field. We emphasise that the logarithmic form proposed in this chapter has played an important role to explain the evolution of the Universe in more better way than power–law form. In the next section, we will show that this form also plays a significant role to resolve the coincidence problem.

5.4 Cosmic Coincidence Problem

Let us investigate the present model on the ground of cosmic coincidence problem [31,32]. Using the conservation Eqs. (5.2.4) and (5.2.5), the evolution of energy density ratio r can be obtained as

$$\dot{r} = 3Hr\left[w_h + \left(\frac{1+r}{r}\right)b^2\right], \tag{5.4.1}$$

which is the same expression as obtained in [73, 218] except we have b^2 at the place of $\Gamma/3H$ because they have chosen Γ as a constant whereas we have taken $\Gamma = 3b^2H$. The nature of evolution of r in our model is different from the mentioned works as we have taken time-dependent Γ and it also depends on the value of w_h.

Using the late time value of w_h given by (5.3.8) into (5.4.1) we obtain $\dot{r} = 0$, i.e., in the late time evolution r has a constant value which is a significant feature to solve the coincidence problem. Pavón and Zimdahl [73] have obtained a constant value of r throughout the evolution for HDE model in the framework of GTR, however, r is expected to be a time–varying value. Therefore, the authors assumed a time–varying value of d_1 to obtain a time–varying r. It has also been suggested to replace the Hubble horizon by the future event horizon to achieve a time–varying value of r [242]. Although, Banerjee and Pavón [218] have obtained time–dependent value of r in BD theory but they achieved a soft coincidence only. They argued that r may vary more slowly in their model than in the conventional ΛCDM model. It have been demonstrated by Zhang et al. [243] that the interacting chaplygin gas model has a better chance to solve the coincidence problem in comparison of interacting quintessence and interacting phantom models. In the literature, many proposals have been made to solve the coincidence problem [244–250] but the problem still exists.

Let us find the value of r for our model to analyse coincidence problem in more detail. From (5.2.1), the value of r is obtained as

$$r = -1 + \frac{1}{d_1^2} + \frac{m\,a}{d_1^2(l+m\,a)\,ln(l+m\,a)} - \frac{\omega\,m^2a^2}{6\,d_1^2(l+m\,a)^2\,[ln(l+m\,a)]^2}. \tag{5.4.2}$$

Here, we obtain a time–dependent value of r which has a constant and finite value $r = -1 + \frac{1}{d_1^2}$ at the beginning of the evolution as the last two terms are zero at $a = 0$. In the late time evolution, we also obtain a constant and finite value $r \approx -1 + \frac{1}{d_1^2}$ as the last two terms converge to zero. The same constant value of r has been obtained through out the evolution in GTR [73] but we have obtained a time–dependent value of r which has constant values in the early and late time evolution.

Here, it will be interesting to quote a paper of Campo et al. [244] “Obviously, a mechanism that makes r tends to a constant today or decrease its rate to a lower value than the scale factor expansion rate ameliorates the coincidence problem significantly, but it does not solve it in full. To do so the said mechanism must also achieve $r_0 \sim \mathscr{O}(1)$". We have obtained $r_0 \sim -1+\frac{1}{d_1^2}$ which excellently satisfies the requirement $r_0 \sim \mathscr{O}(1)$ as most of the observational constraints on d_1 obtain $0.5 < d_1 < 1$ [251–253]. In the other words, the cosmic coincidence problem has been resolve completely. In fact, we observe $r \sim \mathscr{O}(1)$ in the early time as well as in the late time evolution in our model. Therefore, the cosmic coincidence problem does not seem a problem because it is not a coincidence that we are living in a time where $r \sim \mathscr{O}(1)$, it have been observed in the early time also. Thus, in conclusion we can say that the cosmic coincidence problem has been resolved in a significantly well manner in the present model.

From (5.3.6) and (5.4.1), it is clear that whatever the value of b^2 be chosen, we obtain $\dot{r} = 0$ in the late time evolution. It means the coupling parameter b^2 between DM and HDE does not play a significant role in the alleviation of coincidence problem. This is also evident from the value of r given by (5.4.2) which has no b^2 term. Also, one may observe that the parameter d_1 is not significant to resolve the coincidence problem. Although, it plays an important role in the value of r at any given time. Actually, the last two terms in Eq. (5.4.2) which come due to the assumption of logarithmic form of BD scalar field, play an important role to decide the coincidence problem. These two terms converse to zero in the late time and may vary sufficiently slow at present due to which r converses to a constant value in the late time and vary sufficiently slow at present. The suitable values of the parameters l and m ensure the slow variation of r at present time. The variation of r is shown in the Fig. 5.5 which clearly verify our claim.

