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To

My Parents

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Preface

The current literature on dynamic entropy has focused mainly on Shannon entropy [109] and Kullback-Leibler relative information [70], perhaps because of their simplicity. In view of this we explore the motivations and implications of using various generalized classes of dynamic entropy measures in this thesis. It has been seen that the use of different dynamic entropy measures based on non-additive entropy, inaccuracy, and weighted information measures may lead to different models or statistical results than those obtained by dynamic Shannon and dynamic Kullback-Leibler information measures. The subject of present study is to introduce the concept of different dynamic entropy measures, including Havrda-Charvat entropy [57], Kerridge inaccuracy measure [67], and weighted information measures in the context of the characterization theorems and characterization of residual and past lifetime distributions. Also we have focused on the dynamic cumulative residual measures problem for residual and past lifetime distributions and have also provided the characterization theorems.

The thesis comprises seven chapters including the first chapter on introduction and literature survey, and the last chapter on conclusion and further scope of work. The thesis has been organized as follows;

Chapter 1 is introductory in nature presenting a brief account of the available literature and the various information measures proposed by the researchers. Some basic concepts of reliability, including that of proportional hazard model (PHM), proportional reversed hazard model (PRHM) and length biased model, have also been discussed.

In **Chapter 2**, we have considered Havrda and Charvat [57] measure of entropy which is a one parameter generalization of the Shannon entropy and is non-additive in nature. We have proposed a residual measure of entropy based on it and have proved a characterization theorem that the proposed measure determines the distribution function uniquely. Also we have characterized some specific probability distributions based on the proposed measure. The work reported in this chapter has been published in the papers entitled, **Non-additive Entropy Measure Based Residual Lifetime Distributions** in *JMI International Journal of Mathematical Sciences*, 2010, 1 (2), 1-9, and, **A Generalized Entropy- Based Residual Lifetime Distribution** in *International Journal of Biomathematics*, 2011, 4 (2), 171-184.

In **Chapter 3**, we have conceptualized the idea of dynamic measure of inaccuracy, both residual and past. In case of residual inaccuracy measure we have studied the characterization result using proportional hazard model; and in case of past inaccuracy measure we have studied this using proportional reversed hazard model. Also we have characterized some specific distributions based on these measures. The work reported in this chapter has been published in the papers entitled, **A Dynamic Measure of Inaccuracy Between Two Residual Lifetime Distributions** in *International Mathematical Forum*, 2009, 4 (25), 1213-1220, and, **A Dynamic Measure of Inaccuracy Between Two Past Lifetime Distributions** in *Metrika*, 2010, 74 (1), 1-10.

The aforementioned information measures do not take into account the qualitative aspect of the random variable. They consider only its probability density. Based on the notion of weighted distribution, Di Crescenzo and Longobardi [31] introduced the concept of weighted entropy, weighted residual entropy and weighted past entropy. In **Chapter 4**, the results of Chapter 3 have been extended to weighted distributions. Taking weights $w(x) = x$, we have introduced length biased measures of residual and past inaccuracies and have studied their respective characterization theorems, and other properties. The results reported in this chapter have been published in the papers entitled, **Length Biased Weighted Residual Inaccuracy Measure** in *Metron*, 2010, LXVIII (2), 153-160, and, **On Length Biased Dynamic Measure of Past Inaccuracy** in *Metrika*, 2012, 75 (1), 73-84. Also some results were presented at *International Conference in Mathematics and Applications* held in *Bangkok* on Dec. 19-21, 2009.

Since the cumulative distribution function based information measures are more stable in comparison to probability density function based measures. Based on that an alternative notation of entropy called *cumulative residual entropy (CRE)* is proposed in Rao et al. [98]. In **Chapter 5**, we have generalized the concept of cumulative residual entropy measure to one parameter and two parameters entropies, and have studied their dynamic versions and characterization results. The exponential, Pareto and finite range distribution, which are commonly used in reliability modeling, have been characterized in terms of generalized cumulative residual entropy measures. The work reported in this chapter has been published in the papers entitled, **On Dynamic Renyi Cumulative Residual Entropy Measure** in *Journal of Statistical Theory and Applications*, 2011, 10 (3), 491-500, and, **Some Characterization Results on Generalized Cumulative Residual Entropy Measure** in *Statistics and Probability Letters*, 2011, 81 (8), 72-77. Also some results were presented at *International Congress of Mathematicians (ICM)* held in *Hyderabad* on Aug. 19-27, 2010.

In **Chapter 6**, we have considered dynamic cumulative inaccuracy measures, both residual and past and have studied the characterization results respectively under proportional hazard model and proportional reversed hazard model. Also we have characterized certain specific probability distributions using relation between different reliability measure. It is expected that dynamic cumulative inaccuracy measures introduced will further extend the scope of study. The work reported in this chapter has been published in the paper entitled, **On Dynamic Cumulative Residual Inaccuracy Measure** in proceedings of the *World Congress on Engineering (WCE)*, held in *London* on July 4-6 2012, and, some results have been communicated for publication.

In **Chapter 7**, we have concluded the findings of the work carried out in this thesis and also have presented further scope of work. During the present investigation, several ideas have originated which have the potential to extend the study further. We can consider the proposed dynamic measures further for discrete cases, since practically discrete cases are suitable from application point of view. Further the discrete measures of the dynamic versions proposed can possibly find wider applications in different areas of interest. The work reported in this thesis can be extended to bivariate and multivariate domains. Also we can employ the concept of order statistics to the different dynamic measures reported in the thesis.

Chapter 1

Introduction and Literature Survey

1.1 An Overview of Information Theory

Information Theory is relatively a new branch of applied mathematics which was made mathematical rigorous only in 1940s. Broadly speaking, information theory deals with the study of problems concerning any system. This includes information processing, information storage, information retrieval and decision-making. In a narrow sense, the theory deals with all theoretical problems connected with the transmission of information over communication channels. This includes the study of uncertainty measures and practical and economical methods of coding information for transmission.

The first studies in this direction were undertaken by Nyquist [88, 89] and Hartley [56] who introduced the entropy of a distribution of equally probable events. In 1948, Shannon published a paper, **The mathematical theory of communica-**

tion in the **Bell System Technical Journal** which laid the foundation of the modern day's information theory. Being an electrical engineer, his goal was to get maximum line capacity with minimum distortion. He showed little interest in the semantic meaning of a message or its pragmatic effect on the listener and was only aimed at solving the technical problems of high-fidelity transfer of sound. Shannon introduced a measure of information or entropy for a general finite complete probability distribution and gave a characterization theorem of the entropy measure introduced by him. Entropy is randomness. How much information a message contains is measured by the extent it combats entropy. The less predictable the message, the more information it carries. Around the same time, Wiener [125] also considered the communication situation from the statistical aspects and came up independently with results similar to those of Shannon [109].

The second half of the 20^{th} Century was characterized by the tremendous development of systems in which the transmitted information (analog signal) is coded in a digital form. By this coding the real nature of the information signal becomes secondary, that is, the same system can transmit simultaneously signals of very different nature: data, audio, video etc. This development has been made possible by the use of more and more powerful integrated circuits. Although it is mainly during the last 30 years that the truly operational digital systems have been developed, the theoretical foundations for all these developments date back to the work of Shannon and others in the mid of the 20^{th} Century which led to the development of information theory as a field of mathematics. Theory is basically concerned with the mathematical laws governing systems designed to communicate or manipulate information. It sets up quantitative measures of information and of the capacity of various systems to transmit, store and process information.

In the past sixty five years, the literature on information theory has grown quite voluminous and apart from its applications in communication theory, it has found deep applications in many social, physical and biological sciences, for example, economics, statistics, psychology, ecology, pattern recognition, fuzzy sets etc. refer to [62, 107, 111, 121]. Another important area is that of reliability, where information theoretic measures have found applications. Many researchers e.g. Ebrahimi [34], Ebrahimi and Kirmani [38], Asadi and Ebrahimi [36], Belzuence et al. [16], Nanda and Paul [85] and, Wells [126] have studied the information-theoretic measures based lifetime distributions of a system. In this thesis, we shall be concerned mainly with this aspect of the information theory.

1.2 Entropy and Its Generalizations

1.2.1 Shannon's Entropy

The concept of entropy is of fundamental importance in the field of information theory. Shannon [109] conceived the statistical nature of the communication signal with that of a random variable $X = \{X_1, X_2, X_3, \ldots, X_n\}$ having probability distribution function $P = \{p_1, p_2, p_3, \ldots, p_n\}$, where $p_i = P\{X = X_i\}$ is the probability of the i^{th} outcome with $0 \leq p_i \leq 1$, $\sum_{i=1}^{n} p_i = 1$, and introduced a measure of average information (or, uncertainty) as

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \ , \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{n} p_i = 1 \tag{1.1}$$

associated with this experiment. Here it is assumed that $0 \log 0 = 0$; and normally the base of the logarithm is taken as 2, and then, the units are *'bits'* a short of the term *'binary digit'*. The measure (1.1) is called the Shannon entropy measure. A few important properties which are usually considered desirable for a measure of uncertainty defined in terms of probability distributions to satisfy are given as follows:

I Non-negativity: $H(P)$ is always non-negative, that is,

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \geq 0 \ . \tag{1.2}$$

Since $-p_i \log p_i \geq 0$ for all i, the result is obvious. It is zero, if one $p_i = 1$ and rest are zeros.

II Maxima: $H(p_1, p_2, \ldots\ldots, p_n) \leq \log n$, with equality when $p_i = \frac{1}{n}$ for all i.

III Continuity: $H(p_1, p_2, \ldots\ldots, p_n)$ is a continuous function of p_i's, that is, a slight change in the probabilities p_i's results in the slight change in the uncertainty measure also.

IV Symmetry: $H(p_1, p_2, \ldots\ldots, p_n)$ is a symmetric function of p_i's, that is, it is invariant with respect to the order of the outcomes.

V Grouping (or, Branching) Property:

$$H\{p_1, p_2, p_3, \cdots, p_n\} = H\{p_1 + \cdots + p_r,\ p_{r+1} + \cdots + p_n\} + (p_1 + \cdots + p_r) \times$$

$$H\left(\frac{p_1}{\sum_{i=1}^{r} p_i}, \cdots, \frac{p_r}{\sum_{i=1}^{r} p_i}\right) + (p_{r+1} + \cdots + p_n) H\left(\frac{p_{r+1}}{\sum_{i=r+1}^{n} p_i}, \cdots, \frac{p_n}{\sum_{i=r+1}^{n} p_i}\right)$$

for $r = 1, 2, \cdots n-1$.

VI Additivity: If $P = (p_1, p_2, \ldots\ldots, p_n)$ and $Q = (q_1, q_2, \ldots\ldots, q_n)$ are two independent probability distributions, then

$$H(P \bullet Q) = H(P) + H(Q),$$

where $P \bullet Q$ is the joint probability distribution, that is, for two independent distributions entropy of the joint distribution is the sum of the entropies of the two marginal distributions.

The continuous analogue of Shannon's entropy takes the form

$$H(X) = -E[\log f(X)] = -\int_{0}^{\infty} f(x) \log f(x) dx \ . \tag{1.3}$$

This form is often referred to as the **differential entropy** of a random variable X with a known probability density function $f(x)$, refer to McEliece [76]. The differential entropy defined above is not always non-negative as in the case of a discrete random variable. For detailed properties of the entropy measure (1.1) one can refer to Aczel and Darocazy [3].

1.2.2 Characterizations and Generalizations

We have seen that Shannon entropy satisfies a number of useful properties like non-negativity, continuity, symmetry, additivity, grouping, etc. Some of these properties have been used as axioms by number of researchers to characterize the Shannon entropy. The most intuitive and compact axioms for characterizing the Shannon entropy function has been given by Khinchin [69], which are known as the Shannon-Khinchin axioms. Many other researchers have also characterized Shannon entropy using different set of axioms. For some further results on characterization and the algebraic properties of Shannon entropy refer to Aczel and Darocozy [3].

