INTRODUCTION

➢ Introduction to shells

Similar to beams and plates, in many branches of engineering, cylindrical shells are the practical elements of various engineering structures such as pipes and ducts, bodies of cars, space shuttles, aircraft fuselages, ship hulls, submarines and construction buildings. However, analysing the dynamic characteristics of cylindrical shells is more complicated than that of beams and plates. This is mainly because unlike beams or plates which are normally one or two dimensional structures, shells can freely vibrate in three directions. This has caused complicated the motions of the shell at resonance frequencies. And the equations of motion of cylindrical shells combined with boundary conditions are more complex.

The literature concerning the vibration of shells is extremely extensive and readers can refer to Leissa [1] or more recently Amabili and Paidoussis [2] for comprehensive reviews of models and results presented in the literature. In the following, some studies, strictly related to the present study are described.

Love [3] modified the Kirchhoff hypothesis for plates and established the assumptions used in the so-called classic theory of thin shells. These assumptions are now commonly known as Love's approximation of the first kind. Love then subsequently formulated a shell theory known as Love's first approximation theory and the assumptions he established soon became the foundations on which many thin shell theories were later developed, such as the Flugge theory [4].

Soedel [5] introduced a set of three closed form solutions for the natural frequencies of cylindrical shells and also obtained mode shape coefficients of a simply supported cylindrical shell by applying normal solutions to the Love theory [3]. Forsberg [6] applied energy methods to the Flugge theory and considered the effects of tangential inertia.

Knowledge of the free-vibration characteristics of thin elastic shells is important both for our general understanding of the fundamentals of shell behaviour and for industrial applications of shell structures. In connection with the later, the natural frequencies of shell structures must be known to avoid the destructive effect of resonance with adjacent rotating or oscillating equipment (such as jet and reciprocating aircraft engines, electrical machinery, marine tubes etc.)

In the present study, a semi-analytical approach is proposed to investigate the free vibration of simply supported cylindrical shells. As cited above, in traditional analysis, beam functions with similar boundary conditions are used to approximate wave numbers in the axial direction. This method is considered as an approximate technique. The approximate method is used to obtain the natural frequencies based on ten different shell theories (Donnell-Mushtari, Love-Timoshenko, Arnold-Warburton, Houghton-Johns, Flugge-Byrne-Lur'ye, Reissner-Naghdi-Berry, Sanders, Vlasov, Kennard-Simplified and Soedel).

- **THIN SHELL**

A thin shell is a three dimensional body which is bounded by two closely curved surfaces, the distance between surfaces being small in comparison with the other dimensions. The locus of points which lie midway between these surfaces is called the middle surface of the shell.

The distance between the surfaces measured along the normal to the middle surface is the thickness of the shell at that point. The thickness need not to be constant in the formulation of a suitable theory of deformation, but constant thickness results in governing equations which are easier to solve; furthermore, certain manufacturing processes naturally yield shells of essentially constant thickness. Here the fundamental equations of thin shell theory are presented in their most simple, consistent form. Thus the material is assumed to be

- linearly elastic, isotropic, and homogeneous;
- displacement are assumed to be small, thereby yielding linear equations;
- shear deformation and rotary inertia effects are neglected;
- and the thickness is taken to be constant

The main purpose here is to present straightforward derivations of the sets of equations of various thin shell theories. It will be seen that differences in the theories result from slight differences in simplifying assumptions or the exact point in a derivation where a given assumption is used. Only those theories which are obtainable from Love's postulates by using a differential element of the middle surface, have been derived for shells of arbitrary curvature, and which have been applied in the literature to shell vibration problems have been considered here.

➢ BRIEF OUTLINE OF THE THEORY OF SURFACES

The deformation of a thin shell will be completely determined by the displacements of its middle surface. Certain relationships relating to the behaviour of a surface will be summarized in this section

Coordinate System

Let the equation of the undeformed middle surface be given in terms of two independent parameters α and β by the radius vector

$$\vec{r} = \vec{r}(\alpha, \beta) \tag{1.1}$$

Equation (1.1) determines the geometric properties of the surface and yields a method for finding points on the surface. Assume that the parameters α and β always vary within a definite region, and that a one-to-one correspondence exists between the points of this region and points on the portion of the surface of interest. Denote

$$\vec{r},_\alpha = \frac{\partial \vec{r}}{\partial \alpha}$$

$$\vec{r},_\beta = \frac{\partial \vec{r}}{\partial \beta}$$

The vectors $\vec{r},_\alpha$ and $\vec{r},_\beta$ are tangent to the α and β curves, respectively. The length of these vectors will be denoted by

$$|\vec{r},_\alpha| = A \qquad |\vec{r},_\beta| = B$$

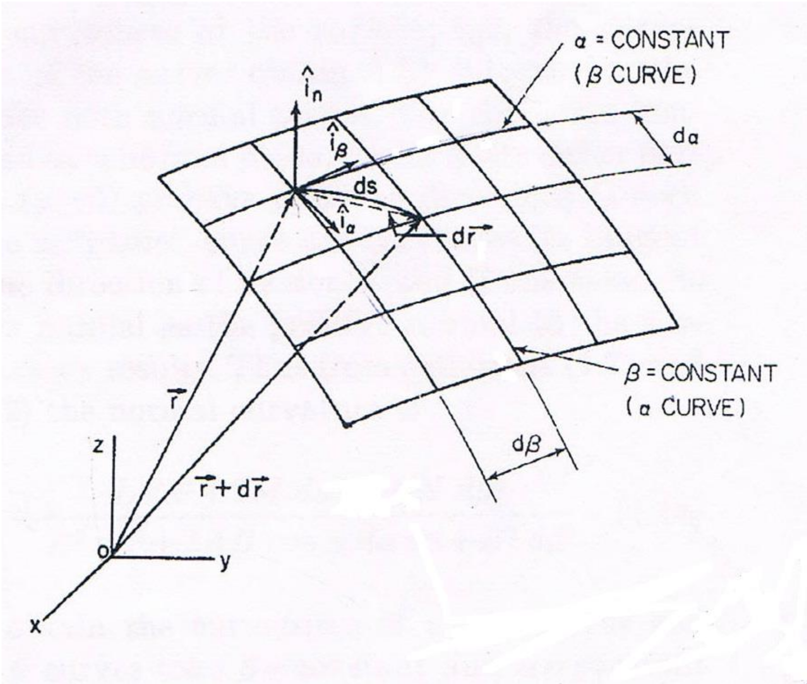

Fig.1. Middle surface coordinates

First Quadratic Form

$$d\vec{r}.d\vec{r} = ds^2 = A^2 d\alpha^2 + 2AB\cos x\, d\alpha\, d\beta + B^2 d\beta^2$$