Let us find the value of d_1 to match with the observational values using the observed value of r. According to recent observations, the present value of r is $\approx \frac{3}{7}$, where matter has $\approx 30\%$ and DE has $\approx 70\%$ of total energy content of the Universe. We obtain the value $d_1 \approx 0.8367$ using the theoretical relation $r \approx -1+\frac{1}{d_1^2}$ from (5.4.2) for late time evolution. Xu at al. [251] have obtained similar value $d_1 = 0.807^{+0.165}_{-0.160}$ for HDE model in the framework of BD theory. The authors in [254] constrained HDE with varying

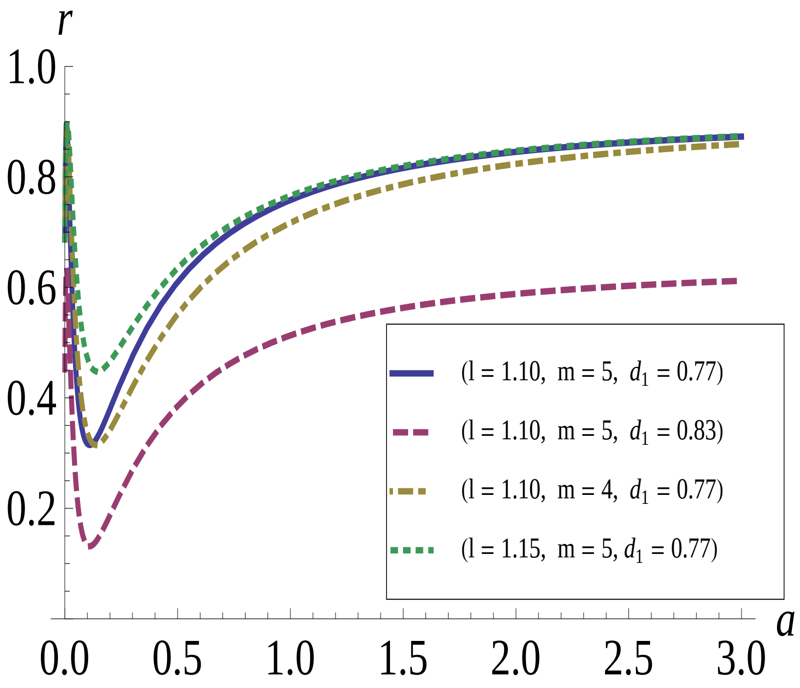

Figure 5.5: The graph of r verses a for different values of l, m and d_1 with $\omega = 12$.

gravitational constant and obtained $d_1 = 0.80^{+0.19}_{-0.14}$. Using the observations from various probes like WMAP, BAO, Planck etc. Li et al. [252] have also obtained values of d_1 which show similarity with the above values, however, all these models are different.

5.5 Conclusion

In this chapter, we have studied interacting HDE model with Hubble horizon as an IR cut–off in the framework of BD theory. We have pointed out a serious problem associated with the assumption of power–law form ($\phi \propto a^n$) of BD scalar field. It has been observed that this form of BD scalar field gives constant and time dependent DP for same model. Therefore, taking into consideration of this problem, we have proposed a logarithmic form of BD scalar field which always gives a time varying DP. We have observed that this form of BD scalar field is consistent with the cosmic evolution of the Universe. We have obtained EoS parameter and DP to discuss the early and late time evolution of the Universe. We have also discussed the cosmic coincident problem. A summary of the main findings is as follows.

In the early time evolution, we observe $w_h < 0$ and $q < 0$ which explains the inflationary era of the cosmic evolution supposed to happen in the very early Universe to resolve the problems of Big Bang cosmology. Further, q changes sign from negative to positive which explains the matter dominated era of the evolution. In the late time evolution, we have observed another sign change of q from positive to negative which explains the late time acceleration of the Universe. Thus, interacting HDE model with logarithmic form of BD scalar field presents a unified model of the Universe. If we consider Γ as a constant then the EoS parameter w_h reduces to its corresponding form in GTR in the late time. The EoS parameter converges to a constant value and for a suitable value of b^2 it may cross the phantom–divide line $w_h = -1$ in the late time evolution. If there is no interaction between HDE and DM, i.e., $b^2 = 0$, it has been observed that the EoS parameter w_h and DP q are always positive, therefore, the accelerated expansion is not possible in this case.

The cosmic coincidence problem, a long standing problem with DE models, has been resolved effectively in the present model. We have observed a time varying energy density ratio r which has a constant value in early and late time evolution of the Universe. The present value r_0 satisfies the condition $r_0 \sim \mathscr{O}(1)$ which is consistent with the observations. Using the present value of r obtained by observations, we get theoretical value $d_1 = 0.8367$ which shows similarity with the observed values of d_1.

In conclusion, the advantage of our model is that we have alleviated the problem associated with the power–law containing the good features of it. In addition, our model presents a unification of early time inflation and late time acceleration including the matter dominated era which have not been obtained through power–law form. In the present model, the cosmic coincidence problem has been alleviated more effectively.