Generalized entropies have also been studied from the mathematical point of view. These entropies are functions of some parameters and tends to Shannon entropy when these parameters approach their limiting values. It started with the work of Renyi [101] who characterized a scalar parametric entropy as entropy of order α, which includes Shannon entropy as a limiting case.

An additive generalization of type α of the entropy (1.1) is the Renyi's entropy [101] given by

$$H^{\alpha}(P) = \frac{1}{1-\alpha} \log \left\{ \sum_{i=1}^{n} {p_i}^{\alpha} \right\}; \quad \alpha \neq 1,\ \alpha > 0. \tag{1.4}$$

It contains additional parameter α which can be used to make it more or less sensitive to the shape of probability distributions. The generalized information measures, after Renyi work, continued to be of interest to many mathematicians. The measure (1.4) is additive in nature. A non-additive generalization of Shannon entropy given by Havrda and Charvat [57] is

$$H^{\alpha}(X) = \frac{1}{1-\alpha}\left[\sum_{i=1}^{n} p_i^{\alpha} - 1\right], \quad \alpha \neq 1,\ \alpha > 0. \tag{1.5}$$

Khinchin [69] generalized (1.1) by choosing a convex function $\phi(x)$ such that $\phi(1) = 0$ and defined the measure

$$H^{\phi}(X) = \int f(x)\phi(f(x))dx. \tag{1.6}$$

For one particular choices of $\phi(x)$, (1.6) becomes, for some fixed $\alpha > 0$ and $\alpha \neq 1$,

$$H^{\alpha}(X) = \frac{1}{1-\alpha}\left\{\int f^{\alpha}(x)dx - 1\right\},$$

which is continuous analogous to (1.5).

A two parameter generalization of different entropy is given by

$$H_{\alpha}^{\beta}(X) = \frac{1}{\beta-\alpha}\log\left[\int_0^{\infty} f^{\alpha+\beta-1}(x)dx\right]; \quad \beta - 1 < \alpha < \beta,\ \beta \geq 1\ . \tag{1.7}$$

When $\beta = 1$, this reduces to the continuous analogous to the Renyi entropy [101] of order α; and in case of $\beta = 1$ and $\alpha \to 1$ then $H_{\alpha}^{\beta}(X)$ reduces to Shannon differential entropy [109]. For some other notable work in this direction, one may refer to Verma [123], Arimoto [5], Ferreri [43], Sharma and Taneja [110], Cover and Thomas [23] etc.

1.3 Kullback's Measure of Relative Information and Kerridge Inaccuracy

Kullback and Leibler [70] studied a measure of information from the statistical aspects involving two probability distributions associated with the same random experiment, and called it the discrimination function. Later different authors named it as cross entropy, relative information, etc.

If $P = \{p_1, p_2, \ldots, p_n\}$ is the actual probability distribution associated with the outcomes $X = \{X_1, X_2, \ldots, X_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$ is the predicted (or, reference) distribution associated the same experiment such that $p_i \geq 0,\ \ q_i \geq 0$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, then Kullback's measure of relative information [70] is given by

$$H(P/Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \ . \tag{1.8}$$

The measure (1.8) aims to quantitize discrimination between two populations. It is assumed that whenever $q_i = 0$, the corresponding p_i is also zero and $0 \log \frac{0}{0} = 0$. Kannappan and Rathie [63] have obtained some characterization results based on the directed divergence. The concept of generalized directed divergence has also been discussed by Kapur [64, 65], Taneja [115, 116] and Rathie [100].

Another important aspect is the notation of inaccuracy as introduced by Kerridge [67]. This can also be viewed as a generalization of the Shannon's entropy. In making statement about probability happening of various events in an experiment, two kinds of errors are possible, namely, one resulting from the lack of enough information or vagueness in experimental results (e.g. missing observation or insufficient data) and the other from incorrect information (e.g. miss-specifying the model). Kerridge [67] proposed an inaccuracy measure that can take accounts for these two types of errors. This is defined as

$$H(P;Q) = -\sum_{i=1}^{n} p_i \log q_i \ , \tag{1.9}$$

where q_i is the predicted probability and p_i is the actual probability of an outcome. Obviously when $q_i = p_i$ for all $i's$, then (1.9) reduces to (1.1), the Shannon entropy measure. The measure of information, discrimination and inaccuracy are associated as

$$H(P;Q) = H(P) + H(P/Q) \ , \tag{1.10}$$

that is, inaccuracy is the sum of entropy and discrimination.

Nath [79] extended Kerridges inaccuracy to the case of continuous situation and discussed some properties. If $f(x)$ is the actual probability density function and $g(x)$ is the function assigned by the experimenter, then the Kerridge inaccuracy measure [67] is defined as

$$H(f;g) = -\int_0^\infty f(x) \log g(x) dx \ , \tag{1.11}$$

and the corresponding measure for relative information [70] is given by

$$H(f/g) = \int_0^\infty f(x) \log \frac{f(x)}{g(x)} dx \ . \tag{1.12}$$

The Shannon's entropy, Kullback-Leibler's relative information and Kerridge's inaccuracy are the three classical measures of information associated with one and two probability distributions. These three measures have found deep applications in the areas of information theory and statistics. In the next section we discuss another class of information measures, the weighted information measures.

1.4 Weighted Information Measures

Let $P = \{p_1, p_2, \ldots, p_n\}, \ 0 \leq p_i \leq 1, \ \sum_{i=1}^n p_i = 1$ be the probability distribution associated with a finite system of events $X = \{X_1, X_2, \ldots, X_n\}$ representing the realization of some experiment. The importance of the different events X_i depend upon the experimenter's goal, or upon some qualitative characteristics of the physical system taken into consideration, that is, they have different weights or utilities. Thus we attach to each event X_i, a number $u_i > 0$ directly proportional to its importance and call u_i the utility of the event X_i. Thus in addition to the probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ associated with an experiment, we have $U = \{u_1, u_2, \ldots, u_n\}$ a utility distribution which accounts for the qualitative aspect of the experiment depending upon the experiment goal.

Taking this into consideration, Belis and Guiasu [15] extended the concept of Shannon entropy to the systems being accounted for both aspects, quantitative as well as qualitative and gave a weighted information measure

$$H(P;U) = -\sum_{i=1}^{n} u_i p_i \log p_i \quad u_i \geq 0, \tag{1.13}$$

where u_i is the utility or the cost factor assigned with the outcome X_i . Obviously when all the u_i's are equal then $H(P;U)$ becomes $H(P)$, the entropy measure. The measure (1.13) is called the *quantitative-qualitative measure of information or 'useful'* information measure. For some further results on characterization, generalization and applications of this measure one may refer to Aggarwal and Picard [4], Bhatia and Taneja [17], Gurdial and Pessoa [53], Jain and Srivastava [59], Parkash et al. [93], Taneja [120] and many others.

A *quantitative-qualitative measure of relative information* as suggested by Taneja and Tuteja [118] is given by

$$H(P/Q;U) = \sum_{i=1}^{n} u_i p_i \log \frac{p_i}{q_i} \,. \tag{1.14}$$

The utility u_i of the outcome X_i is independent of probability p_i or q_i and depends only on the qualitative characteristics of the physical system into account. In sequel to these measures, by considering the aspect, analogous to (1.10), we have

$$\begin{aligned} H(P;U) + H(P/Q;U) &= -\sum_{i=1}^{n} u_i p_i \log p_i + \sum_{i=1}^{n} u_i p_i \log \frac{p_i}{q_i} \,, \\ &= -\sum_{i=1}^{n} u_i p_i \log q_i = H(P,Q;U). \end{aligned} \tag{1.15}$$

Taneja and Tuteja [119] defined (1.15) as the *quantitative qualitative measure of inaccuracy*. They have also characterized this measure using a set of axioms. This extends the concept of Kerridge inaccuracy [67] given by (1.9) to a system with qualitative concept. When the utilities are ignored, that is $u_i = 1$ for each i, then (1.13), (1.14) and (1.15) reduces respectively to measures of Shannon entropy [109], Kullback relative information [70], and Kerridge inaccuracy [67] .

Generalizations of these *'useful'* information measures and their applications to coding theory have been studied extensively, refer to Aggarwal and Picard [4], Gurdial and Pessoa [53], Taneja and Tuteja [118] and Kapur [65] and many other researchers.

Since we shall be dealing with the aspect information theoretic measures in lifetime distributions, next we consider some basic concepts in reliability used in the work reported.

1.5 Basic Concepts in Reliability

Let X be a continuous non-negative random variable with distribution function $F(x)$, which represents the lifetime of a unit or system. There are several functions which completely specify the distribution of the random variable X, for example, the reliability (or, survival) function, the hazard rate function, and the mean residual life function. Each of these functions completely describe the distribution function of lifetime of a unit. In fact for a random variable X, each determines the other two uniquely. The nature and scope of information provided by these functions differ and so does their relevance in specific situations. We give a few definitions and some basic properties associated with these concepts.

1.5.1 Reliability (or, Survival Function) of a Component

Reliability is defined as the probability that a given component or a system will perform its required function without failure for a given period of time, when used under stated operating conditions. Mathematically, if X represents the lifetime of a component, then reliability (or, survival function) is defined by

$$\overline{F}(x) = Pr(X > x) = \int_x^{\infty} f(x)dx \ ,$$

where $f(x)$ is the probability density function (p.d.f.) of X. We note that $\bar{F}(x) = 1 - F(x)$, where $F(x)$ is the distribution function of X. It is decreasing function of x satisfying $\overline{F}(0) = 1$ and $\lim_{x\to\infty} \bar{F}(x) = 0$. The probability density function $f(x)$ of X is obtained from its survival function $\overline{F}(x)$ by the relationship

$$f(x) = -\frac{d}{dx}\overline{F}(x).$$

1.5.2 Hazard Rate Function

The *hazard rate function*, also known as the *conditional failure rate* in reliability, is a non-negative function defined as

$$\lambda(x) = \lim_{\Delta x \to 0} \frac{P(x < X < x + \Delta x | X \geq x)}{\Delta x} \ .$$

This is the conditional probability of an item failing in the interval x to $(x + \Delta x)$ given that it has not failed by time x.

In case of a continuous random variable X it is given by

$$\lambda(x) = \frac{f(x)}{\overline{F}(x)} = -\frac{d}{dx} \log \overline{F}(x). \tag{1.16}$$

Obviously

$$\lambda(x) \geq 0 \ \ \forall \ \ x, \ \ \text{and} \int_0^{\infty} \lambda(x)dx = \infty \ .$$

In the discrete set up, Xekalaki [129] defines the failure rate for a random variable X, with non-negative integral support, as

$$\lambda(x) = \frac{P(X = x)}{P(X \geq x)} \ . \tag{1.17}$$

In the discrete case the hazard rate can be interpreted as a probability which is not the case in the continuous case.

If X represents the lifetime of a component, then $\lambda(x)$ is the probability that the component will fail at time $X = x$ given that it has survived up to the time before x. The units of $\lambda(x)$ are probability of failure per unit of time, distance or cycle. In reliability analysis, a life distribution can be classified according to the shape of its hazard rate function $\lambda(x)$. Taking the bathtub curve, the early failure period has a decreasing hazard function as time goes by; the useful life period has a constant hazard function, and the wear-out period has an increasing hazard function. The hazard rate function uniquely determines the survival function $\overline{F}(.)$ of the random variable X through the relationship

$$\overline{F}(x) = \exp\{-\int_0^x \lambda(t)dt\} \ . \tag{1.18}$$

1.5.3 Reversed Hazard Rate Function

The concept of reversed hazard rate introduced by Keilson and Sumita [68] has attracted considerable interest of researchers in survival analysis and reliability, especially in study on parallel systems. For a non-negative random variable X, the *reversed hazard rate* is defined as

$$\mu(x) = \lim_{\Delta x \to 0} \frac{P(x - \Delta x < X < x | X \leq x)}{\Delta x} \ . \tag{1.19}$$

Here $\mu(x)\Delta x$ provides the probability of failing in the interval $(x - \Delta x, \ x)$, when a unit has been found failed at time x.