It determines the *infinitesimal lengths, the angle between the curves, and the area on the surface.*

Second Quadratic Form

$$\frac{\cos \emptyset}{\rho} = \frac{L\, d\alpha^2 + 2M\, d\alpha\, d\beta + N\, d\beta^2}{ds^2}$$

It gives *the curvatures of curves on the surface.*

Thus by using first and second quadratic form, to obtain the normal curvature of the curves α and the β curves take β = constant and α = constant respectively, thus

$$\frac{1}{R_\alpha} = -\frac{L}{A^2}$$

$$\frac{1}{R_\beta} = -\frac{N}{B^2}$$

Principal Curvature

At this point assume that the curves α = constant and β= constant are the lines of principal curvature of the undeformed middle surface. The coordinates α and β are then called principal coordinates.

Weatherburn shows that the necessary and sufficient condition for the parametric curves (α curves and β curves) to be lines of principal curvature on a surface are that

$M = 0$ (Orthogonal Condition)

$\cos x = 0$ (Conjugate System Condition)

Gauss Characteristic Equation

The four fundamental quantities for principal coordinates A, B, L and N are not functionally independent, but are connected by three differential relations. One of these, due to Gauss, is an expression for (LN) in terms of A and B and their derivatives, and is given by,

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = -\frac{AB}{K} = -\frac{LN}{AB} \qquad (2)$$

Mainardi-Codazzi Relations

In addition to the Gauss Characteristic equation, there are two other independent relations i.e.

$$\frac{\partial}{\partial\beta}\left(\frac{A}{R_\alpha}\right)=\frac{1}{R_\beta}\frac{\partial A}{\partial\beta} \qquad (3)$$

$$\frac{\partial}{\partial\alpha}\left(\frac{B}{R_\beta}\right)=\frac{1}{R_\alpha}\frac{\partial B}{\partial\alpha} \qquad (4)$$

When A, B,R_α, and R_β are given, satisfying the Gauss Characteristic equation and the Mainardi-Codazzi relations, they determine a surface uniquely, except to position and orientation in space.

Shell coordinates and the fundamental shell element

- The position vector of an arbitrary point in the space occupied by a thin shell is defined as

$$\vec{R}(\alpha,\beta,z) = \vec{r}(\alpha,\beta) + z\hat{\imath}_n$$

Where z measures the distance of the point from the corresponding point on the middle surface along normal and varies over the thickness

$$(-h/2 \leq z \leq h/2)$$

- The magnitude of an arbitrary infinitesimal change in the vector $\vec{R}(\alpha,\beta,z)$ is determined by

$$(ds)^2 = d\vec{R}.d\vec{R} = (d\vec{r} + z\,d\hat{\imath}_n + \hat{\imath}_n\,dz)\,(d\vec{r} + z\,d\hat{\imath}_n + \hat{\imath}_n\,dz)$$

From this one obtains

$$(ds)^2 = d\vec{R}.d\vec{R} = g_1 d\alpha^2 + g_2 d\beta^2 + g_3 dz^2$$

Where,

$$g_1 = \left[A\left(1 + \frac{z}{R_\alpha}\right)\right]^2$$

$$g_2 = \left[B\left(1 + \frac{z}{R_\beta}\right)\right]^2$$

$$g_3 = 1$$

Now *Gauss equation (2)* and the *Mainardi-Codazzi equations (3&4)* generalized for a surface at a distance z from the middle surface are

$$\frac{\partial}{\partial\alpha}\left\{\frac{1}{A\left(1+\frac{z}{R_\alpha}\right)}\frac{\partial}{\partial\alpha}\left[B\left(1+\frac{z}{R_\beta}\right)\right]\right\}+\frac{\partial}{\partial\beta}\left\{\frac{1}{B(1+z/R_\beta)}\frac{\partial}{\partial\beta}\left[A\left(1+\frac{z}{R_\alpha}\right)\right]\right\}=\frac{AB}{R_\alpha R_\beta} \quad \text{..... (A)}$$

$$\frac{1}{A\left(1+\frac{z}{R_\alpha}\right)}\frac{\partial}{\partial\alpha}\left[B\left(1+\frac{z}{R_\beta}\right)\right]\frac{\partial}{\partial z}\left[A\left(1+\frac{z}{R_\alpha}\right)\right] \quad \text{........ (B)}$$

$$\frac{1}{B\left(1+\frac{z}{R_\beta}\right)}\frac{\partial}{\partial\beta}\left[A\left(1+\frac{z}{R_\alpha}\right)\right]\frac{\partial}{\partial z}\left[B\left(1+\frac{z}{R_\beta}\right)\right]=\frac{\partial^2}{\partial\beta\partial z}\left[A\left(1+\frac{Z}{R_\alpha}\right)\right] \quad \text{........(C)}$$

Having established the coordinate system of the shell space, the fundamental three dimensional element of a thin shell will be defined now. The fundamental shell element is the differential element bounded by two surfaces dz apart at a distance z from the middle surface and four ruled surfaces whose generators are the normals to the middle surface along the parametric curves $\alpha = \alpha_0$, $\alpha = \alpha_0 + d\alpha$, $\beta = \beta_0$ and $\beta = \beta_0 + d\beta$. The assumptions that the parametric curves are lines of principal curvature ensures that the ruled surfaces will be plane surfaces and, furthermore, that these planes intersect with each other at right angles.