Chapter 6

New Agegraphic Dark Energy in Brans-Dicke Theory

In the present chapter[1], the cosmological evolution of a NADE model within the framework of FRW Universe is analyzed in BD theory by assuming the logarithmic form of BD scalar field. We derive the EoS parameter w_d and DP q of NADE model. It is observed that NADE mimics cosmological constant in the late time evolution. The NADE model shows a phase transition from matter dominated phase in early time to accelerated phase in late time. We further extend NADE model by including the interaction between DM and NADE. In this case, w_d definitely crosses the phantom divide line ($w_d = -1$) in the late time evolution. The phase transition from matter dominated to NADE dominated may be achieved at early stage in interacting model. Further, we show that the interacting NADE model resolves the cosmic coincidence problem as the energy density ratio may evolve sufficiently slow at present.

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[1]This chapter is based on a research paper entitled "New agegraphic dark energy in Brans-Dicke theory with logarithmic form of scalar field, *Astrophysics and Space Science*, DOI 10.1007/s10509-017-3032-0 (2017); [arXiv: gr-qc/1609.02751]".

6.1 Introduction

In the literature, NADE has also been presented as a possible candidate of DE. It is a generalization of ADE model which is based on the quantum fluctuations of the space–time metric. Using the concept of quantum fluctuations of space–time metric, Cai [81] proposed the original ADE model in which the energy density of ADE is given by

$$\rho_d = \frac{3d_2^2 {M_p}^2}{T^2}, \tag{6.1.1}$$

where T is age of the Universe. The numerical factor $3d_2^2$ has been introduced to parameterize some uncertainties like the species of quantum fields in the Universe, the effects of curved space–time, etc. Here, we assume the units $\hbar = k_b = c = 1$. A number of papers on ADE models are available which deal with various aspects of the evolution of the Universe [82–84]. Wei and Cai [85] have proposed a new version of this model referred as NADE model by replacing the cosmic age T with the cosmic conformal age η for the time scale in (6.1.1). Thus, the density of NADE is given by

$$\rho_d = 3d_2^2 M_p^2\, \eta^{-2}, \tag{6.1.2}$$

where conformal time η is defined by Eq. (1.4.8).

In the paper [86], the authors have found constraints on NADE. The authors have also observed that if the parameter d_2 present in model is of order unity then coincidence problem may be avoided in NADE model. Setare [87] has studied NADE model in $f(R)$ gravity and has developed a reconstruction scheme of modified gravity models using $f(R)$ action. Sun and Yue [255] have investigated NADE for inflationary era and they found that NADE is not compatible with inflation era. A number of authors have discussed various aspects of NADE models in the literature [88–90].

The NADE models have also been studied in the framework of BD theory to discuss the evolution of the Universe. Sheykhi [223] have studied interacting NADE in non-flat BD cosmology and found that the phantom divide crossing may be observed for both non-interacting and interacting NADE models. The NADE models with chameleon scalar field cosmology have been studied to explain accelerated expansion in the framework of BD theory [216, 256]. Many other authors have studied NADE model in BD theory [257, 258].

To the best of our knowledge, In the framework of BD theory all the papers available in the literature on NADE models have assumed that the BD scalar field evolves as a power–law of scale factor, $\phi \propto a^n$, where n is a constant. In previous chapter, we have discussed the drawback of this relation of BD scalar field in BD theory and have proposed a new relation between BD scalar field and scale factor, namely, a logarithmic form $\phi \propto ln(l+m\,a)$ which is free from this problem. We have demonstrated that the logarithmic form of BD scalar field is not only suitable to explain the phase transition of the Universe but also it resolves the cosmic coincidence problem in an effective way.

As the logarithmic form of BD scalar field presents some interesting results in HDE model, therefore, it is worthwhile to discuss the evolution of NADE model with this form of BD scalar field. We derive the EoS parameter w_D and DP q of NADE model to discuss the evolution of the Universe. We further extend NADE model by including the interaction between DM and NADE in BD theory. We show that the interacting NADE model resolves the cosmic coincidence problem as the energy density ratio may evolve sufficiently slow at present.

6.2 NADE in BD Theory with Logarithmic Scalar Field

We consider a homogeneous and isotropic FRW space–time defined in (1.1.4) to discuss the evolution of the Universe. We assume that the Universe is filled with pressureless DM and NADE for which the energy–momentum tensor is given by

$$T_{\mu\nu} = (\rho_m + \rho_d + p_d)u_\mu u_\nu - p_d g_{\mu\nu}, \tag{6.2.1}$$

where ρ_d and p_d are the energy density and pressure of NADE, respectively. We exclude baryonic matter and radiation due to their negligible contribution to total energy budget in the late time evolution.