The reversed hazard rate of a continuous random variable X with distribution function $F(x)$, denoting the lifetime of a component, is given by

$$\mu(x) = \frac{f(x)}{F(x)} = \frac{d}{dx} \log F(x) \ ,$$

where $f(x)$ is the probability density function (p.d.f.) of the random variable X. The reversed hazard rate function for a discrete random variable X with non-negative integral support is defined by

$$\mu(x) = \frac{P(X = x)}{P(X < x)} \ . \tag{1.20}$$

The reversed hazard rate uniquely determines the distribution function $F(x)$ through the relation

$$F(x) = \exp\left(- \int_x^\infty \mu(t)dt \right) \ .$$

We note that the hazard rate and reversed hazard rate are functionally related through the relationship

$$\mu(x) = \frac{\lambda(x)\overline{F}(x)}{F(x)} \ . \tag{1.21}$$

Finkelstein [44] has shown that

$$\mu(x) = \frac{\lambda(x)}{\exp\left(- \int_0^x \lambda(t)dt\right) - 1} \ . \tag{1.22}$$

The reversed hazard rate function is quite useful in forensic sciences, where exact time of failure of a unit is of importance. For more properties and applications of reversed hazard rate function, refer to Block et al. [18], Di Crescenzo [28], Gupta and Nanda [51], Gupta and Wu [52], Nair et al. [81] and Sengupta et al. [108].

1.5.4 Mean Residual Life Function

The *mean residual life* (MRL) of a system or a component is another important aspect in reliability studies. It provides an idea of how long a device of any particular age can be expected to survive. Indeed, if the goal is to improve the average system lifetime, then the mean residual life is the relevant measure.

For a continuous random variable X with $E(X) < \infty$, the *mean residual life function* is defined as

$$\delta(t) = E[X - t | X > t] = \frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)} \ . \tag{1.23}$$

The expected remaining life of the component gives an indication whether to replace or to re-schedule and this can be more useful than the failure rate to formulate maintenance policies. For the various properties and application of mean residual life function one can refer to Swartz [114], Tennakoon [127], Asadi and Bayramoglu [7], Barlow and Proschan [13] and Muth [78].

The reliability function can be represented as a function of the mean residual life, as

$$\overline{F}(t) = \frac{\delta_F(0)}{\delta_F(t)} \exp\left[-\int_0^t \frac{dx}{\delta_F(x)} \right]. \tag{1.24}$$

Further the relationship between the failure rate and the mean residual life function is given by

$$\lambda_F(t) = \frac{\delta'_F(t) + 1}{\delta_F(t)} \ . \tag{1.25}$$

Several characterizations of probability models have been obtained based on the mean residual life (MRL) function, refer to Mukharjee and Roy [77], Sullo and Rutherford [111] and Sunoj et al. [113].

1.6 Hazard Models

In this section we discuss two types of dependence structures between two probability distributions; one the proportional hazard model, and second, the proportional reversed hazard model which have been extensively used in survival analysis.

1.6.1 Proportional Hazard Model

Cox [26] introduced and studied a dependence structure among two distributions, which is referred to as the *proportional hazard model* (PHM). In literature, this model has been used to model failure time data. The PHM, commonly known as Cox PH Model, was basically introduced by Lehmann [74]. In survival analysis, the PHM has been applied to continuous as well as to discrete random variables. It has been used for estimating the risk of failure associated with a vector of covariates.

If X and Y are two non-negative continuous random variables with the same support representing the time to failure of two systems with $\lambda_F(x) = \frac{f(x)}{\bar{F}(x)}$ and $\lambda_G(x) = \frac{g(x)}{\bar{G}(x)}$ as their hazard rates respectively, and if

$$\lambda_G(x) = \beta \lambda_F(x), \tag{1.26}$$

where β is a positive constant, then the model is called proportional hazard model (PHM).

We can easily see that the PHM model (1.26) is equivalent to the model

$$\bar{G}(x) = [\bar{F}(x)]^{\beta}, \ \beta > 0. \tag{1.27}$$

This model finds application in variety of fields such as reliability, survival analysis, medicine, economics etc. Ebrahimi and Kirmani [38] and Nair and Gupta [80] looked into the problem of characterization of specific probability distributions using information theoretic measures under the proportional hazards model assumption.

Sometimes the hazard rates need not be proportional uniformly over the whole time interval, but may be proportional differently in different intervals. In order to take care of this kind of problems, Nanda and Das [82] introduced the *dynamic proportional hazard model* (DPHM), and studied their properties for different aging classes. If we replace β in the above model by some non-negative function of some parameter t, then the corresponding model becomes

$$\lambda_G(x) = \beta(t) \ \lambda_F(x), \ \forall \, t > 0, \tag{1.28}$$

called the dynamic proportional hazard model (DPHM), which accounts for different proportionality in different time intervals.

1.6.2 Proportional Reversed Hazard Model

Gupta et al. [50] proposed another model called the *proportional reversed hazard model* (PRHM) to analyze the failure time data. Sengupta et al. [108] illustrated that proportional reversed hazard model leads to a better fit for some data set rather than proportional hazard model (PHM).

If X and Y are two non-negative continuous random variables with the same support and with reversed hazard rates $\mu_X(x) = \frac{f(x)}{F(x)}$ and $\mu_Y(x) = \frac{g(x)}{G(x)}$ respectively, and if

$$\mu_Y(x) = \beta\ \mu_X(x)\ , \quad \beta > 0 \tag{1.29}$$

then the model is called proportional reversed hazard model (PRHM).
The PRHM is equivalent to the model

$$G(x) = [F(x)]^{\beta}\ , \tag{1.30}$$

where $F(x)$ is the baseline distribution function and $G(x)$ can be considered as some reference distribution function.

Proportional reversed hazard model is useful in the analysis of left censored or right truncated data. The structure and the properties of the PRHM in contrast to the PHM have been studied by Gupta and Gupta [46] and Gupta and Wu [52]. Di Crescenzo [28] has obtained some results on proportional reversed hazard model concerning aging characteristics and stochastic orders. Recently Nanda and Das [82] have proposed the *dynamic proportional reversed hazard model* (DPRHM), defined as

$$\mu_Y(x) = \beta(t)\ \mu_X(x)\ \ , \tag{1.31}$$

which accounts for different proportionality in different time intervals.

1.7 Length Biased Model

The concept of weighted distribution introduced by Rao [96] is widely used in statistics and other applications. Jain et al. [61], Gupta and Kirmani [48] and Nanda and Jain [83] have used the weighted distributions in many practical problems to model unequal sampling probabilities. Such distributions arise when the observations generated from a stochastic process are recorded with some weight function.

Let X be a non-negative continuous random variable with probability density function (p.d.f.) $f(x)$, and let X^w be a weighted random variable corresponding to X with weight function $w(x)$, which is positive for all value of $x \geq 0$. Then the probability density function $f^w(x)$ of the weighted random variable X^w is given by

$$f^w(x) = \frac{w(x)f(x)}{E[w(X)]}, \quad 0 \leq x < \infty, \tag{1.32}$$

with $0 < E[w(X)] < \infty$. Obviously $f^w(x) \geq 0$ and $\int_0^\infty f^w(x)dx = 1$.

When $w(x) = x$, X^w is said to be a *length biased (or, a size biased) random variable* and the p.d.f. (1.32) in this case becomes

$$f^L(x) = \frac{xf(x)}{E[X]} \ . \tag{1.33}$$

Length-biased sampling situations may occur in clinical trials, reliability, queuing models, survival analysis and population studies where a proper sampling frame is absent. In such situations, items are sampled at rate proportional to their length so that larger values of the quantity being measured are sampled with higher probabilities. The statistical interpretation of the length biased distribution was originally identified by Cox [25] in the context of renewal theory. But the same idea has originally been conceived much before as evident from Daniels [27] who discussed length biased sampling in the analysis of the distribution of fiber lengths in wool.

Gupta and Kirmani [48] have shown how length biased sampling affects the original distribution and how the corresponding reliability characteristics change under such a scheme of sampling. While comparing the distribution under length biased sampling with the parent model, it will be of some definite advantage if the original distribution keeps the same form under length biased sampling also, except possibly for a change in the parameters. Substantial work on various aspects of length-biased sampling has been contributed by Patil and Rao [94], Oluyede [90, 91, 92] and Sankaran and Nair [106]. Gupta and Keating [47] have proposed some standard relationships between original and length biased random variables using reliability concepts. We shall apply the concept of length biased random variable to inaccuracy measures in Chapter 4.

1.8 Dynamic Information-Theoretic Measures

Study of the duration of a system is a subject of interest common to reliability, survival analysis, actuary, economics, business, and many other fields. In this section we discuss two dynamic information theoretic measures; one the residual, and the second, the past. The residual information theoretic measures arise when the data is left truncated, and the past information measures arise when the data is right truncated.

1.8.1 Residual Information-Theoretic Measures

In life testing situations, the additional lifetime given that the component has survived up to time t is called the residual lifetime of the component. More specifically, if X is the lifetime distribution of a component, then the random variable $[X-t|X>t]$ is called the *residual lifetime* of the component.

Since Shannon's differential entropy (1.3) is not appropriate to measure the remaining uncertainty for the lifetime of a system which has survived for some unit of time

t, the concept of residual entropy has been developed in the literature. In such a situation, Ebrahimi [34] proposed a dynamic measure of entropy based on Shannon entropy known as *residual entropy*, given by

$$H(f;t) = -E[\log f_t(X_t)] = \int_t^{\infty} -f_t(x)\log f_t(x)dx, \tag{1.34}$$

where $f_t(x)$ denotes the probability density function of the random variable $X_t = [X - t|X > t]$, the remaining lifetime of a unit of age t, given as

$$f_t(x) = \begin{cases} \frac{f(x)}{\bar{F}(t)} & ; \quad \text{if } \ x > t \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$H(f;t)$ basically measures the expected uncertainty contained in the conditional density of $X - t$ given $X > t$ about the predictability of remaining lifetime of the unit.

The Shannon residual entropy (1.34) for the residual lifetime can be expressed as

$$\begin{aligned} H(f;t) &= -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \qquad (1.35) \\ &= \log \bar{F}(t) - \int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log f(x)dx \\ &= 1 - \int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \lambda_F(x) \ , \qquad (1.36) \end{aligned}$$

where $\lambda_F(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate function.

Various results concerning the Shannon residual entropy have been obtained by many researchers. Ebrahimi [34] showed that the dynamic measure (1.35) determines the underlying distribution function uniquely. Rajesh and Nair [99] gave a similar result for the discrete case. By considering a relationship between dynamic entropy and mean residual life of a component, Asadi and Ebrahimi [8] have characterized three specific lifetime distributions namely exponential, Pareto and finite range. Similar results in case of a generalized residual entropy have been derived by Belzunce et al. [16].

Further results concerning residual entropies have been obtained in recent years by Ebrahimi and Pellerey [40], Ebrahimi [35], Sankaran and Gupta [105] and Ebrahimi [36]. Nanda and Paul [85] have obtained some characterization results for distributions based on a generalized residual entropy function. Some other result in this reference in context with the Renyi entropy and Verma entropy have been given by Asadi et al. [9], Abraham and Sankaran [2], Baig and Dar [12] and Abbasnejada et al. [1].