- The lengths of the edges of this fundamentally element are

$$ds_\alpha^{(x)} = A\left(1+\frac{z}{R_\alpha}\right)\, d\alpha$$

$$ds_\beta^{(x)} = B\left(1+\frac{z}{R_\beta}\right)\, d\beta$$

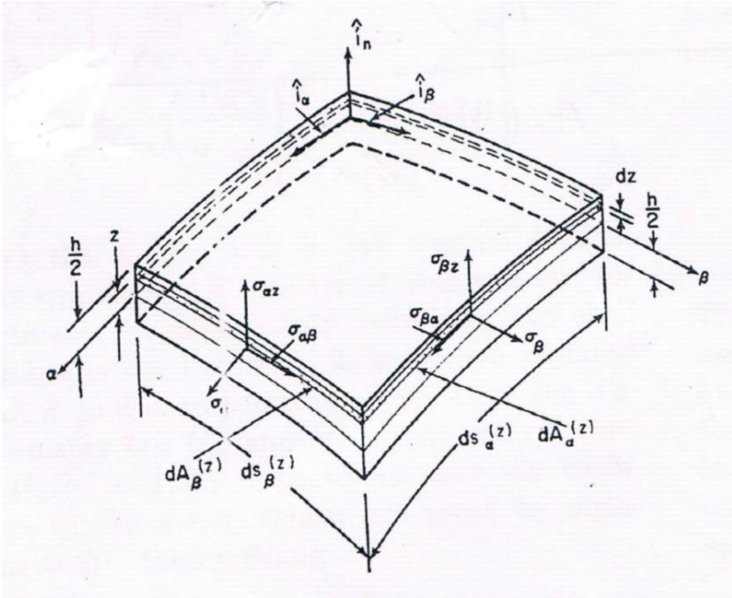

Fig.2. Notation and positive directions of stress in shell coordinates

- And the differential areas of the edge faces of the fundamental element are

$$dA_\alpha^{(x)} = A\left(1 + \frac{z}{R_\alpha}\right) d\alpha\, dz$$

$$dA_\beta^{(x)} = B\left(1 + \frac{z}{R_\beta}\right) d\beta\, dz$$

- While the volume of the fundamental element is

$$dV^{(x)} = \left[A\left(1 + \frac{z}{R_\alpha}\right)\right]\left[B\left(1 + \frac{z}{R_\beta}\right)\right] d\alpha\, d\beta\, dz$$

➢ LOVE'S FIRST APPROXIMATION

In the classical theory of small displacement of thin shells following assumptions were made by Love

- The thickness of the shell is small compared with the other dimensions.
- Strains and displacements are sufficiently small.
- The transverse normal stress is small compared with the other normal stress components and may be neglected
- Normals to the undeformed middle surface remain straight and normal to the deformed middle surface and suffer no extension.

The first assumption defines what is meant by “thin shells”
and sets the stage for the entire theory. Denoting the thickness of the shell by h and the smallest radius of curvature by R, then it will be convenient at various places in the subsequent derivation of shell theories to neglect higher powers of z/R or h/R in comparison with unity. The second assumption permits one to refer all calculations to the original configuration of the shell and ensures that the differential equations will be linear. The fourth assumption is known as Kirchhoff’s hypothesis and categorizes the shell theories that will be discussed here.

As a consequence of these assumptions

$$\gamma_{\alpha z} = 0$$

$$\gamma_{\beta z} = 0$$

$$e_z = 0$$

And therefore the transverse shear stresses

$$\sigma_{\alpha z} = \sigma_{\beta z} = 0$$

LITERATURE REVIEW

A comprehensive summary and discussion of shell theories including natural frequencies and mode shape information has been done by Liessa[1] in 1973. More recently, Amabili and Paidoussis[2], Amabili[3] and Kurylov and Amabili[4] have presented noteworthy reviews with a non-linear point of view. Many investigations followed the pioneering work of Love[5] using his first approximation theory, such as Flugge[6]. The Flugge theory is based on Kirchhoff-Love hypothesis for thin elastic shells. By using this theory, the strain-displacement relations and changes of curvatures of the middle surface of a cylindrical shell can be obtained. The simplified Donnell's theory would be achieved by neglecting few terms in Flugge equations Livanov[7] applied love's assumption and used displacement functions to solve the problem of axisymmetrical vibrations of simply supported cylindrical shells. Rinehart and Wang[8] investigated the vibration of simply supported cylindrical shells stiffened by discrete longitudinal stiffeners using Donnell's approximate theory, Flugge's more exact theory and Love's assumption for longitudinal wave numbers. Thesis theories are not only concerned with simply supported end conditions, but they have also applied other boundaries, such as cantilever cylindrical shells[9], fixed free circular cylindrical shells[10], clamped-clamped shells[11] and infinite length shells[12].

Most researchers and those cited above, use beam function as an approximation for the simply supported boundary conditions and find natural frequencies of vibration by the approximate method. This approximation is also useful for finite element analysis of cylindrical shells by using Hermitain polynomial of beam function type[13]. In addition to the approximate method, there are other approaches to find natural frequencies, like the computer based numerical method[14],[15] to avoid cumbersome computational effort and the wave propagation technique [16]. More recently Farshidianfar etl.[17] used the advantage of acoustical excitation to find natural frequency of long cylindrical shells.

MATHEMATICAL FORMULATION

STRAIN-DISPLACEMENT EQUATION

- By the use of well-known strain-displacement equations of the three-dimensional theory of elasticity in orthogonal curvilinear coordinates, we get

$$e_\alpha = \frac{1}{(1+z/R_\alpha)}\left(\frac{1}{A}\frac{\partial U}{\partial \alpha} + \frac{V}{AB}\frac{\partial A}{\partial \beta} + \frac{W}{R_\alpha}\right) \qquad \ldots\ldots\ldots\ldots\text{(I)}$$

$$e_\beta = \frac{1}{(1+z/R_\beta)}\left(\frac{U}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial V}{\partial \beta} + \frac{W}{R_\beta}\right) \qquad \ldots\ldots\ldots\ldots\text{(II)}$$

$$\gamma_{\alpha\beta} = \frac{A(1+z/R_\alpha)}{B(1+z/R_\beta)}\frac{\partial}{\partial \beta}\left[\frac{U}{A(1+z/R_\alpha)}\right] + \frac{B(1+z/R_\beta)}{A(1+z/R_\alpha)}\frac{\partial}{\partial \alpha}\left[\frac{V}{B(1+z/R_\beta)}\right]$$