The BD field equations (1.4.21) for the line element (1.1.4) and energy–momentum tensor (6.2.1) yield

$$H^2 + \frac{k}{a^2} + H\frac{\dot{\phi}}{\phi} - \frac{\omega}{6}\frac{\dot{\phi}^{\,2}}{\phi^{\,2}} = \frac{\rho_m + \rho_d}{3\phi}, \tag{6.2.2}$$

$$2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} + 2H\frac{\dot{\phi}}{\phi} + \frac{\omega}{2}\frac{\dot{\phi}^{\,2}}{\phi^{\,2}} + \frac{\ddot{\phi}}{\phi} = -\frac{p_d}{\phi}, \tag{6.2.3}$$

The wave equation (1.4.22) for scalar field in BD theory gives

$$\ddot{\phi}+3H\dot{\phi}=\frac{\rho_m+\rho_d-3p_d}{2\omega+3}, \tag{6.2.4}$$

Let us assume the logarithmic form of BD scalar field as proposed in chapter 5 in Eq. (5.3.1), which is given by

$$\phi=\phi_0\, ln(l+m\, a), \tag{6.2.5}$$

where ϕ_0, $l>1$ and $m>0$ are constants. Substituting (6.2.5) into (6.2.2) we find

$$H^2\left(1+\frac{k}{a^2}+\frac{m\, a}{(l+m\, a)\, ln(l+m\, a)}-\frac{\omega}{6}\frac{m^2a^2}{(l+m\, a)^2\,\{ln(l+m\, a)\}^2}\right)=\frac{\rho_m+\rho_d}{3\phi}. \tag{6.2.6}$$

It is easy to observe from (6.2.6) that the standard cosmology of GTR will be recovered in the limit of $m\to 0$.

In BD theory, BD scalar field is taken as $\phi=(8\pi G)^{-1}$ which implies $\phi=M_p^2$. Therefore, in this theory the energy density of NADE in (6.1.2) now becomes

$$\rho_d=3d_2^2\phi\;\eta^{-2}. \tag{6.2.7}$$

The critical energy density ρ_{cr} and energy density of the curvature ρ_k can be defined as follows

$$\rho_{cr}=3\phi H^2, \qquad \rho_k=3k\phi a^{-2}. \tag{6.2.8}$$

It has been observed that the representation of quantities and equations in the terms of fractional energy densities are useful from the point of view of calculation as well as from the point of view of physical interpretation of the results. The fractional energy densities can be defined in their usual forms as

$$\Omega_m=\frac{\rho_m}{\rho_{cr}}=\frac{\rho_m}{3\phi H^2}, \tag{6.2.9}$$

$$\Omega_d=\frac{\rho_d}{\rho_{cr}}=\frac{d_2^2}{\eta^2H^2}, \tag{6.2.10}$$

$$\Omega_k=\frac{\rho_k}{\rho_{cr}}=\frac{k}{a^2H^2}. \tag{6.2.11}$$

In what follows, we consider the non-interacting and interacting NADE models, and discuss their physical behaviors.

6.2.1 Non-interacting NADE Model

In this section, we assume that DM and NADE do not interact and their energy densities are conserved separately. The conservation equation of both the contents are given as

$$\dot{\rho}_m + 3H\rho_m = 0, \tag{6.2.12}$$

$$\dot{\rho}_d + 3H(1+w_d)\rho_d = 0, \tag{6.2.13}$$

where $w_d = p_d/\rho_d$ denotes the EoS parameter of NADE. Using (6.2.5), (6.2.7) and (6.2.10) into (6.2.13), the EoS parameter w_d is given by

$$w_d = -1 - \frac{m\,a}{3(l+m\,a)\,ln(l+m\,a)} + \frac{2}{3\,d_2\,a}\sqrt{\Omega_d}. \tag{6.2.14}$$

It is to be noted that the EoS parameter w_d of NADE reduces to its respective form in GTR [85] for $m = 0$, that is,

$$w_d = -1 + \frac{2}{3\,d_2\,a}\sqrt{\Omega_d}. \tag{6.2.15}$$

From (6.2.15) we observe that the w_d does not cross the phantom divide line $w_d = -1$ in the framework of GTR.

The behavior of the term $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ present in (6.2.14) has already been discussed in Sect. 5.3 of chapter 5. We observe that max$\{\frac{m\,a}{(l+m\,a)ln(l+m\,a)}\} \to 1$ as $l \to 1$ and for large value of l we get a small maximum value of the term. The time when the term will attain its maximum value depends on the values of l and m. It occurs in the late time for a large value of l and for a large value of m, it occurs in the early time, and vice-versa.

Now, from (6.2.14) one can observe that $w_d \to -1$ in the late time as the last two terms converge to zero, i.e., the NADE mimics the cosmological constant in the late time evolution. Wei and Cai [85] have also observed in GTR that NADE mimics cosmological constant in the late time evolution. Indeed, NADE with logarithmic form of BD scalar field in the BD theory behaves like NADE in GTR in the late time evolution due to the term $\frac{ma}{(l+m\,a)ln(l+m\,a)}$ which converses to zero as $a \to \infty$. However, in the power–law form of BD scalar field NADE crosses the phantom divide line in the late time evolution because the second term is a constant $\frac{2n}{3}$ for power–law form $\phi \propto a^n$ [216]. It is also interesting to note that w_d may cross the phantom divide line if

$\sqrt{\Omega_d} < [\frac{m\, d_2\, a^2}{2(l+m\,a)\, ln(l+m\,a)}]$ during the evolution but ultimately it will mimic the cosmological constant in the late time evolution. However, for a sufficiently large value of l and a sufficiently small value of d_2 the phantom divide line crossing is not possible. If l has a small value ($l \to 1$) and d_2 a large value then the value of m is crucial to observe the phantom divide crossing and therefore for sufficiently small value of m the possibility of $w_d < -1$ may be observed.