Next, if X and Y are two absolutely continuous, non-negative random variables with the same supports that describe the lifetimes of two systems, then measure of discrepancy between two residual-life distributions has been proposed by Ebrahimi and Kirmani [37], analogous to the Kullback-Leibler relative information measure (1.12) . It is defined as

$$\begin{aligned} H(f/g;t) &= \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx \qquad (1.37) \\ &= \log G(t) - H(f;t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log g(x) dx \ . \end{aligned}$$

This is obtained by replacing $F(.)$ and $G(.)$ by distributions of the corresponding residual lifetimes. If we have a system with true survival function $\overline{F}(.)$ then $H(f/g;t)$ can also be interpreted as a measure of distance between $G_t(x)$ and the true distribution $F_t(x)$. This measure has been used for the classification and ordering of survival function. Ebrahimi and Kirmani [38] and Asadi et al. [9] have studied the aspects of residual Kullback-Leibler information. We note that for each fixed $t > 0$, $H(f/g;t)$, has all the properties of the Kullback-Leibler discrimination information measure $H(f/g)$. Recently, Navarro et al. [87] have extended the measure $H(f/g;t)$ proposed by Ebrahimi and Kirmani [37] to conditionally specified models.

This extension has been used to characterize some bivariate distributions. These distributions are also characterized in terms of proportional hazard rate models and weighted distributions.

1.8.2 Past Information-Theoretic Measures

In many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. For instance if at time t a system which is observed only at certain preassigned inspection times, is found to be down, then the uncertainty of the system's life relies on the past, that is, at which instant in $(0,\ t)$ the system has failed. To be more specific, in a periodic replacement policy where the system is observed at times T, 2T, 3T, $\ldots$ for some preassigned time T, it is possible that at time $(n-1)T$ the system is functioning, but at time nT the system is found to be down, where n is a positive integer. Then, if X is the failure time of the system, the variable of interest is $[nT - X|X \leq nT]$. By writing $nT = t$, we have the random variable ${}_tX = [t - X|X \leq t]$, known as the *inactivity time or the past lifetime* . For various results on the past lifetime random variable, one may refer to Chandra and Roy [20, 21] and Kayid and Ahmad [66].

Based on this idea, Di Crescenzo and Longobardi [29] have considered measure of past entropy over $(0,\ t)$ given by

$$H^*(f;t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx \tag{1.38}$$

$$= 1 - \int_0^t \frac{f(x)}{F(t)} \log \mu_F(x) dx, \tag{1.39}$$

where $\mu_F(x)$ is the reversed hazard rate function of X and, $\frac{f(x)}{F(t)} = f_t^*(x)$ for $X \leq t$ is the probability density function of the past lifetime random variable ${}_tX$, analogous to $f_t(x)$ in case of residual lifetime X_t.

Given that at time t a component is found to be down, $H^*(f;t)$ measures the uncertainty about its past lifetime. In forensic sciences where the knowledge of exact time of failure is important, this type of measures are of added value. Nanda and Paul [84], have proposed some ordering properties based on this measure. Recently, Kundu et al. [73] have characterized some specific continuous and discrete distributions based on certain relationships among past entropy, reversed hazard rate and mean inactivity time.

Di Crescenzo and Longobardi [30] have studied a measure of divergence which constitutes a distance between past lifetimes distributions. The discrimination measure between past lifetimes is

$$H^*(f/g;t) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)/F(t)}{g(x)/G(t)} dx. \tag{1.40}$$

If we have a system with true distribution function $F(.)$ and reference distribution $G(.)$, then $H^*(f/g;t)$ can also be interpreted as a measure of distance between $G_t^*(x)$ and the true distribution $F_t^*(x)$. Di Crescenzo and Longobardi [30] further proved that $H^*(f/g;t)$ is constant if and only if X and Y satisfy the proportional reversed hazard model (PRHM). Recently, Hooda and Saxena [58] have defined a generalized measure of discrimination between two past lifetime distributions of a system and have studied some of its important properties.

1.9 Distribution Function Based Information -Theoretic Measures

Shannon entropy plays an important role in various fields. However, Shannon differential entropy (1.3) raises the following concerns:

1) It is based on the density of the random variable, which in general may or may

not exist. That is, for the case when the cumulative distribution function (CDF) is not differentiable, it would not be possible to define the differential entropy.

2) Shannon entropy of a discrete distribution is always non-negative, while the differential entropy of a continuous variable may take any value on the extended real line.
3) Shannon entropy computed from samples of a random variable lacks the property of convergence to the differential entropy, that is, even when the sample size goes to infinity, the Shannon entropy estimated from these samples will not converge to differential entropy. For further details refer to Rao [97].

Taking note of these limitations, Rao et al. [98] developed another measure of randomness called *Cumulative Residual Entropy (CRE)*, which is based on distribution rather than the density function of a random variable X. The CRE of a non-negative random variable X with distribution $F(x)$ is defined as

$$\xi(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx \ . \tag{1.41}$$

This measure parallels the well-known Shannon entropy but has the following advantages over the differential entropy measure:

1. Cumulative residual entropy has consistent definitions in both the continuous and discrete domains.

2. Cumulative residual entropy is always non-negative.

3. Cumulative residual entropy can be easily computed from sample data and these computations asymptotically converge to the true values.

The basic idea is to replace the density function with the cumulative distribution in Shannon's definition . The distribution function is more regular than the density function, because the density is computed as the derivative of the distribution.

Moreover, in practice what is of interest and/or measurable is the distribution function. For example, if the random variable is the life span of a machine, then the event of interest is not whether the life span equals a specific instant, but rather whether the life span exceeds that instant. This definition also preserves the well-established principle that the logarithm of the probability of an event should represent the information content in the event. Further Rao et al. [98] have obtained several properties of this measure and have provided some applications of it in reliability engineering and computer vision. Rao [97] have developed some more mathematical properties of cumulative residual entropy (CRE) and have been an alternate formula for this measure.

Asadi and Zohrevand [11] have proposed a dynamic cumulative residual entropy and have obtained some of its properties. The cumulative residual entropy (CRE) for the residual lifetime distribution of a system with survival function $\overline{F_t}(x) = P(X - t > x | X > t) = \frac{\overline{F}(x+t)}{\overline{F}(t)}$, is given as

$$\xi(X;t) = -\int_0^\infty \bar{F}_t(x) \log \bar{F}_t(x) dx, \tag{1.42}$$

which can be rewritten as

$$\xi(X;t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} dx \ . \tag{1.43}$$

Zografos and Nadarajah [132] have proposed two new broad classes of measures of uncertainty based on the survival function, called the survival exponential entropy and the generalized survival exponential entropy and studied it. The cumulative residual entropy is a particular case of these class of measures. Recently, a dynamic form of the survival entropy of order α has been proposed by Abbasnejada et al. [1].

1.10 Motivation and Plan of Work

In view of the above discussion and literature review, we were motivated to consider the dynamic entropy measures based on non-additive entropy, since non-additivity rather than additivity is more prevalent in many physical situations. Also, we found considerable interest in studying dynamic (residual and past both) and weighted (length biased) dynamic inaccuracy measures, since this aspect was not explored much. Considering the importance of entropy measures based on distribution function over density function, we considered it worthwhile to study cumulative entropy measures based on generalized information measure, inaccuracy measure, and their dynamic versions. Thesis comprises seven chapters including the current chapter on introduction and literature survey and a bibliography. The work reported is organized as follows;

In **Chapter 2**, we have considered Havrda and Charvat [57] measure of entropy which is a one parameter generalization of the Shannon entropy and is non-additive in nature. We have proposed a residual measure of entropy based on it and have proved a characterization theorem that the proposed measure under some conditions determines the distribution function uniquely. Also, we have characterized some specific probability distributions based on the residual measure proposed. The work reported in this chapter has been published in the papers entitled, **Non-additive Entropy Measure Based Residual Lifetime Distributions** in *JMI International Journal of Mathematical Sciences*, 2010, 1 (2), 1-9, and; **A Generalized Entropy- Based Residual Lifetime Distribution** in *International Journal of Biomathematics*, 2011, 4 (2), 171-184.

In **Chapter 3**, we have conceptualized the idea of dynamic measure of inaccuracy both residual and past. In case of residual inaccuracy measure we have studied the characterization result using proportional hazard model; and in case of past inaccuracy measure we have studied this result using proportional reversed hazard model. Also we have characterized some specific distributions based on these measures. The work reported in this chapter has been published in the papers entitled, **A Dynamic Measure of Inaccuracy Between Two Residual Lifetime Distributions** in *International Mathematical Forum*, 2009, 4 (25), 1213-1220, and; **A Dynamic Measure of Inaccuracy Between Two Past Lifetime Distributions** in *Metrika*, 2010, 74 (1), 1-10.

In **Chapter 4**, the results of Chapter 3 have been extended to weighted distributions, a concept of considerable importance as reviewed already. Taking weights $w(x) = x$, we have introduced length biased measures of residual and past inaccuracies and have studied their respective characterization theorems and other properties. The results reported in this chapter have been published in the papers entitled, **Length Biased Weighted Residual Inaccuracy Measure** in *Metron*, 2010, LXVIII (2), 153-160, and; **On Length Biased Dynamic Measure of Past Inaccuracy** in *Metrika*, 2012, 75 (1), 73-84. Also some results were presented at *International Conference in Mathematics and Applications* held in *Bangkok* on Dec. 19-21, 2009.

In **Chapter 5**, we have generalized the concept of cumulative residual entropy measure to one parameter and two parameters entropies, studied their dynamic versions and characterization results. The exponential, the Pareto and the finite range distribution which are commonly used in the reliability modeling have been characterized in terms of the proposed generalized dynamic cumulative entropy measures. The work reported in this chapter has appeared in the papers entitled, **On Dynamic Renyi Cumulative Residual Entropy Measure** in *Journal of Statistical Theory and Applications*, 2011, 10 (3), 491-500, and; **Some Characterization Results on Generalized Cumulative Residual Entropy Measure** in *Statistics and Probability Letters*, 2011, 81 (8), 72-77. Also some results were presented at *International Congress of Mathematicians (ICM) 2010* held in *Hyderabad* on Aug. 19-27, 2010.

In **Chapter 6**, we have considered dynamic cumulative inaccuracy measures, both residual and past and have studied the characterization results respectively under proportional hazard model and proportional reversed hazard model. The work reported in this chapter has been published in the paper entitled, **On Dynamic Cumulative Residual Inaccuracy Measure** in proceeding of the *World Congress on Engineering (WCE)*, held in London on July 4-6 2012, and, some results have been communicated for publication.

The **Chapter 7**, presents the conclusion of the work reported in the thesis and further scope of work. In the end, we have given bibliography and the complete list of publications from this thesis.

Chapter 2

Generalized Dynamic Entropy Measure

2.1 Introduction

The description of the behavior of biological and engineering systems normally requires the use of concepts of information theory, and in particular of entropy. Shannon's entropy [109] is probably the most widely used index of alpha diversity in ecology, also the Kullback's relative information measure [71] has received scant attention from ecologists as dissimilarity measure between two communities. Shannon's theory has been used to study genomic sequences by calculating the amount of information contributed by individual nucleotides during these encoding and decoding processes, refer to [107]. Novel applications of Shannon [109] and Kullback-Leibler [71] information measures are promoting increased understanding of the mechanisms by which genetic information is converted to work and order. More recently it has been used in the context of theoretical neurobiology, refer to Johnson and Glantz [62].

Let X be a non-negative continuous random variable which denote the lifetime of a device or a system with probability density function $f(x)$ and survival function $\bar{F}(x) = 1 - F(x)$, where $F(.)$ is the failure distribution function of X. Then the average amount of uncertainty associated with the random variable X is given by the differential entropy [76]

$$H(f) = -\int_0^\infty f(x) \log f(x) dx \ , \tag{2.1}$$

which is the continuous analogous of the Shannon entropy measure for the discrete probability distribution $P = (p_1, p_2, \cdots, p_n)$ given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \ , \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{n} p_i = 1. \tag{2.2}$$

In life testing experiments, normally the experimenter has information about the current age of the system under consideration. Obviously the measure like (2.1) is not suitable in such situations and needs to be modified to take into account the current age also. Accordingly Ebrahimi [34] proposed a dynamic measure of uncertainty known as *residual entropy* for the residual lifetime distribution and defined the residual entropy $H(f;t)$ based on the measure (2.1) as

$$H(f;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \ , \tag{2.3}$$

where $\bar{F}(x)$ is the survival function of X. We note that the measure (2.3) is the Shannon entropy of the random variable $X_t = (X - t | X > t)$, and also, when $t = 0$, (2.3) becomes (2.1).