$$\ldots\ldots\ldots\text{..(III)}$$

Now in order to satisfy the Kirchhoff hypothesis i.e. the fourth assumption, the class of displacement is restricted to the following linear relationships:

$$U(\alpha, \beta, z) = u(\alpha, \beta) + z\theta_\alpha(\alpha, \beta)$$

$$V(\alpha, \beta, z) = v(\alpha, \beta) + z\theta_\beta(\alpha, \beta)$$

$$W(\alpha, \beta, z) = w(\alpha, \beta)$$

$$\ldots\ldots\ldots\ldots\ldots\text{(IV)}$$

Where u, v, and w are the components of displacement at the middle surface in the α, β, and normal directions, respectively, and $\theta_\alpha$and θ_β are the rotations of the normal to the middle surface during deformation about the β and α axes, respectively; i.e.,

$$\theta_\alpha = \frac{\partial U(\alpha,\beta,Z)}{\partial Z}$$

$$\theta_\beta = \frac{\partial V(\alpha,\beta,Z)}{\partial Z}$$

➢ EQUATIONS OF BYRNE, FLŰGGE, GOLDENVEIZER, LUR'YE & NOVOZHILOV

- Substituting equations (IV) into equations (I, II, III) yields

$$e_{\alpha} = \frac{1}{(1 + z/R_{\alpha})}(\epsilon_{\alpha} + z\kappa_{\alpha})$$

$$e_{\beta} = \frac{1}{(1 + z/R_{\beta})}(\epsilon_{\beta} + z\kappa_{\beta})$$

$$\gamma_{\alpha\beta} = \frac{1}{(1 + z/R_{\alpha})(1 + z/R_{\beta})}\left[\left(1 - \frac{z^2}{R_{\alpha}R_{\beta}}\right)\epsilon_{\alpha\beta} + z\left(1 + \frac{z}{2R_{\alpha}} + \frac{z}{2R_{\beta}}\right)\tau\right]$$

………. (V)

- Where ϵ_{α} , ϵ_{β}, and $\epsilon_{\alpha\beta}$ are the normal and shear strains in the middle surface (z = 0) given by

$$\epsilon_{\alpha} = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{v}{AB}\frac{\partial A}{\partial \beta} + \frac{w}{R_{\alpha}}$$

$$\epsilon_{\beta} = \frac{u}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{w}{R_{\beta}}$$

$$\epsilon_{\alpha\beta} = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)$$ ..……. (VI)

And κ_{α} and κ_{β} are the middle surface changes in curvature and τ the midsurface twist, given by

$$\kappa_{\alpha} = \frac{1}{A}\frac{\partial \theta_{\alpha}}{\partial \alpha} + \frac{\theta_{\beta}}{AB}\frac{\partial A}{\partial \beta}$$ .... (VII a)

$$\kappa_{\beta} = \frac{\theta_{\alpha}}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial \theta_{\beta}}{\partial \beta}$$ .... (VII b)

$$\tau = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{\theta_{\alpha}}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{\theta_{\beta}}{B}\right) + \frac{1}{R_{\alpha}}\left(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{v}{AB}\frac{\partial B}{\partial \alpha}\right) + \frac{1}{R_{\beta}}\left(\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{u}{AB}\frac{\partial A}{\partial \beta}\right)$$

.... (VII c)

EQUATIONS OF LOVE AND TIMOSHENKO

- If in equation (V) one neglects the terms z/R_α and z/R_β and their products as being small in comparison with unity one obtains

$$e_\alpha = \epsilon_\alpha + z\kappa_\alpha$$

$$e_\beta = \epsilon_\beta + z\kappa_\beta$$

$$\gamma_{\alpha\beta} = \epsilon_{\alpha\beta} + z\tau$$

- With strains, curvature and the twist given by equations (VI & VII)

EQUATIONS OF REISSNER, NAGHDI, AND BERRY

If one make the simplification of Love and Timoshenko (i.e., z/R_α and z/R_β) earlier in the derivation, then doing so in equation (I, II, III) reduces them to

$$e_\alpha = \frac{1}{A}\frac{\partial U}{\partial \alpha} + \frac{V}{AB}\frac{\partial A}{\partial \beta} + \frac{w}{R_\alpha}$$

$$e_\beta = \frac{U}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial V}{\partial \beta} + \frac{w}{R_\beta}$$

$$\gamma_{\alpha\beta} = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{U}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{V}{B}\right)$$

- Terms for strain, curvature remain same as in equation (VI, VII a, b),except that equation for midsurface twist changes to become

$$\tau = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{\theta_\alpha}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{\theta_\beta}{B}\right)$$

EQUATIONS OF VLASOV

- Since for a shell z/R_i (i =α,β) is less than unity, therefore

$$\frac{1}{1+z/R_i} = \sum_{n=0}^{\infty}\left(-\frac{z}{R_i}\right)^n \qquad \text{………… (VIII)}$$

- Substituting equations (IV) and (VIII) into equations(I, II, III)

$$e_\alpha = \epsilon_\alpha + \sum_{n=1}^{\infty} \kappa_{\alpha n} z^n$$

$$e_\beta = \epsilon_\beta + \sum_{n=1}^{\infty} \kappa_{\beta n} z^n$$

$$\gamma_{\alpha\beta} = \epsilon_{\alpha\beta} + \sum_{n=1}^{\infty} \tau_n z^n$$

Where

$$\kappa_{\alpha n} = \left(-\frac{1}{R_\alpha}\right)^{n-1}\left(\kappa_\alpha - \frac{\epsilon_\alpha}{R_\alpha}\right)$$

$$\kappa_{\beta n} = \left(-\frac{1}{R_\beta}\right)^{n-1}\left(\kappa_\beta - \frac{\epsilon_\beta}{R_\beta}\right)$$

$$\tau_n = (-1)^n\left\{\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\left[\left(\frac{1}{R_\beta}\right)^{n-1}\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) - \left(\frac{1}{R_\alpha}\right)^{n-1}\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)\right] - \frac{1}{AB}\left[\left(\frac{1}{R_\alpha}\right)^{n-1} + \left(\frac{1}{R_\beta}\right)^{n-1}\right]\right\}$$