Let us discuss the evolution of the Universe using DP. Using (6.2.5), (6.2.7), (6.2.10) and (6.2.14) in (6.2.3), the value of q can be obtained as

$$q = \frac{1+3w_d\Omega_d+\Omega_k - \frac{m^2a^2}{(l+m\,a)^2\, ln(l+m\,a)} + \frac{2\,m\,a}{(l+m\,a)\, ln(l+m\,a)} + \frac{\omega\, m^2a^2}{2(l+m\,a)^2\,\{ln(l+m\,a)\}^2}}{2+\frac{m\,a}{(l+m\,a)\, ln(l+m\,a)}}. \quad (6.2.16)$$

The value of BD parameter ω plays an important role in the value of q. The solar system experiment Cassini gave a very stringent high bound result on ω as $|\omega| > 40000$ [236, 237]. However, the cosmological observations put relatively lower bounds on ω [238–240]. The term $\frac{m^2a^2}{(l+m\,a)^2\, ln(l+m\,a)}$ present in (6.2.16) has same behavior as discussed in Sect. 5.3. Due to a large value of ω suggested by the observations, the last term of the numerator containing ω will dominate in the early phase of the evolution of the Universe. Thus, we observe a positive value of q, i.e., the Universe evolves under the decelerated rate of the expansion. It means that the matter dominates the early phase of the expansion of the Universe. Thus, the NADE model with logarithmic form $\phi \propto ln(l+m\,a)$ explains the matter dominated phase of the Universe. Since, in the late time evolution $w_d \to -1$, and last three terms of numerator and last term of denominator in (6.2.16) converge to zero, therefore, we obtain the late time value of q as

$$q \approx \frac{1-3\Omega_d+\Omega_k}{2}. \quad (6.2.17)$$

Observations and available theoretical models suggest that $\Omega_d \to 1$ in the late time evolution. We have three possibilities for Ω_k: (i) $\Omega_k > 0$ for close Universe ($k > 0$), (ii) $\Omega_k = 0$ for flat Universe ($k = 0$), and (iii) $\Omega_k < 0$ for open Universe ($k < 0$). However, the observations suggest that our Universe is almost flat ($k \approx 0$) at present time. Therefore, if we take $k = 0$, i.e., $\Omega_k = 0$ for late time evolution then we have $q \approx \frac{1-3\Omega_d}{2}$. Thus, we may obtain an accelerated expansion for $\Omega_d > 1/3$. If we take present value $\Omega_d = 0.70$ obtained by resent observations then we get $q = -0.55$ which shows the accelerated expansion. One can observe that the accelerated expansion occurs more easily if we consider the open geometry of the Universe ($\Omega_k < 0$). In the case of close

Universe ($\Omega_k > 0$), the accelerated expansion is possible but a relatively large value of Ω_d is required, i.e, a large value of d_2^2 is required to achieve it.

So far, we have obtained decelerated expansion in the early phase of the evolution which explains the matter dominated phase and accelerated expansion in the late time evolution which explains the NADE dominated phase. Thus, the present model successfully explains the recent phase transition of the Universe from decelerated expansion to accelerated expansion. The time of phase transition depends on the values of the parameters of the model l, m, d_2^2 and ω. It may be observed from (6.2.16) that if phase transition occurs at time t_1 for value ω_1 then it will happen at time t_2 for ω_2 where $t_1 < t_2$ and $\omega_1 < \omega_2$. It may also be observed that it occurs at earlier stage for a large value of d_2^2. In the similar way, one can see the effects of l and m depending on their effects on the terms $\frac{m\,a}{(l+m\,a)\,ln(l+m\,a)}$ and $\frac{m^2 a^2}{(l+m\,a)^2\,ln(l+m\,a)}$.

Let us discuss the cosmic coincidence problem which has not been addressed for NADE models in the framework of BD theory. Steinhardt [31] addressed first time the coincidence problem- why are the matter and DE densities comparable at present time? In the conventional ΛCDM model very specific conditions of very early Universe are required to observe the matter and DE densities of the same order today. There are two ways to resolve the coincidence problem. In one of the ways, we should observe the energy density ratio $r_0 \sim \mathcal{O}(1)$ for a wide range of the initial conditions. In the other way, either r should converse to a constant value or evolve very slowly in the late time evolution(including present time) such that $r_0 \sim \mathcal{O}(1)$. In the ΛCDM model, the energy density ratio r evolves as $|\frac{\dot{r}}{r}|_0 = 3H_0$. Using Eqs. (6.2.12) and (6.2.13) one can obtain the evolution of r as