Ebrahimi [34] has showed that the dynamic (residual) measure (2.3) uniquely determines the survival function $\overline{F}(.)$. Sankaran and Gupta [105] have characterized some specific residual lifetime distributions using (2.3) in terms of hazard rate function and mean residual life function.

The measure (2.2) is additive in nature in the sense that if X and Y are two independent random variables, then

$$H(X \bullet Y) = H(X) + H(Y)\,. \tag{2.4}$$

With ever increasing applications of information theoretic measures, sub-additivity rather than additivity has become an acceptable basis. In many social and physical systems the additivity does not quite prevail. For instance, in biological systems the interactions between the various drugs call for non-additivity of the individual effects rather than additivity. Thus non-additive entropy measures are of vital importance from applications point of view. An important non-additive entropy measure given by Havrda and Charvat [57] is

$$H^{\alpha}(P) = \frac{1}{(2^{1-\alpha}-1)}\left[\sum_{i=1}^{n} p_i^{\alpha} - 1\right], \quad \alpha \neq 1,\ \alpha > 0. \tag{2.5}$$

It satisfies the non-additivity

$$H(X \bullet Y) = H(X) + H(Y) + (2^{1-\alpha}-1)H(X)H(Y). \tag{2.6}$$

The continuous analogous to the measure (2.5) is

$$H^{\alpha}(f) = \frac{1}{(2^{1-\alpha}-1)}\left[\int_0^{\infty} f^{\alpha}(x)dx - 1\right],\ \alpha \neq 1,\ \alpha > 0\,. \tag{2.7}$$

When $\alpha \to 1$, the measure (2.7) tends to the differential entropy (2.1).

Among the existing Shannon-Like entropies, the Havrda and Charvat entropy is perhaps the best known and most widely used entropy. This is mainly because Havrda and Charvat entropy has a number of desirable properties which are crucial in many applications. It is more general than the Shannon entropy and simpler than the Renyi entropy [101]. Further the importance of this measure arises from the fact that it is frequently employed in other fields with slight variations. One such variations is q-entropy

$$H_q(f) = \frac{1}{(1-q)}\left[\int_0^{\infty} f^{q}(x)dx - 1\right], \tag{2.8}$$

where q can be seen as measuring the degree of nonextensivity. This is also the well known Tsallis entropy [121, 122]. In recent years, authors have shown more interest in studying the properties and applications of Tsallis entropy, refer to Boghosian [19], Compte and Jou [22], Hamity and Barraco [55] and, Ion and Ion [60].

In this chapter we propose a dynamic (residual) measure of entropy, based on the non-additive entropy (2.7) and study it. The chapter is organized as follows. In Section 2.2, the generalized dynamic measure of entropy is proposed and a characterization result that $H^{\alpha}(f;t)$ uniquely determines the survival function $\overline{F}(.)$ has been studied. By considering a relation between dynamic entropy measure and hazard rate function, some specific residual lifetime distributions have been characterized in Section 2.3. Section 2.4 deals with some properties, like upper bound, monotonicity etc. of the measure prescribed. The chapter ends with the concluding remarks.

2.2 Generalized Dynamic Entropy Measure

Let X be a non-negative random variable representing the lifetime of a system with the average uncertainty given by the non-additive entropy (2.7). Suppose that the system has survived up to time t, then the measure of uncertainty of the remaining lifetime denoted by the random variable $X_t = [X - t \mid X > t]$, based on the generalized entropy (2.7) is proposed as

$$H^{\alpha}(f;t) = \frac{1}{(2^{1-\alpha} - 1)} \left[\int_0^{\infty} f_t^{\alpha}(x) dx - 1 \right], \ \alpha \neq 1, \ \alpha > 0, \tag{2.9}$$

where $f_t(x)$ is the p.d.f. of the random variable $X_t = (X - t|X > t)$ given by

$$f_t(x) = \begin{cases} \frac{f(x)}{\bar{F}(t)} & , \quad \text{if} \;\; x > t \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The measure (2.9) may be considered as the residual measure of entropy. This can be rewritten as

$$H^{\alpha}(f;t) = \frac{1}{(2^{1-\alpha}-1)} \left[\frac{\int_t^{\infty} f^{\alpha}(x)dx}{\bar{F}^{\alpha}(t)} - 1 \right], \; \alpha > 0, \; \alpha \neq 1. \tag{2.10}$$

Obviously $H^{\alpha}(f;0) = H^{\alpha}(f)$ is the Havrda and Charvat information measure (2.7), and when $\alpha \to 1$, then (2.10) reduces to (2.3), the residual entropy $H(f;t)$.

2.2.1 Characterization Result

A natural question arises that whether the proposed generalized residual measure of entropy $H^{\alpha}(f;t)$ determines the lifetime distribution $F(.)$ uniquely . In this context we prove the following Theorem.

Theorem 2.1 *Let X be a non-negative continious random variable with probability density function $f(x)$. If $H^{\alpha}(f;t) < \infty, \forall \; \alpha > 0, \; \alpha \neq 1$ and is increasing in t, then $H^{\alpha}(f;t)$ determines the distribution function $F(.)$ uniquely.*

Proof Rewriting the residual entropy (2.10) as

$$(2^{1-\alpha}-1)H^{\alpha}(f;t) + 1 = \frac{\int_t^{\infty} f^{\alpha}(x)dx}{\bar{F}^{\alpha}(t)} \quad . \tag{2.11}$$

Differentiating (2.11) with respect to t, we obtain

$$(2^{1-\alpha}-1)H'^{\,\alpha}(f;t) = -[\lambda_F(t)]^{\alpha} + \alpha\lambda_F(t)\frac{\int_t^{\infty} f^{\alpha}(x)dx}{\bar{F}^{\alpha}(t)} \; , \tag{2.12}$$

where $\lambda_F(t) = \frac{f(t)}{\overline{F}(t)}$ is the hazard rate of the random variable X.

Using (2.11), it can be rewritten as

$$(2^{1-\alpha}-1)H'^{\ \alpha}(f;t) = -[\lambda_F(t)]^\alpha + \alpha\lambda_F(t) + \alpha\lambda_F(t)(2^{1-\alpha}-1)H^\alpha(f;t). \quad (2.13)$$

This gives

$$[\lambda_F(t)]^\alpha = \alpha\lambda_F(t) + \alpha\lambda_F(t)(2^{1-\alpha}-1)H^\alpha(f;t). - (2^{1-\alpha}-1)H'^{\ \alpha}(f;t) \quad (2.14)$$

Hence for fixed $t > 0,\ \lambda_F(t)$ is a solution of the equation

$$g(x) = x^\alpha - \alpha x - \alpha x(2^{1-\alpha}-1)H^\alpha(f;t) + (2^{1-\alpha}-1)H'^{\ \alpha}(f;t). \quad (2.15)$$

Differentiating it both sides with respect to x, we have

$$g'(x) = \alpha x^{\alpha-1} - \alpha - \alpha(2^{1-\alpha}-1)H^\alpha(f;t). \quad (2.16)$$

For extreme value of $g(x)$, we must have $g'(x) = 0$, which gives

$$x_t = \left[1 + (2^{1-\alpha}-1)H^\alpha(f;t)\right]^{\frac{1}{\alpha-1}} .$$

Further

$$g''(x) = \alpha(\alpha-1)x^{\alpha-2}.$$

Case I: Let $\alpha > 1$, then $g''(x_t) > 0$. Thus $g(x)$ attains minimum at x_t. Also, $g(0) < 0$ and $g(\infty) = \infty$. Further $g(x)$ decreases for $0 < x < x_t$ and increases for $x > x_t$, so $x = \lambda_F(t)$ is the unique solution to $g(x) = 0$.

Case II: Let $\alpha < 1$, then $g''(x_t) < 0$. Thus $g(x)$ attains maximum value at x_t. Also, $g(0) > 0$ and $g(\infty) = -\infty$. Further it can be easily seen that $g(x)$ decreases for $x > x_t$, and increases for $0 < x < x_t$, so $x = \lambda_F(t)$ is the unique solution to $g(x) = 0$.

Thus the generalized dynamic entropy measure (2.10) determines the hazard rate function, and hence, the distribution function uniquely. This completes the proof.

2.3 Characterizing Some Specific Lifetime Distribution Functions

In this section, by considering a relationship between the non-additive residual entropy $H^\alpha(f;t)$ and the hazard rate function $\lambda_F(t)$, we characterize some specific lifetime distributions based on the generalized dynamic entropy measure (2.10). We prove the following theorem:

Theorem 2.2 *Let X be a non-negative continuous random variable with survival function $\bar{F}(.)$, hazard rate $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ and non-additive residual entropy $H^\alpha(f;t)$, then*

$$H^\alpha(f;t) = \frac{c}{\alpha} + \frac{\lambda_F^{\alpha-1}(t) - \alpha}{\alpha(2^{1-\alpha} - 1)}\ , \tag{2.17}$$

if, and only if for

(i) $c = 0$, X has exponential distribution for $\alpha \neq 1,\ \ \alpha > 0$,

(ii) $c > 0$, X has distribution with p.d.f.

$$f(t) = Apqe^{A}(1+pt)^{q-1}\exp[-A(1+pt)^q],\ \ t \geq 0,\ 0 < \alpha < 1, \tag{2.18}$$

(iii) $c < 0$, X has distribution with p.d.f.

$$f(t) = Apqe^{-A}(1-pt)^{q-1}\exp[A(1-pt)^q],\ \ t \geq 0,\ 0 < \alpha < 1, \tag{2.19}$$

where

$$p = \frac{kc\alpha}{d},\ \ q = \frac{\alpha-1}{\alpha-2},\ A = \left[\frac{d}{q}\right]^q \frac{1}{kc\alpha},\ k = (2^{1-\alpha} - 1)\ \text{and}\ \ d > 0$$

are constants.

Proof (i) Let X be an exponential random variable with parameter $\theta > 0$, then its p.d.f. is given by

$$f(x) = \theta e^{-\theta x} \tag{2.20}$$

and the failure rate function is $\lambda_F(t) = \theta$. The residual entropy $H^\alpha(f;t)$ in this case becomes

$$H^\alpha(f;t) = \frac{1}{(2^{1-\alpha}-1)}\left[\frac{\int_t^\infty f^\alpha(x)dx}{\bar{F}^\alpha(t)} - 1\right]$$

$$= \frac{1}{(2^{1-\alpha}-1)}\left[\frac{\int_t^\infty (\theta e^{-\theta x})^\alpha dx}{e^{-\theta\alpha t}} - 1\right]$$

$$= \frac{1}{(2^{1-\alpha}-1)}\left[\frac{\theta^{\alpha-1}-\alpha}{\alpha}\right]$$

$$= \left[\frac{\lambda_F^{\alpha-1}(t)-\alpha}{\alpha(2^{1-\alpha}-1)}\right], \tag{2.21}$$

which is (2.17) for $c = 0$.

Conversely, consider

$$\frac{1}{(2^{1-\alpha}-1)}\left[\frac{\int_t^\infty f^\alpha(x)dx}{\bar{F}^\alpha(t)} - 1\right] = \frac{1}{(2^{1-\alpha}-1)}\left[\frac{\lambda_F^{\alpha-1}(t)-\alpha}{\alpha}\right],$$

Substituting for $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ and simplifying, we obtain

$$\alpha\int_t^\infty f^\alpha(x)dx = \overline{F}(t)f^{\alpha-1}(t)\ . \tag{2.22}$$

Differentiating (2.22) w.r.t. t both sides, we obtain

$$f'(t)\overline{F}(t) + f^2(t) = 0\ ,$$

which further gives

$$\lambda'_F(t) = 0 \Rightarrow \lambda_F(t) = a, \text{ a constant.}$$

Since exponential distribution is the only distribution with failure rate as a constant, thus X follows the exponential distribution.