➢ EQUATIONS OF SANDERS

Sanders developed an eighth order shell theory from the principle of virtual work. The principle is written as

$$\int_\alpha \int_\beta \Bigg[\left(\frac{\partial BN_\alpha}{\partial\alpha} + \frac{\partial AN_{\beta\alpha}}{\partial\beta} + N_{\alpha\beta}\frac{\partial A}{\partial\beta} - N_\beta\frac{\partial B}{\partial\alpha} + Q_\alpha\frac{AB}{R_\alpha}\right)\delta u$$

$$+\left(\frac{\partial AN_\beta}{\partial\beta} + \frac{\partial BN_{\alpha\beta}}{\partial\alpha} + N_{\beta\alpha}\frac{\partial B}{\partial\alpha} - N_\alpha\frac{\partial A}{\partial\beta} + Q_\beta\frac{AB}{R_\beta}\right)\delta v$$

$$+\left(-N_\alpha\frac{AB}{R_\alpha} - N_\beta\frac{AB}{R_\beta} + \frac{\partial BQ_\alpha}{\partial\alpha} + \frac{\partial AQ_\beta}{\partial\beta}\right)\delta w$$

$$+\left(\frac{\partial BM_\alpha}{\partial\alpha} + \frac{\partial AM_{\beta\alpha}}{\partial\beta} + M_{\alpha\beta}\frac{\partial A}{\partial\beta} - M_\beta\frac{\partial B}{\partial\alpha} - ABQ_\alpha\right)\delta\theta_\alpha$$

$$+\left(\frac{\partial AM_\beta}{\partial\beta} + \frac{\partial BM_{\alpha\beta}}{\partial\alpha} + M_{\beta\alpha}\frac{\partial B}{\partial\alpha} - M_\alpha\frac{\partial A}{\partial\beta} + ABQ_\beta\right)\delta\theta_\beta$$

$$+AB\left(N_{\alpha\beta} - N_{\beta\alpha} + \frac{M_{\alpha\beta}}{R_\alpha} - \frac{M_{\beta\alpha}}{R_\beta}\right)\delta\theta_n\Bigg]\, d\alpha\, d\beta = 0$$

By the use of this equation we found that the strain-displacement equations of the sanders theory are given by equations (VI), (VII a and b), and

$$\tau = \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{\theta_\alpha}{A}\right) + \frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{\theta_\beta}{B}\right) + \frac{1}{2AB}\left(\frac{1}{R_\beta} - \frac{1}{R_\alpha}\right)\left(\frac{\partial Bv}{\partial\alpha} - \frac{\partial Au}{\partial\beta}\right)$$

EQUATIONS OF DONNELL AND MUSHTARI

- The tangential displacements and their derivatives are neglected, they simplify to

$$\kappa_\alpha = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\ \frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}$$

$$\kappa_\beta = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\ \frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}$$

$$\tau = -\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{B^2}\ \frac{\partial w}{\partial\beta}\right) - \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{A^2}\ \frac{\partial w}{\partial\alpha}\right)$$

- The strains are given by strain equation of Love and Timoshenko.
- The ϵ_α ,ϵ_β and $\epsilon_{\alpha\beta}$ are given by equations(VI)

After deriving strain displacement equation we form equation of motion according to different theories. These equation of motions are used to find out the natural free vibration of a thin cylindrical shell The cylindrical shell under consideration is with constant thickness *h*, mean radius *R*, axial length *L*, Poisson's ratio υ, density ρ and Young's modulus of elasticity *E*. Here the respective displacements in the axial, circumferential and radial directions are denoted by $u(x,\theta,t)$, $v(x,\theta,t)$ and $w(x,\theta,t)$ as shown in Figure 3.

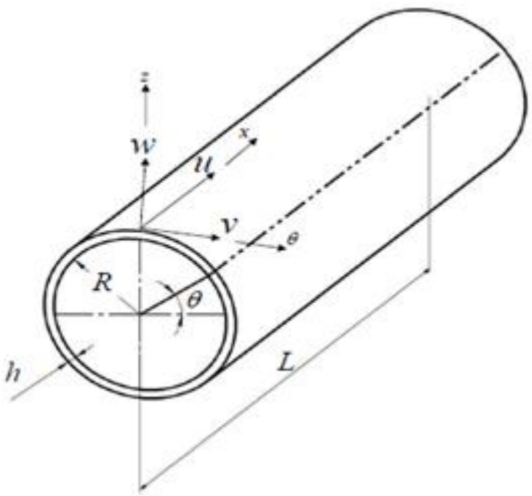

Fig.3. Thin Cylindrical Shell

In order to study free vibration of a cylindrical shell, the equations of motion can be written in matrix form as follows:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ -L_{13} & -L_{23} & L_{33} \end{bmatrix} \begin{Bmatrix} u(x,\theta,t) \\ v(x,\theta,t) \\ w(x,\theta,t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (1)$$

where $L_{ij}(i,j = 1,2,3)$ are differential operators with respect to $x, \theta \ and \ t$.

Different systems of equations are used to model the vibration behaviour of circular cylindrical shells. In this paper ten theories namely: 1) Donnell-Mushtari[1], 2) Love-Timoshenko[1], 3) Arnold-Warburton[1], 4)Houghton-Johns[1], 5) Flugge-Byrne-Lur'ye[1], 6)Reissner-Naghdi-Berry[1], 7) Sanders[1], 8) Vlasov[1], 9)Kennard-Simplified[1] and 10) Soedel[19], are used to find natural frequencies for various boundary conditions.

The first attempt in solving (1) is the assumption of a synchronous motion:

$$\begin{cases} u(x,\theta,t) = U(x,\theta)f(t) \\ v(x,\theta,t) = V(x,\theta)f(t) \\ w(x,\theta,t) = W(x,\theta)f(t) \end{cases} \quad (2)$$

where $f(t)$ is the scalar model coordinate corresponding to the mode shapes $U(x,\theta)$, $V(x,\theta)$ and $W(x,\theta)$.