$$\dot{r} = 3rHw_d. \tag{6.2.18}$$

As we have observed that NADE mimics cosmological constant in the late time when $a \to \infty$. Therefore, if we take $w_{d0} = -1$, we get $|\frac{\dot{r}}{r}|_0 = 3H_0$ which is same as in the ΛCDM model. Thus, there is no reduction in the acuteness of coincidence problem. However, we have a time–dependent w_d, therefore, it is possible to have a less acute coincidence problem if we have $w_{d0} > -1$. The quintessence like EoS ($w_d > -1$) may be achieved for $\sqrt{\Omega_d} > \frac{m\,d_2\,a^2}{2(l+m\,a)\,ln(l+m\,a)}$ as can be observed from (6.2.14). It is to be noted that Eq. (6.2.14) has a time–dependent second term which converses to zero in the late time whereas the second term is a constant in case of power–law form. Therefore, the quintessence like EoS may be achieved more easily with logarithmic form in comparison to the power–law form of BD scalar field in the present time. Thus, the logarithmic form is more suitable to achieve a less acute coincidence problem. The observations also suggest quintessence like EoS of NADE [86]. Let us assume $w_{d0} = -2/3$, we obtain $|\frac{\dot{r}}{r}|_0 = 2H_0$ which is less acute coincidence problem in comparison to the ΛCDM model. On the other hand, the problem is more acute if w_d has phantom like EoS ($w_d < -1$). In this case, the coincidence problem becomes more worse than the ΛCDM model.

In conclusion, the coincidence problem may not be resolved significantly for non–interacting NADE model in the framework of BD theory, however, there are possibilities to achieve a less acute problem. The logarithmic form of BD scalar field is more suitable to have a less acute coincidence problem than power–law form.

6.2.2 Interacting NADE Model

In the present scenario, interest in interacting DE models is growing as it presents a way to resolve some unsolved issues of modern cosmology. A number of authors [227–231] have studied interaction between DM and DE, and recent cosmological observations [232–235] also support it. The interacting DE models have also been studied as a possible candidate to resolve the coincidence problem [259–261]. Therefore, in this subsection we extend NADE model by considering the interaction between DM and NADE.

Assuming that DM and NADE exchange energy through the interaction term Q, the continuity equations become

$$\dot{\rho}_m + 3H\rho_m = Q, \tag{6.2.19}$$

$$\dot{\rho}_d + 3(1+w_d)H\rho_d = -Q. \tag{6.2.20}$$

The sign of interaction term Q is crucial and defines the direction of the energy transfer, i.e., for Q > 0, there is an energy transfer from NADE to DM, and for Q < 0, there is an energy transfer from DM to NADE. The interaction between DM and NADE is a quantum scale phenomenon, therefore, the interaction term Q should be defined by the theory of quantum gravity. We do not have a satisfactory theory of quantum gravity, therefore, the dimension of the conservation equation has been used to choose a suitable form of Q in the literature. The conservation equations of DM and NADE imply that the interaction term should has the units of energy density $\times$ $(time)^{-1}$. Therefore, we may have the interaction term of the following forms: (i) $Q \propto H\rho_d$, (ii) $Q \propto H\rho_m$ or (iii) $Q \propto H(\rho_m + \rho_d)$. In this paper, we use the last assumption which is more general than first two. This form can be rewritten as

$$Q = 3b^2 H(\rho_m + \rho_d), \tag{6.2.21}$$

where b^2 is a coupling constant. Using (6.2.5), (6.2.7), (6.2.10) and (6.2.21) in (6.2.20), we get

$$w_d = -1 - b^2(1+r) - \frac{m\,a}{3(l+m\,a)\,ln(l+m\,a)} + \frac{2}{3\,d_2\,a}\sqrt{\Omega_d}. \tag{6.2.22}$$

It is easy to observe that the EoS parameter of NADE achieves its respective form in GTR for $m = 0$. We observe that w_d contains the third term as time–dependent whereas it is constant term in this place in power–law form of BD scalar field. Therefore, the evolution of w_d in our model will be different from that of power–law form. As discussed in Sect. 6.2.1, analyzing the terms in w_d, one can observe that w_d will definitely cross the phantom divide in the late time evolution, where $w_d \to -1 - b^2(1+r)$ as $a \to \infty$. The interaction between DM and NADE is the deriving force behind the phantom crossing in the late time. The late time value of w_d depends on the values of coupling constant b^2 and density ratio r. It means that it depends on how DM and NADE interact and what is the ratio of their densities. In the interacting NADE model, we observe that w_d may cross the phantom divide line more easily and earlier in the framework of BD theory than GTR.

The expression of q remains same as in the case of non–interacting NADE which is given by Eq. (6.2.16). However, the evolution of q is different due to different value of w_d for interacting NADE given in (6.2.22). From (6.2.14) and (6.2.22), we observe that the EoS parameter of interacting NADE has an extra term $b^2(1+r)$ with negative sign than non–interacting NADE. Therefore, it is easy to observe from (6.2.16) that the recent phase transition of the Universe from matter dominated phase to NADE dominated phase is more easily achieved and we may achieve it at earlier stage in the interacting case.