(ii) Let X be a random variable with p.d.f. as given in (2.18), then

$$(2^{1-\alpha}-1)H^{\alpha}(f;t)=[\frac{\int_t^{\infty} f^{\alpha}(x)dx}{\bar{F}^{\alpha}(t)}-1],\ \alpha>0,\ \alpha\neq 1,$$

becomes

$$(2^{1-\alpha}-1)H^{\alpha}(f;t)=\frac{(Apq)^{\alpha-1}}{\alpha}(1+pt)^{q}+\frac{(Apq)^{\alpha-1}}{A\alpha^{2}}-1\ ,$$

or,

$$H^{\alpha}(f;t)=\frac{[(Apq)(1+pt)^{q-1}]^{\alpha-1}-\alpha}{\alpha(2^{1-\alpha}-1)}+\left[\frac{(Apq)^{\alpha-1}}{A\alpha^{2}(2^{1-\alpha}-1)}\right]. \tag{2.23}$$

Further since the hazard rate function of the p.d.f. (2.18) is

$$\lambda_F(t)=(Apq)(1+pt)^{q-1},$$

thus (2.23) can be rewritten as

$$H^{\alpha}(f;t)=\frac{[\lambda_F^{\alpha-1}(t)-\alpha]}{\alpha(2^{1-\alpha}-1)}+\frac{c}{\alpha}\ \ ,\ \ 0<\alpha<1, \tag{2.24}$$

where $c=\left[\frac{(Apq)^{\alpha-1}}{A\alpha(2^{1-\alpha}-1)}\right]>0$, and this proves the if part.

To prove the 'only if' part consider (2.17) to be valid. This is equivalent to

$$\frac{\int_t^{\infty} f^{\alpha}(x)dx}{\bar{F}^{\alpha}(t)}=\frac{kc}{\alpha}+\frac{\lambda_F^{\alpha-1}(t)}{\alpha}\ ,$$

or,

$$\alpha\int_t^{\infty} f^{\alpha}(x)dx=kc\bar{F}^{\alpha}(t)+f^{\alpha-1}(t)\overline{F}(t)\ . \tag{2.25}$$

Differentiating both sides of this equation with respect to t, we get

$$\frac{\alpha-1}{\alpha}\lambda_F^{\alpha-3}(t)\left[\lambda_F^{2}(t)+\frac{f'(x)}{\bar{F}(t)}\right]=kc\ . \tag{2.26}$$

Using the fact that

$$\lambda_F^{'}(t)=\frac{f'(t)}{\bar{F}(t)}+\lambda_F^{2}(t)\ \ ,$$

Eq.(2.26) becomes

$$\frac{\lambda_F'(t)}{\lambda_F^{3-\alpha}(t)} = \frac{kc\alpha}{\alpha - 1} \; . \tag{2.27}$$

Solving this for $\lambda_F(t)$, we obtain

$$\lambda_F(t) \;=\; \left[\left(\frac{2-\alpha}{1-\alpha}\right)(kc\alpha t + d)\right]^{\frac{1}{\alpha-2}} \tag{2.28}$$

$$\;=\; \left[\left(\frac{d}{q}\right)(1+pt)\right]^{(q-1)} \; ; \; p\;, q\;, t > 0,$$

which is the hazard rate function of the probability density function (2.18), and this concludes the proof for part (ii).

(iii) The proof for the case $c < 0$ is similar to that of (ii) except that the signs of p and A become negative.

2.3.1 Behavior of Hazard Rate Function Versus Time

We know that a lifetime distribution $F(.)$ is classified according to the shape of its hazard rate function $\lambda_F(t)$ as follows. *Distribution $F(.)$ is increasing failure rate(IFR)(or, decreasing failure rate(DFR)), if its hazard rate function $\lambda_F(t)$ is non-decreasing (or, non-increasing) in t; bathtub (BT) (or, upside bath tub (UBT)) curve, if $\lambda_F(t)$ has a bath tub (or, upside-down bath tub) shape.*

The patterns of failures over time are normally classified as *infant mortality, useful life,* and *wear-out* recognized respectively by decreasing, constant, and increasing hazard rate functions. The three patterns combine to produce the well known *bath tub curve.* The bath tub shaped failure rate functions play an important role in reliability applications, such as human life and electronic devices.

The graph of the hazard rate function (2.28) for some specific values of the parameters have been shown in Figs. 2.1-2.4.

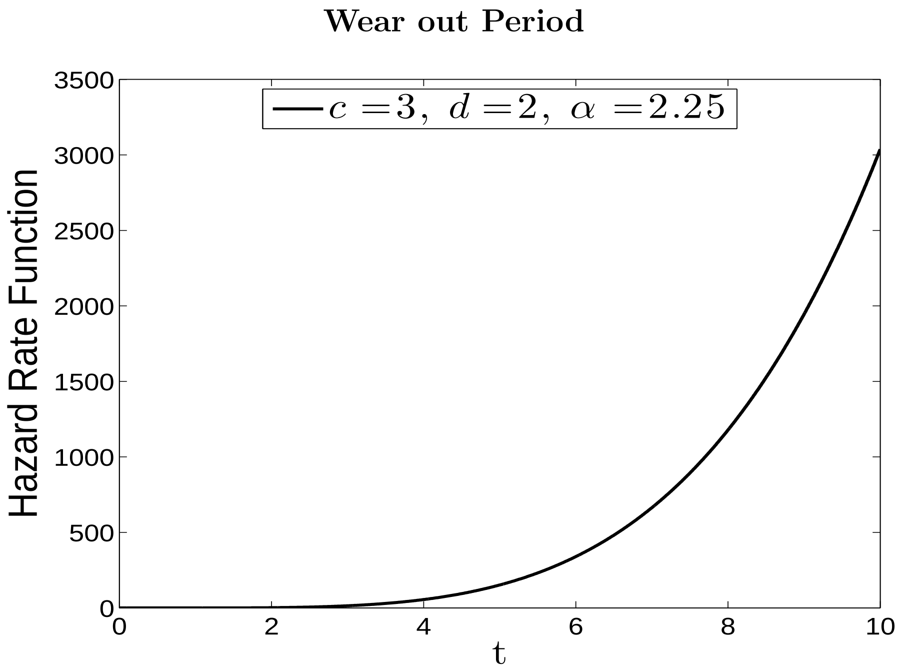

Fig. 2.1: Plot of $\lambda_F(t)$ versus t

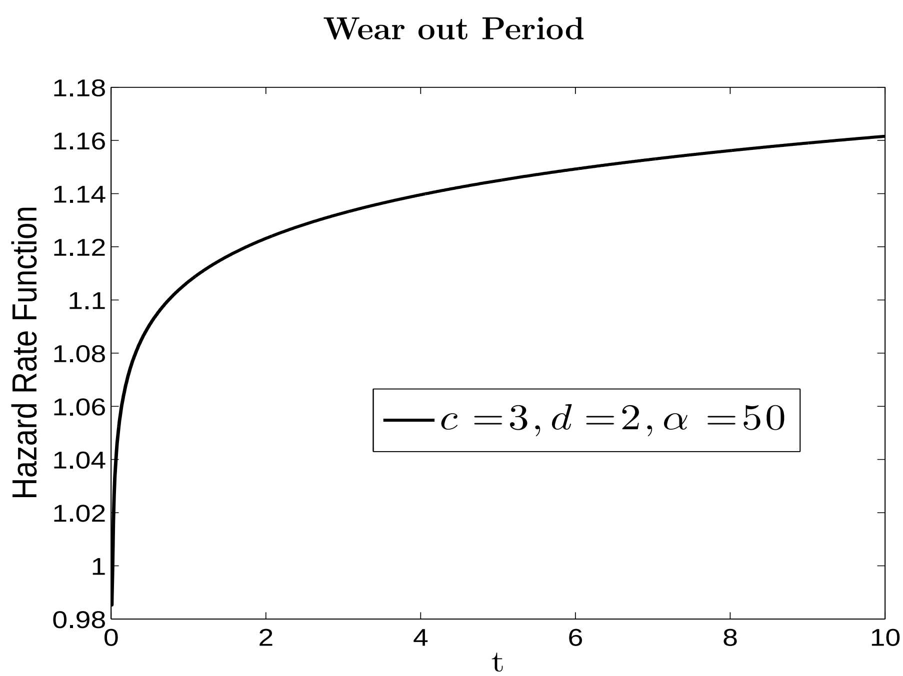

Fig. 2.2: Plot of $\lambda_F(t)$ versus t

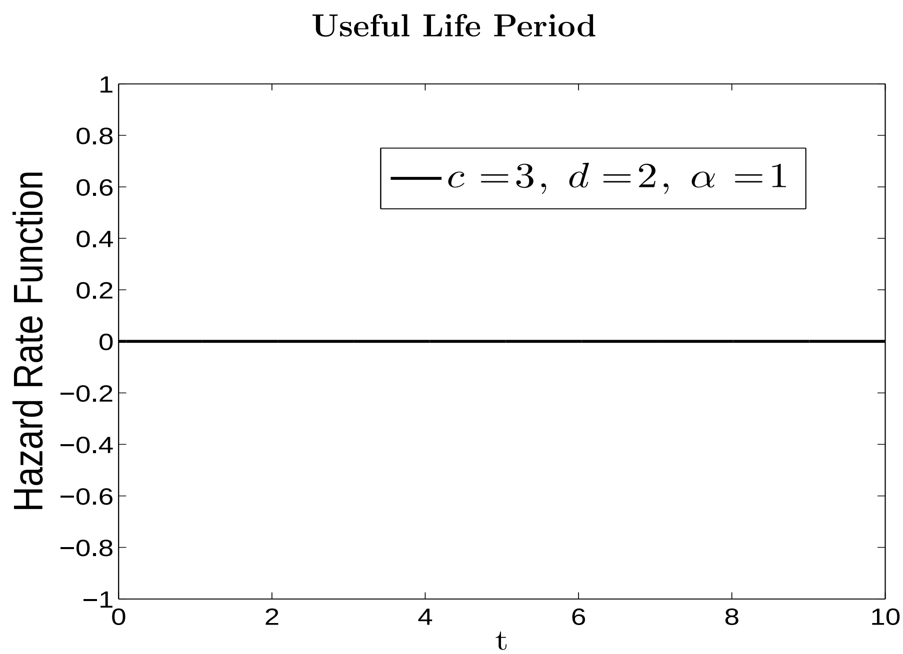

Fig. 2.3: Plot of $\lambda_F(t)$ versus t

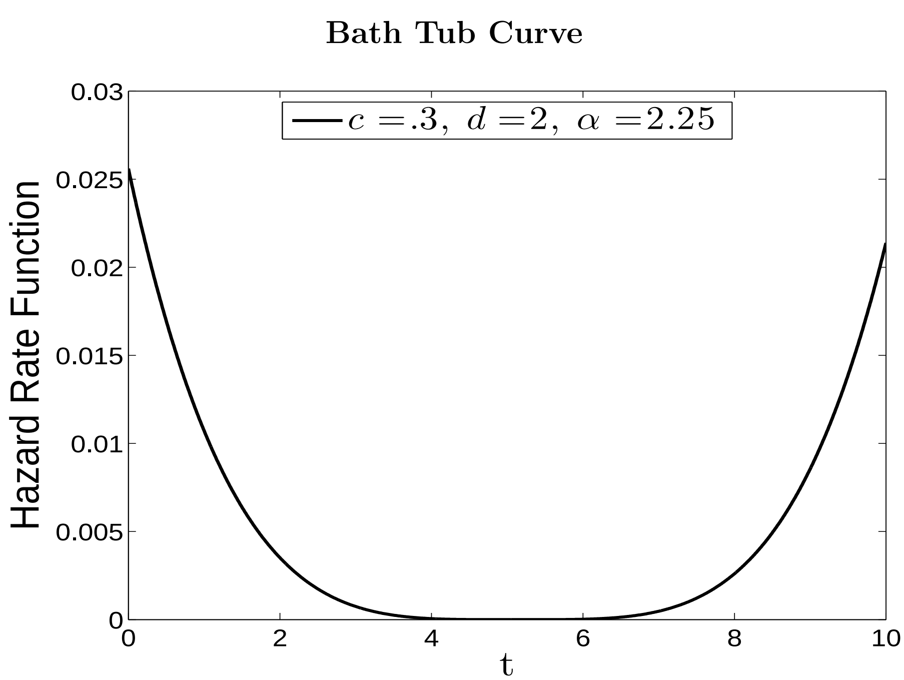

Fig. 2.4: Plot of $\lambda_F(t)$ versus t

Figs. 2.1 and 2.2 has increasing hazard rate function a case of wear out. In Fig. 2.3, the period is characterized by a relatively constant failure rate. The length of this period is referred to as useful life of a unit. In Fig. 2.4, the life of a unit can be divided into three distinct periods. The first period is of infant mortality period, the next period is of useful life and the third period, which begins at the point where the slope begins to increase and extends to the end of the graph, is wear out period.