The next step is to use the separation of variables method in order to separate the spatial dependence of the modal shape between longitudinal and circumferential directions. Hence the axial, tangential and radial displacements of the wall vary according to

$$\begin{cases} u(x,\theta,t) = Ae^{\lambda_m x}\sin(n\theta)\cos(\omega t) \\ v(x,\theta,t) = Be^{\lambda_m x}\cos(n\theta)\cos(\omega t) \\ w(x,\theta,t) = Ce^{\lambda_m x}\sin(n\theta)\cos(\omega t) \end{cases} \quad (3)$$

in which λ_m and n are the axial wave number and the circumferential wave parameter, respectively. A, B and C are the undetermined constants, and ω is the circular frequency of the natural vibration.

Substituting (3) into (1), using any of the shell theories, leads to a set of homogenous equations having the following matrix form:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ -C_{12} & C_{22} & C_{23} \\ -C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4)$$

in which $|C_{ij}|(i,j = 1,2,3)$ are functions of n, λ_m and a frequency parameter Ω that is defined as follows:

$$\Omega^2 = \frac{(1-\nu^2)\rho}{E}\omega^2 R^2 \quad (5)$$

The coefficient matrix, $|C_{ij}|$ for the ten shell theories is obtained as follows:

Donnell-Mushtari

$$\begin{bmatrix} \Omega^2 + \lambda_m^2 - \frac{1-\nu}{2} n^2 & -\frac{1+\nu}{2} n\lambda_m & \nu\lambda_m \\ \frac{1+\nu}{2} n\lambda_m & \Omega^2 + \frac{1-\nu}{2}\lambda_m^2 - n^2 & n \\ -\nu\lambda_m & n & \Omega^2 - [1 + k(\lambda_m^2 - n^2)^2] \end{bmatrix} \tag{6}$$

Love-Timoshenko

$$\begin{bmatrix} \Omega^2 + \lambda_m^2 - \frac{1-\nu}{2} n^2 & -\frac{1+\nu}{2} n\lambda_m & \nu\lambda_m \\ \frac{1+\nu}{2} n\lambda_m & \Omega^2 + (1+2k)\frac{1-\nu}{2}\lambda_m^2 - (1+k)n^2 & n + nk(n^2 - \lambda_m^2) \\ -\nu\lambda_m & n + nk(n^2 - \lambda_m^2) & \Omega^2 - [1 + k(\lambda_m^2 - n^2)^2] \end{bmatrix} \tag{7}$$

Arnold-Warburton

$$\begin{bmatrix} \Omega^2 + \lambda_m^2 - \frac{1-\nu}{2} n^2 & -\frac{1+\nu}{2} n\lambda_m & \nu\lambda_m \\ \frac{1+\nu}{2} n\lambda_m & \Omega^2 + (1+4k)\frac{1-\nu}{2}\lambda_m^2 - (1+k)n^2 & n + nk[n^2 - (2-\nu)\lambda_m^2] \\ -\nu\lambda_m & n + nk[n^2 - (2-\nu)\lambda_m^2] & \Omega^2 - [1 + k(\lambda_m^2 - n^2)^2] \end{bmatrix} \tag{8}$$

Houghton-Johns

$$\begin{bmatrix} \Omega^2 + \lambda_m^2 - \frac{1-\nu}{2} n^2 & -\frac{1+\nu}{2} n\lambda_m & \nu\lambda_m \\ \frac{1+\nu}{2} n\lambda_m & \Omega^2 + \frac{1-\nu}{2}\lambda_m^2 - n^2 & n + nk[n^2 - (2-\nu)\lambda_m^2] \\ -\nu\lambda_m & n + nk[n^2 - (2-\nu)\lambda_m^2] & \Omega^2 - [1 + k(\lambda_m^2 - n^2)^2] \end{bmatrix} \tag{9}$$

Flugge-Byrne-Lur'ye

$$\begin{bmatrix} \Omega^2+\lambda_m^2-(1+k)\frac{1-\nu}{2}n^2 & -\frac{1+\nu}{2}n\lambda_m & \nu\lambda_m-k\lambda_m\times[\lambda_m^2+(1-\nu)n^2] \\ \frac{1+\nu}{2}n\lambda_m & \Omega^2-n^2+(1+3k)\frac{1-\nu}{2}\lambda_m^2 & n\times\left[1-\frac{3-\nu}{2}k\lambda_m^2\right] \\ -\nu\lambda_m+k\lambda_m\times[\lambda_m^2+(1-\nu)n^2] & n\times\left[1-\frac{3-\nu}{2}k\lambda_m^2\right] & \Omega^2-(1+k)-k[(\lambda_m^2-n^2)^2-2n^2] \end{bmatrix} \tag{10}$$

Reissner-Naghdi-Berry

$$\begin{bmatrix} \Omega^2+\lambda_m^2-\frac{1-\nu}{2}n^2 & -\frac{1+\nu}{2}n\lambda_m & \nu\lambda_m \\ \frac{1+\nu}{2}n\lambda_m & \Omega^2+(1+k)\left(\frac{1-\nu}{2}\lambda_m^2-n^2\right) & n\times[1+k(n^2-\lambda_m^2)] \\ -\nu\lambda_m & n\times[1+k(n^2-\lambda_m^2)] & \Omega^2-[1+k(\lambda_m^2-n^2)^2] \end{bmatrix} \tag{11}$$

Sanders

$$\begin{bmatrix} \Omega^2+\lambda_m^2-\left(1+\frac{k}{4}\right)\frac{1-\nu}{2}n^2 & -n\lambda_m\times\left[\frac{1+\nu}{2}-\frac{3k(1-\nu)}{8}\right] & \lambda_m\times\left(\nu-\frac{1-\nu}{2}kn^2\right) \\ n\lambda_m\times\left[\frac{1+\nu}{2}-\frac{3k(1-\nu)}{8}\right] & \Omega^2-(1+k)n^2+\left(1+\frac{9k}{4}\right)\frac{1-\nu}{2}\lambda_m^2 & n\times\left[1+k\left(n^2-\frac{3-\nu}{2}\lambda_m^2\right)\right] \\ \lambda_m\left(\frac{1-\nu}{2}kn^2-\nu\right) & n\times\left[1+k\left(n^2-\frac{3-\nu}{2}\lambda_m^2\right)\right] & \Omega^2-[1+k(\lambda_m^2-n^2)^2] \end{bmatrix} \tag{12}$$