Let us study the evolution of r to check whether the interaction between DM and NADE is able to alleviate the coincidence problem or not. One can obtain the evolution of r using (6.2.19), (6.2.20) and (6.2.21) as

$$\dot{r} = 3rH\left[w_d + b^2\frac{(1+r)^2}{r}\right]. \tag{6.2.23}$$

we observe that the value of r contains sum of the terms w_d and $b^2\frac{(1+r)^2}{r}$ which have negative and positive value respectively. Thus, the resultant value of terms within bracket will be less than $|w_d|$. Moreover, for a suitable value of b^2, we may achieve $|w_d + b^2\frac{(1+r)^2}{r}| << |w_d|$. Now, it is evident from (6.2.18) and (6.2.23) that the energy density ratio r may evolve more slowly in interacting NADE model than the ΛCDM model and non–interacting NADE model. Therefore, the interaction between dark components of the Universe may play an important role in the explanation of the coincidence problem.

To achieve a soft coincidence we should observe $|\frac{\dot{r}}{r}|_0 \leq H_0$ [244]. The condition of soft coincidence may be easily achieved in our model if b^2 satisfies the condition, $b^2 \leq \frac{(1-3w_{d0})r_0}{3(1+r_0)^2}$. If we use the present observational values $r_0 \simeq 3/7$ and $w_{d0} \simeq -1$, we obtain the constraint $b^2 \leq 7/25$ to achieve soft coincidence problem. Even for a suitably small values of b^2, r may evolve more slowly at present which may resolve the coincidence problem more effectively. For example, if we have $b^2 = \frac{1}{5}$ and $w_{d0} = -1$ then we obtain $|\frac{\dot{r}}{r}|_0 \leq \frac{H_0}{7}$ which shows significantly slow variation of r compare to conventional ΛCDM model and condition mentioned above for soft coincidence.

Feng et al. [221] found that a small coupling constant b^2 is compatible with observation and also required to alleviate the coincidence problem. It also required to satisfy the second law of thermodynamics (also, see references within [221]). We also observe that the coupling constant b^2 should be a small quantity to avoid the coincidence problem which shows compatibility with the observations. Thus, the interaction between DM and NADE in the framework of BD theory with logarithmic BD scalar field may resolve the coincidence problem effectively.

6.3 Conclusion

In this chapter, we have investigated NADE model in BD theory with the logarithmic form of BD scalar field with in framework of FRW line element. We have discussed the dynamics of non–interacting and interacting NADE models using EoS parameter and DP. We have also discussed the cosmic coincidence problem. The model with this form explains the recent phase transition of the Universe and it may also resolve the cosmic coincidence problem. The results of these two models are summarized as follows:

In non–interacting case, NADE model successfully explains the recent phase transition from decelerated to accelerated expansion. NADE mimics cosmological constant in the late time in this model whereas it achieves phantom like behavior for power–law form [216]. It has also been observed that w_d may cross the phantom divide line under certain constraint but ultimately it will mimic the cosmological constant in late time evolution. Further, we have shown that a less acute coincidence problem than conventional ΛCDM model may be achieved. The logarithmic form of BD scalar field is more suitable to achieve a less acute coincidence problem than power–law form.

The recent phase transition from decelerated to accelerated expansion may be achieved earlier in interacting NADE model than non–interacting model. In the late time evolution, it has been analyzed that w_d definitely crosses the phantom divide line. We have shown that the energy density ratio r may evolve sufficiently slow in the present time. Thus, the interacting NADE model may be able to alleviate the coincidence problem effectively.

In conclusion, we can say that the logarithmic form of BD scalar field is suitable to explain the recent accelerated expansion of the Universe in NADE model. This model shows the phase transition of the evolution of the Universe. It may also resolve the cosmic coincidence problem effectively. Also, in the limit $m \to 0$, the NADE model in general relativity is recovered. Therefore, this form of BD scalar field may play an important role in explaining the present day Universe.

Chapter 7

Conclusion and Future Scope

7.1 Conclusion

In the thesis work, we have carried out a study on the accelerating cosmological models of the Universe which has been predicted by many cosmological observations. We have worked on modified theories of gravity to explain it. In chapter 2, the concept of bulk viscosity has been analyzed in $f(R,T)$ gravity to explain the recent accelerated expansion of the Universe. We have observed that the accelerated expansion may be possible for perfect fluid in our model of $f(R,T)$ gravity but the phase transition is not possible. The introduction of bulk viscosity makes the phase transition possible and presents a wide range of possible evolutions of the Universe depending on parameters of the model.

Further, chapter 3 deals with the bulk viscosity and matter creation together in $f(R,T)$ gravity. In the literature, both the phenomena have been treated as same cosmological phenomena and some papers treat both as different phenomena. In our study, we have found that although both the phenomena generate effective negative pressure which may explain the accelerated expansion but both have considerably different effects which make both the concepts separate and they exist independently.