2.4 Properties of Generalized Dynamic Entropy Measure

In this section we study some properties of the non-additive residual entropy measure (2.10) . We recall, refer to Section 1.5.4, that if X is a random variable with distribution function $F(.)$, then the mean residual life of X is given by

$$\begin{aligned} \delta_F(t) = E[X - t|X > t] &= \int_t^\infty x \frac{f(x)}{\overline{F}(t)} dx, \\ &= t + \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x) dx \ . \end{aligned}$$

This represents the expected time a system will work further provided that it has survived to a certain point of time t, refer to [75].

1. Upper Bound to $H^\alpha(f;t)$**:** We have the following result:

Theorem 2.3 *If X is the lifetime of a system with probability density function $f(x)$, survival function $\overline{F}(x)$, then*

$$H^\alpha(f;t) \le \frac{\left[\delta_F^{1-\alpha}(t) - \alpha\right]}{\alpha(2^{1-\alpha} - 1)}, \ \ \forall \ t \ge 0, \ \alpha > 0, \ \alpha \ne 1 \ , \tag{2.29}$$

where $\delta_F(t)$ is the mean residual life function of the exponential distribution.

Proof For a given t, let the random variable Y_t be defined as $[Y_t = Y|Y > t]$ and $g_t(y)$ be its probability density function. Then

$$g_t(y) = \frac{d}{dy}P(Y_t \leq y) = \frac{d}{dy}[P(Y \leq y|Y > t)]$$

$$= \begin{cases} \frac{f(y)}{\overline{F}(t)} & ; \quad \text{if } \; y > t \\ 0 & ; \quad \text{if } \; y \leq t \end{cases}$$

It is easy to see that $\int_t^\infty y g_t(y)dy = \delta_F(t) + t$. If we define $Z_t = Y_t - t$, then the probability density function of Z_t is $h_t(\eta)$, where $h_t(\eta) = g_t(\eta+t)$ and $E[Z_t] = \delta_F(t)$. Thus the Havrda and Charvat entropy of Z_t is

$$\begin{aligned} H^\alpha(Z_t) &= \frac{1}{(2^{1-\alpha}-1)}\left[\int_0^\infty h_t^\alpha(\eta)d\eta - 1\right] \\ &= \frac{1}{(2^{1-\alpha}-1)}\left[\int_0^\infty g_t^\alpha(\eta+t)d\eta - 1\right] \\ &= \frac{1}{(2^{1-\alpha}-1)}\left[\int_t^\infty g_t^\alpha(\eta)d\eta - 1\right] \\ &= H^\alpha(g;t). \end{aligned}$$

Under the assumption that $\delta_F(t) < \infty$, and if the support of a random variable is $[0, \; \infty)$, then the exponential distribution with mean residual life $\delta_F(t)$ has the maximum entropy, refer to [10]. Now the dynamic Havrda and Charvat entropy is

$$H^\alpha(f;t) = \frac{1}{(2^{1-\alpha}-1)}\left[\int_0^\infty f_t^\alpha(x)dx - 1\right],$$

and for exponential distribution $\delta_F(t) = \frac{1}{\lambda_F(t)} = \frac{1}{\theta}$, thus from (2.21) we have

$$H^\alpha(f;t) \leq \frac{\left(\delta_F^{1-\alpha}(t) - \alpha\right)}{\alpha(2^{1-\alpha}-1)}. \tag{2.30}$$

This completes the proof.

Remark 2.1 Further if $\delta_F(t)$ is a decreasing function of t, then

$$H^{\alpha}(f;t) \leq \frac{(\mu^{1-\alpha}-\alpha)}{\alpha(2^{1-\alpha}-1)} \ ,$$

where $\delta_F(0) = E[X] = \mu$ is the mean lifetime of the unit.

2. Monotonicity of $H^{\alpha}(f;t)$: In reliability and life testing situations, a number of non-parametric classes of lifetime distributions are considered to model the life times of individuals as well as of mechanical systems or components. Most of these classes characterize the aging properties of the underlying phenomenon. Some of the most commonly used classes are the ones defined in terms of failure rate and mean residual life functions. Here we identify the conditions under which the residual entropy measure $H^{\alpha}(f;t)$ given by (2.10) is monotone. First we give the following definitions.

Definition 2.1 *A distribution function $F(.)$ has increasing (or, decreasing) residual entropy of order α (IREO(α)) (or, DREO(α)), if $H'^{\alpha}(f;t)$ is increasing (or, decreasing) in t, $t>0$, where $H'^{\alpha}(f;t)$ is the derivative of $H^{\alpha}(f;t)$ w.r.t. t.*
This implies that $F(.)$ has IREO(α) (DREO(α)) if

$$H'^{\alpha}(f;t) \geq (\leq)\ 0.$$

When $F(.)$ is both IREO(α) and DREO(α), then $H'^{\alpha}(f;t) = 0$ and consequently the distribution is exponential . This means that the exponential distribution is the only distribution which is both IREO(α) and DREO(α).

Definition 2.2 *A distribution function $F(.)$ is said to be decreasing (or, increasing) mean residual life DMRL (or, IMRL) if its mean residual life function is decreasing (or, increasing) in $t\ \geq 0$.*

Next we prove the following results in context with the monotonicity of $H^{\alpha}(f;t)$.

Theorem 2.4 *(a) If $F(.)$ is DMRL, then it is DREO(α).*
(b) If $F(.)$ is IREO(α), then it is IMRL.

Proof From Eq. (2.13), we have

$$H'^{\,\alpha}(f;t) = -\frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha}-1)} + \frac{\alpha\lambda_F(t)}{(2^{1-\alpha}-1)} + \alpha\lambda_F(t)H^\alpha(f;t).$$

Using Theorem 2.3, we get

$$\begin{aligned} H'^{\,\alpha}(f;t) &\leq -\frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha}-1)} + \frac{\alpha\lambda_F(t)}{(2^{1-\alpha}-1)} + \frac{\lambda_F(t)[\delta_F^{1-\alpha}(t)-\alpha]}{(2^{1-\alpha}-1)} \\ &= \frac{\lambda_F(t)}{(2^{1-\alpha}-1)}\left\{\delta_F^{1-\alpha}(t) - \lambda_F^{\alpha-1}(t)\right\} \\ &= \frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha}-1)}\left\{[\lambda_F(t)\delta_F(t)]^{1-\alpha} - 1\right\}. \qquad (2.31) \end{aligned}$$

Using the relationship $\lambda_F(t)\delta_F(t) = 1 + \delta'_F(t)$, we obtain

$$H'^{\,\alpha}(f;t) \leq \frac{[\lambda_F(t)]^\alpha}{(2^{1-\alpha}-1)}\left\{-1 + [1+\delta'_F(t)]^{(1-\alpha)}\right\}. \qquad (2.32)$$

We consider the following two cases.

Case I: Let $0 < \alpha < 1$, then $(2^{1-\alpha}-1) > 0$ and thus $H'^{\,\alpha}(f;t) \leq 0$.

Case II: Let $\alpha > 1$, then $(2^{1-\alpha}-1) < 0$ and thus $H'^{\,\alpha}(f;t) \leq 0$.

This completes the proof.

(b) The proof is similar to that of part (a), and hence omitted.

Remark 2.2 When $\alpha \to 1$, then (2.32) reduces to

$$H'(f;t) \leq \lambda_F(t)\left[\log(1+\delta'_F(t))\right],$$

a result given by Ebrahimi and Kirmani [39].

Theorem 2.5 *(a) If X is IREO(α) and if ϕ is non-negative, increasing and convex, then $\phi(X)$ is DREO(α).*

(b) If X is DREO(α) and if ϕ is non-negative, increasing and convex, then $\phi(X)$ is IREO(α).

Proof (a) The probability density function of $Y = \phi(X)$ is $g(y) = \frac{f(\phi^{-1}(y))}{\phi\,'(\phi^{-1}(y))}$. Thus

$$H^{\alpha}(g;t) = \frac{1}{(2^{1-\alpha}-1)}\left[\frac{\int_t^{\infty} g^{\alpha}(y)dy}{\bar{G}^{\alpha}(t)} - 1\right].$$

This gives

$$H^{\alpha}(g;t) = \frac{1}{(2^{1-\alpha}-1)}\left(\frac{1}{\overline{F}^{\alpha}(\phi^{-1}(t))}\frac{\int_t^{\infty} f^{\alpha}(\phi^{-1}(y))dy}{\phi\,'\,^{\alpha}(\phi^{-1}(y))} - 1\right). \tag{2.33}$$

By taking $x = \phi^{-1}(t))$, we have

$$H^{\alpha}(g;t) = \frac{1}{(2^{1-\alpha}-1)}\left(\frac{1}{\overline{F}^{\alpha}(\phi^{-1}(t))}\int_{\phi^{-1}(t)}^{\infty} f^{\alpha}(x)\phi'\,^{1-\alpha}(x)dx - 1\right). \tag{2.34}$$

Differentiating w.r.t. t under the integral sign, we obtain

$$(2^{1-\alpha}-1)\frac{d}{dt}H^{\alpha}(g;t) = -\frac{f^{\alpha}(\phi^{-1}(t))\phi\,'^{1-\alpha}(\phi^{-1}(t))}{\phi\,'(t)\overline{F}^{\alpha}(\phi^{-1}(t))}$$

$$+\int_{\phi^{-1}(t)}^{\infty} f^{\alpha}(x)\phi'\,^{1-\alpha}(x)dx\left[\frac{\alpha f(\phi^{-1}(t))}{\phi'(t)\overline{F}^{\alpha+1}(\phi^{-1}(t))}\right]. \tag{2.35}$$

This gives

$$(2^{1-\alpha}-1)\frac{d}{dt}H^{\alpha}(g;t) = -\frac{\lambda_F^{\alpha}(\phi^{-1}(t))\phi\,'^{1-\alpha}(\phi^{-1}(t))}{\phi'(t)}$$

$$+\frac{\alpha\lambda_F(\phi^{-1}(t))}{\phi'(t)}\left(\frac{1}{\overline{F}^{\alpha}(\phi^{-1}(t))}\int_{\phi^{-1}(t)}^{\infty} f^{\alpha}(x)\phi'\,^{1-\alpha}(x)dx - 1\right). \tag{2.36}$$

Let $\alpha > 1$. $\phi'(x)$ is increasing function because $\phi(x)$ is a convex function and so, $\phi'^{\,1-\alpha}(x)$ is a decreasing function , that is,

$$\phi'^{\,1-\alpha}(x) \leq \phi'^{\,1-\alpha}(\phi^{-1}(t)), \quad \forall \;\; x > \phi^{-1}(t).$$