Vlasov

$$\begin{bmatrix} \Omega^2+\lambda_m^2-\frac{1-\nu}{2}n^2 & -\frac{1+\nu}{2}n\lambda_m & \lambda_m\times\left[\nu-k\left(\frac{1-\nu}{2}n^2+\lambda_m^2\right)\right] \\ \frac{1+\nu}{2}n\lambda_m & \Omega^2+\frac{1-\nu}{2}\lambda_m^2-n^2 & n\times\left(1-\frac{3-\nu}{2}k\lambda_m^2\right) \\ \lambda_m\times\left[k\left(\frac{1-\nu}{2}n^2+\lambda_m^2\right)-\nu\right] & n\times\left(1-\frac{3-\nu}{2}k\lambda_m^2\right) & \Omega^2-(1+k)-k[(\lambda_m^2-n^2)^2-2n^2] \end{bmatrix} \tag{13}$$

Kennard-Simplified

$$\begin{bmatrix} \Omega^2+\lambda_m^2-\frac{1-\nu}{2}n^2 & -\frac{1+\nu}{2}n\lambda_m & \nu\lambda_m \\ \frac{1+\nu}{2}n\lambda_m & \Omega^2+\frac{1-\nu}{2}\lambda_m^2-n^2 & n\times\left(1+\frac{3k\nu}{2(1-\nu)}(1-n^2)\right) \\ -\nu\lambda_m & 0 & \Omega^2-\left[1+\frac{2+\nu}{2(1-\nu)}\right]-k\left[(\lambda_m^2-n^2)^2-\frac{4-\nu}{2(1-\nu)}n^2\right] \end{bmatrix}$$

(14)

For nontrivial solution the determinant of the coefficient matrix in (4) must be zero:

$$det(|C_{ij}|)=0 \ \ ; \ i,j=1,2,3 \tag{15}$$

The expansion of (15) will give the following two eigenvalue problems:

• For a given value of λ_m there exists one or more proper values for ω so that the (15) vanishes.

• For a given value of there exists one or more proper values for λ_m so that the (15) vanishes.

Solving (15) leads to a cubic equation in terms of the non dimensional frequency parameterΩ^2. Thus for a fixed value of *n* andλ_m, three positive roots and three negative roots are yield for the non dimensional frequency. The three positive roots are the natural frequencies of the cylindrical shell that can be classified as primarily axial, circumferential or radial. The lowest frequency is usually associated with a motion that is primarily radial (or flexural).

➢ BEAM FUNCTION METHOD

In general, solving the roots of the characteristic equation of (15) for λ_m is not possible in closed form. Beam functions can be used to obtain natural frequencies and approximate displacements for closed circular cylindrical shells. This method is an assimilation of the flexural vibration of cylindrical shell with a transversely vibrating beam of the same boundary conditions. According to the approximate method, for a simply supported shell at both ends and clamped-clamped shell the nature of the axial mode can be defined respectively as:

$$\lambda_m = \frac{m\pi R\sqrt{-1}}{L}$$

$$\lambda_m = \frac{(2m+1)\pi R\sqrt{-1}}{2L} \quad (16)$$

By substituting (16) into (15), the only unknown of the characteristic equation will be the frequency parameter Ω^2 for a fixed combination of m and n

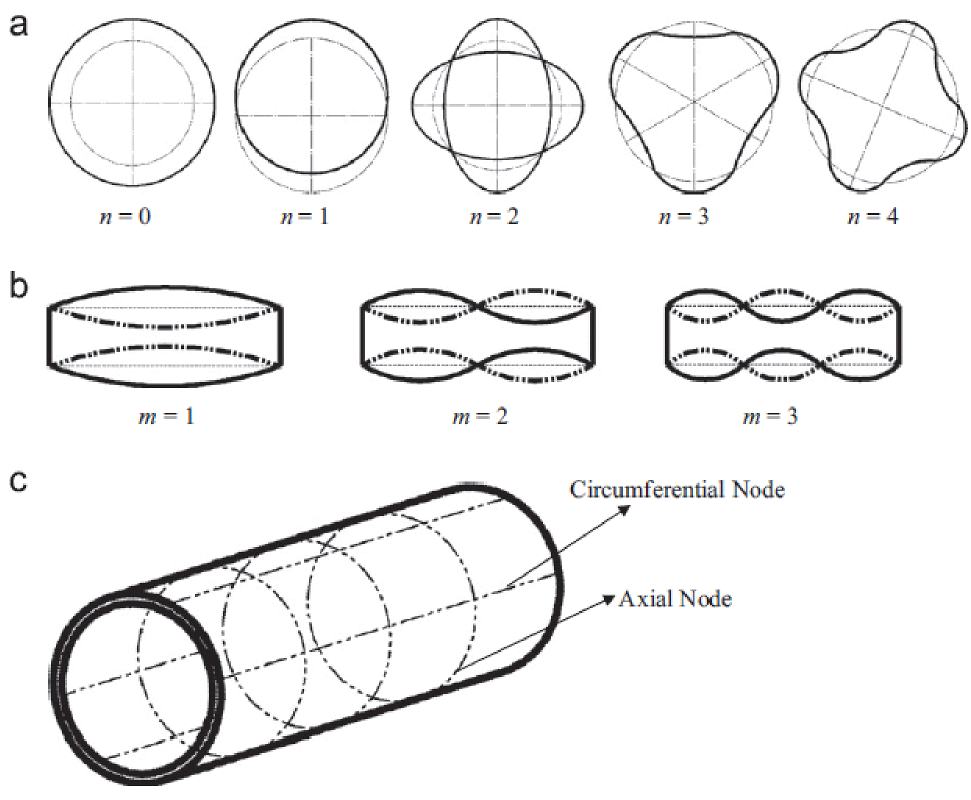

Fig.4. Mode shapes of cylindrical shell: (a) circumferential mode shapes; (b) longitudinal and radial mode shapes and; (c) nodal arrangement of a cylindrical shell for n=2 and m=4.