It has been shown in chapter 4 that HDE model with Hubble horizon as an IR cut–off may explain the accelerated expansion in $f(R,T)$ gravity without taking interaction between DM and HDE which is not possible in GTR. Further, we have shown that if we consider the viscous HDE then the phase transition may also be achieved. Using statefinder parameters, it has ben found that our model shows the similar behavior as ΛCDM model and quintessence model for different values of the parameters. Thus, we have observed that $f(R,T)$ gravity has potential to explain the recent accelerated expansion of the Universe. The concept of bulk viscosity presents a mechanism to observe accelerated expansion as well as phase transition of the Universe.

In chapter 5, we have studied HDE model in well motivated and established BD theory. We have pointed out that the power–law form of BD scalar field gives a constant as well as time–dependent DP for HDE models in BD theory. It means that the same model describes two types of behavior of the Universe, evolution without phase transition and evolution with phase transition. Therefore, we have proposed a logarithmic form of BD scalar field which is free from this problem. We have shown that HDE model with Hubble horizon as IR cut–off in BD theory with this form of BD scalar field explains early time inflation, medieval time deceleration and late time acceleration of the Universe. This form also plays an important role to avoid cosmic coincidence problem.

We have extended our study to NADE model in chapter 6 and have shown that this model explains the evolution and coincidence problem more effectively with logarithmic form of BD scalar field in comparison of power–law form. Thus, first advantage of this form is that it is free from the constant deceleration parameter problem of power–law, and second is that it explains the evolution of the Universe more effectively and may explain the cosmic coincidence problem.

7.2 Future Scope

Our motive is to develop some cosmological models in the framework of modified theories of gravity which may be helpful in order to explain accelerated expansion of the Universe. In this thesis, we have studied $f(R,T)$ gravity because it presents a maximal coupling between matter and geometry which may be useful to explain accelerated expansion of the Universe. The concepts of bulk viscosity, particle creation and HDE have been analyzed in this theory. Further, we have considered BD theory which is a simple extension of GTR to discuss accelerated expansion. We have studied HDE and NADE models in this theory.

As we have mentioned, the $f(R,T)$ theory of gravitation presents a maximal coupling between matter and geometry of the Universe, therefore, it has potential to explain the problems of modern cosmology. Although, a wide range of models have been studied in this theory but still there is a good scope to do work. We have studied only first order Eckart theory of bulk viscosity in $f(R,T)$ gravity, therefore, it may be worthy to discuss the full causal theory of bulk viscosity which may provide better and more general results. The observational cosmology have not been studied considerably in this theory, therefore, there is a lot of scope in this area. The study of structure formation and perturbation theory of $f(R,T)$ gravity are another main fields where a considerable work can be done.

We have assumed bulk viscous coefficient as $\zeta = \zeta_0 + \zeta_1 H$, however a second order bulk viscous coefficient as $\zeta = \zeta_0 + \zeta_1 H + \zeta_2 H^2$ may be considered to describe the late time evolution. In forthcoming paper, we will try to assume such form of ζ in modified $f(R,T)$ gravity theory. The conservation of energy–momentum tensor is one of the main problem in $f(R,T)$ gravity. Harko has tried to explain the conservation equation through thermodynamics of particle creation. We will try to resolve this problem in future.

The BD theory is an important scalar–tensor theory which provides a natural extension of GTR. We have proposed a logarithmic form of BD scalar field and have shown that HDE model with the logarithmic form of BD scalar field provides some significant results in BD theory. Therefore, it opens door to study the cosmological implications of this form in other aspects of the evolution. It may be interesting to study the behavior of perturbation of BD theory with this form of BD scalar field. It will be worthwhile to analyze whether this form of BD scalar field may provide a mechanism for structure formation to take place. There are many other area like Black hole study, DM problem etc. to discuss in BD theory with this form of BD scalar field. We have considered only Hubble horizon as an IR cut–off to study HDE and NADE models in the framework of BD theory. Therefore, these models may be studied with other IR cut–off candidates like future event horizon, particle event horizon, Grand–Olivers cut–off etc.

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List of Publications

1. C. P. Singh and **Pankaj Kumar**; *Friedmann models with viscous cosmology in modified $f(R,T)$ gravity theory*, The European Physical Journal C **74**, 3070 (2014).

2. **Pankaj Kumar** and C. P. Singh; *Viscous cosmology with matter creation in modified $f(R,T)$ gravity*, Astrophysics and Space Science **357**, 120 (2015).

3. C. P. Singh and **Pankaj Kumar**; *Statefinder diagnosis for holographic dark energy models in modified $f(R,T)$ gravity*, Astrophysics and Space Science **361**, 157 (2016).

4. C. P. Singh and **Pankaj Kumar**; *Holographic dark energy in Brans-Dicke theory with logarithmic scalar field*, International Journal of Theoretical Physics (accepted).

5. **Pankaj Kumar** and C. P. Singh; *New agegraphic dark energy in Brans-Dicke theory with logarithmic scalar field*, Astrophysics and Space Science **362**, 52 (2017).