RESULTS AND DISCUSSION

Since the beam function method is an approximation to obtain natural frequencies for thin circular cylindrical shells, it is important to check the accuracy of this method. Hence, the natural frequency for simply supported boundary conditions, calculated by using the beam functions via ten common theories of cylindrical shells has been compared with experimental results.

In Table 1, results calculated by the approximate method according to the ten theories, are given for simply supported circular cylindrical shell and

The shell investigated in Table 1 is made of aluminium with material properties;

E=68.2GPa,

ρ=2700 Kg/m^3 and

ν= 0.33

The dimensions of the shell are:

L=1.7272 m,

R=0.0762 m and

h=0.00147 m.

For simply supported cylindrical shell

m	n	Experimental Results	Donnell-Mushtari(Hz)	Love-Timoshenko(Hz)	Arnold-Warburton(Hz)	Houghton-Johns(Hz)	Flugge-Byrne-Lur’ye(Hz)	Reissner-Naghdi-Berry(Hz)	Sanders(Hz)	Vlasov(Hz)	Kennard-Simplified (Hz)	Soedel(Hz)
1	1	138.40	148.642	141.668	141.578	134.151	141.498	141.623	141.578	141.407	145.637	141.446
1	2	190.30	231.881	176.728	176.524	167.113	176.501	176.706	176.524	176.558	209.101	176.692
1	3	502.20	541.841	481.813	481.722	477.946	481.733	481.813	481.722	481.779	515.852	481.811
1	4	884.40	983.559	922.114	922.057	920.000	922.103	922.114	922.057	922.125	956.539	922.112
2	1	464.70	530.105	528.121	528.019	525.944	527.962	528.064	528.019	527.860	529.300	527.358
2	2	310.50	292.056	249.876	249.297	242.573	249.240	249.830	249.308	249.388	274.424	249.663
2	3	477.00	551.332	491.995	491.632	487.890	491.632	491.984	491.644	491.780	525.853	491.960
3	2	496.60	458.874	432.613	431.865	427.874	431.820	432.557	431.899	432.001	447.932	432.010
3	3	558.90	582.173	525.615	524.867	521.295	524.855	525.592	524.878	525.150	558.169	525.499
4	2	679.80	718.773	701.754	700.937	698.363	700.937	701.697	700.994	701.107	711.902	700.852
4	3	638.30	650.082	599.091	597.912	594.703	597.923	599.057	597.957	598.354	628.742	598.809
5	3	782.00	764.458	720.678	719.147	716.403	719.193	720.633	719.238	719.737	746.463	720.163

It can be seen from the results obtained that the Soedel and Kennard-Simplified theories revealed better results compared to other theories, when these were compared with experimental results. Some theories (Love-Timoshenko, Arnold-Warburton, Flugge Byrne-Lur’ye, Reissner-Naghdi-Berry, Sanders, Vlasov, and Soedel) reveal same results. Dunnell-Mushtari and Houghton-Johns theory are not precise compared to other theories.

CONCLUSION

The free vibration of circular cylindrical shells with simply supported boundary conditions has been studied using ten different thin shell theories: Donnell-Mushtari, Love-Timoshenko, Arnold-Warburton, Houghton-Johns, Flugge-Byrne-Lur’ye, Reissner-Naghdi-Berry, Sanders, Vlasov, Kennard-Simplified and Soedel. The scope of the investigation was focused upon using the beam function as an approximation for boundary condition to find the natural frequencies of a shell.

It is also concluded that some theories (Love-Timoshenko, Arnold-Warburton, Flugge Byrne-Lur’ye, Reissner-Naghdi-Berry, Sanders, Vlasov, and Soedel) reveal same results. Dunnell-Mushtari and Houghton-Johns theory are not precise compared to other theories.

Next, in order to check the accuracy of the theories, a comparison was carried out with experimental results and it shows good agreement. Moreover, the approximate method based on the Soedel and Kennard-Simplified theories reviled better results compared to other theories.

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APPENDIX

1. MATLAB PROGARM

For Donnell-Mushtari Theory

```
syms  p  E  v L R h m n x k s

 p=2700
 E=(68.2*10^9)
 v=0.33
 L=1.7272
 R= 0.0762
 h=0.00147
 m=1
 n=1

 x=(m*pi*R*(-1^(1/2)))/(L)
 x=0.1385996i
 k=(h^2)/(12*(R^2))
 k= 0.0000310


 % s=(((1-(v^2))*(p)*(w^2)*(r^2))/(E))^(1/2)
 m =[((s^2)+(x^2)-(((1-v)/2)*(n^2))) -(((1+v)/2)*(n)*(x))
 (v*x);
       (((1+v)/2)*(n)*(x)) ((s^2)+(((1-v)/2)*(x^2))-(n^2)) n;
        -(v*x) n ((s^2)-(1+k*((x^2)-(n^2))^2))]
  d=(det(m))
  s=solve(vpa(d))
  z=((1-(v^2))*(p)*((R-h)^2))/(E)
  w=solve(vpa(d))/(z^(1/2))
 vpa (w/(2*pi))
```

For Love-Timoshenko Theory

```
syms  p  E  v L R h m n x k s
p=2700
E=(68.2*10^9)
v=0.33
L=1.7272
R= 0.0762
h=0.00147
m=1
n=3

x=(m*pi*R*(-1^(1/2)))/(L)
x=0.1385996i
k=(h^2)/(12*(R^2))
k= 0.0000310


% s=(((1-(v^2))*(p)*(w^2)*(r^2))/(E))^(1/2)
m =[((s^2)+(x^2)-(((1-v)/2)*(n^2))) -(((1+v)/2)*(n)*(x))
(v*x);
      (((1+v)/2)*(n)*(x)) ((s^2)+((1+(2*k))*((1-v)/2)*(x^2))-
((1+k)*(n^2))) (n+(n*k*((n)^2-(x)^2)));
       -(v*x) (n+(n*k*((n)^2-(x)^2))) ((s^2)-(1+k*((x^2)-
(n^2))^2))]
 d=(det(m))
 s=solve(vpa(d))
 z=((1-(v^2))*(p)*((R-h)^2))/(E)
 w=solve(vpa(d))/(z^(1/2))
vpa (w/(2*pi))
```