Hence, (2.36) becomes

$$(2^{1-\alpha}-1)\frac{d}{dt}H^{\alpha}(g;t) \leq -\frac{\lambda_F^{\alpha}(\phi^{-1}(t))\phi'^{1-\alpha}(\phi^{-1}(t))}{\phi'(t)}$$

$$+\alpha\frac{\lambda_F(\phi^{-1}(t))\phi'^{1-\alpha}(\phi^{-1}(t))}{\phi'(t)}\left(\frac{\int_{\phi^{-1}(t)}^{\infty} f^{\alpha}(x)dx}{\overline{F}^{\alpha}(\phi^{-1}(t))}-1\right). \tag{2.37}$$

Using (2.11), we obtain

$$(2^{1-\alpha}-1)\frac{d}{dt}H^{\alpha}(g;t) = -\frac{\lambda_F^{\alpha}(\phi^{-1}(t))\phi'^{1-\alpha}(\phi^{-1}(t))}{\phi'(t)}$$

$$+\alpha\frac{\lambda_F(\phi^{-1}(t))\phi'^{1-\alpha}(\phi^{-1}(t))}{\phi'(t)}\left\{((2^{1-\alpha}-1)H^{\alpha}(f;\phi^{-1}(t))+1\right\}$$

$$=\frac{\phi'^{1-\alpha}(\phi^{-1}(t))}{\phi'(t)}\left[-\lambda_F^{\alpha}(\phi^{-1}(t))+\alpha\lambda_F(\phi^{-1}(t)\left\{(2^{1-\alpha}-1)H^{\alpha}(f;\phi^{-1}(t))+1\right\}\right].$$

Using (2.13), we get

$$\frac{d}{dt}H^{\alpha}(g;t) = \frac{\phi'^{1-\alpha}(\phi^{-1}(t))}{\phi'(t)}\left[H'^{\,\alpha}(f;\phi^{-1}(t))\right] \leq 0.$$

A similar result follows for $0 < \alpha < 1$.

(b) The proof is similar to that of part (a), and hence omitted.

The next theorem gives upper (lower) bound to the hazard rate function $\lambda_F(t)$ in terms of increasing (decreasing) $H^{\alpha}(f;t)$.

Theorem 2.6 *Let $F(.)$ be a IREO(α), (DREO(α)), then*

$$\lambda_F(t) \leq (\geq)\{\alpha + [(2^{1-\alpha}-1)\alpha H^{\alpha}(f;t)]\}^{\frac{1}{\alpha-1}} . \tag{2.38}$$

This bound can be obtained using (2.13). The proof is simple and hence omitted.

Remark 2.3 Since the distribution function and the hazard rate function are equivalent in the sense that one can be obtained from the other uniquely, thus using the relationship

$$\bar{F}(t) = \exp\left[-\int_0^t \lambda_F(x)dx\right] ,$$

Theorem 2.6 can give a bound to the distribution function also. The result is stated as follows.

Corollary 2.1 *Let $F(.)$ be an IREO(α), (DREO(α)), then*

$$\bar{F}(t) \geq (\leq) \ \exp\left[-\int_0^t \left\{\alpha + (2^{(1-\alpha)} - 1)\alpha H^{\alpha}(f;u)du\right\}^{\frac{1}{\alpha-1}}\right] \forall t \geq 0 \ .$$

2.5 Conclusion

The concept of entropy $H(f)$ introduced by Shannon [109] in the literature measures the average uncertainty associated with a random variable X with probability density function $f(.)$. For a component, which has survived up to time t, $H(f;t)$ measures the uncertainty about the remaining lifetime $[X|X \geq t]$. Considering the importance of non-additive entropy measure we have proposed one parameter generalized residual entropy measure $H^{\alpha}(f;t)$ and have observed that the proposed measure determines the distribution uniquely. Further we have seen that it characterizes three specific lifetime distributions. Some properties like upper bound to the measure proposed, and monotonicity etc. have been studied. In the subsequent chapters we extend the scope of dynamic entropy measures to the concept of inaccuracy given by Kerridge [67].

Chapter 3

Dynamic Inaccuracy Measures

3.1 Introduction

Most of the work on characterization of lifetime distribution function of a system in the reliability context centers around the hazard rate or the mean residual life function. In a variant approach, Ebrahimi [34] proposed the residual entropy function as a useful tool to analyze the stability of a component or a system . In the preceding chapter we have considered a one parameter non-additive residual information measure and based on that we have characterized a few specific lifetime distributions.

Several researchers, refer to [8, 9, 12], have employed information measures like time dependent Kullback-Leibler directed divergence [71] and its generalizations in characterizing lifetime distributions. The Kerridge inaccuracy measure [67] can be viewed as a generalization of Shannon's entropy [109] in the sense that when the predicted probability distribution of a random variable X coincides with the actual probability distribution, then the Kerridge inaccuracy measure reduces to the

Shannon entropy measure . Therefore, there is a scope for extending the results based on Shannon's entropy and its generalizations to the inaccuracy measures. Motivated by this, in the present chapter we extend the definition of the inaccuracy to the truncated situation and propose dynamic measures of inaccuracy, both residual and past. The chapter is organized as follows. In Section 3.2, we introduce a residual inaccuracy measure by considering the measures of residual entropy and residual discrimination. In Section 3.3, we prove a characterization result that if the proposed and actual probability distributions satisfy the proportional hazard model then the residual inaccuracy measure determines the underlying probability distribution uniquely. Section 3.4 introduces the concept of past inaccuracy and in the subsequent Section 3.5 we prove a characterization result. Also we have derived some specific properties of the measures introduced. The chapter ends with conclusion.

3.2 Residual Inaccuracy Measure

Let X and Y be two non-negative random variables representing time to failure of two systems with p.d.f. respectively $f(x)$ and $g(x)$. Let $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ be failure distributions, $\lambda_F(x) = \frac{f(x)}{\bar{F}(x)}$ and $\lambda_G(x) = \frac{g(x)}{\bar{G}(x)}$ be hazard rates, and $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ be survival functions of X and Y respectively. Shannon's measure of uncertainty [109] associated with the random variable X and Kullback's measure of discrimination [71] of X about Y are given respectively by

$$H(f) = -\int_0^\infty f(x)\log f(x)dx \ , \tag{3.1}$$

and

$$H(f/g) = \int_0^\infty f(x)\log \frac{f(x)}{g(x)}dx \ . \tag{3.2}$$

In survival analysis and in life testing, since the current age of the system under consideration is also taken into account, thus for calculating the uncertainty of a system or the discrimination between two systems, the measures (3.1) and (3.2) are not suitable. Given that the system has survived up to time t, the corresponding dynamic measure of uncertainty [34], and of discrimination [37, 38] are given by

$$H(f;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \ , \tag{3.3}$$

and

$$H(f/g;t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx \tag{3.4}$$

respectively.

When $t = 0$, then (3.3) reduces to (3.1), and (3.4) reduces to (3.2).

Adding (3.3) and (3.4) , we obtain

$$H(f;t) + H(f/g;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx + \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx$$

$$= -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx,$$

$$= H(f,g;t), \ \textit{say}. \tag{3.5}$$

In case we have a system with true survival function $\bar{F}(.)$ and the reference survival function $\overline{G}(.)$, then the measure $H(f,g;t)$ can be interpreted as a measure of inaccuracy associated with the density functions f_t and g_t, where $f_t = \frac{f(x)}{\bar{F}(t)}$ and $g_t = \frac{g(x)}{\bar{G}(t)}$.

We define the measure

$$H(f,g;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx \ , \tag{3.6}$$

as a *dynamic measure of inaccuracy associated with two residual lifetime distributions $F(.)$ and $G(.)$* analogous to the Kerridge inaccuracy [67] given by

$$H(f,g) = -\int_0^\infty f(x) \log g(x) dx \ . \tag{3.7}$$

Obviously, at $t = 0$, (3.6) reduces to (3.7).

When $g(x) = f(x)$, then (3.6) becomes (3.3), the dynamic measure of uncertainty given by Ebrahimi [34].

3.3 Characterization Problem For Residual Inaccuracy Measure

The general characterization problem is to determine when the dynamic inaccuracy measure determines the distribution functions uniquely. We study characterization problem for the dynamic inaccuracy measure under the assumption that the distribution functions of the random variables X and Y satisfy the proportional hazard model. Under this model, refer to [24] and [42], their survival functions $\bar{F}(.)$ and $\bar{G}(.)$ are related by

$$\bar{G}(x) = [\bar{F}(x)]^{\beta},\ \beta > 0\ . \tag{3.8}$$

We note that based on the proportional hazard model (3.8), the hazard rate functions $\lambda_F(.)$ and $\lambda_G(.)$ satisfy the relation

$$\lambda_G(x) = \beta\lambda_F(x). \tag{3.9}$$

Next, we prove the following characterization result.

Theorem 3.1 *Let X and Y be two non-negative random variables satisfying the proportional hazard model (3.8), and let $H(f, g; t) < \infty, \forall\, t \geq 0$, then $H(f, g; t)$ determines the survival function $\bar{F}(.)$ uniquely.*

Proof Let f_1, g_1 and f_2, g_2 be two sets of the probability density functions satisfying the proportional hazard model, that is, $\lambda_{G_1}(x) = \beta\lambda_{F_1}(x)$, and $\lambda_{G_2}(x) = \beta\lambda_{F_2}(x)$, and let

$$H(f_1, g_1; t) = H(f_2, g_2; t)\ ,\ \forall\ \ t \geq 0. \tag{3.10}$$

Consider

$$H(f,g;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx \ , \tag{3.11}$$

$$= \log \bar{G}(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log g(x) dx \ . \tag{3.12}$$

Differentiating (3.12) w.r.t. t and using (3.9), we obtain

$$H^{'}(f,g;t) = -\lambda_G(t) + \lambda_F(t) \log g(t) - \lambda_F(t) \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log g(x) dx \tag{3.13}$$

$$= \lambda_F(t)[-\beta + \log g(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log g(x) dx]$$

$$= \lambda_F(t)[-\beta + \log g(t) - \log \bar{G}(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx]$$

$$= \lambda_F(t)[-\beta + \log g(t) - \log \bar{G}(t) + H(f,g;t)] \ .$$

This gives

$$H'(f,g;t) = \lambda_F(t)[-\beta + \log \beta + \log \lambda_F(t) + H(f,g;t)] \ . \tag{3.14}$$

Using (3.14), from (3.10) we obtain

$$\lambda_{F_1}(t)[-\beta + \log \beta + \log \lambda_{F_1}(t) + H(f_1,g_1;t)]$$
$$= \lambda_{F_2}(t)[-\beta + \log \beta + \log \lambda_{F_2}(t) + H(f_2,g_2;t)] \tag{3.15}$$

To prove that (3.10), under the assumption of proportional hazard model (3.8), implies $\overline{F}_1(t) = \overline{F}_2(t)$, it is sufficient to prove that

$$\lambda_{F_1}(t) = \lambda_{F_2}(t), \forall \ \ t \geq 0. \tag{3.16}$$

Define a set

$$A = \{t : t \geq 0, \text{and } \ \lambda_{F_1}(t) \neq \lambda_{F_2}(t)\} \tag{3.17}$$

and assume the set A to be non empty. Thus for some $t_0 \in A$, $\lambda_{F_1}(t_0) \neq \lambda_{F_2}(t_0)$. Without loss of generality suppose that $\lambda_{F_1}(t_0) > \lambda_{F_2}(t_0)$ and since (3.15) holds, then either

$$-\beta+\log\beta+\log\lambda_{F_1}(t_0)+H(f_1,g_1;t_0) < -\beta+\log\beta+\log\lambda_{F_2}(t_0)+H(f_2,g_2;t_0) \quad (3.18)$$

or

$$\begin{aligned} -\beta + \log\beta + \log\lambda_{F_1}(t_0) + H(f_1,g_1;t_0) &= \\ -\beta + \log\beta + \log\lambda_{F_2}(t_0) + H(f_2,g_2;t_0) &= 0. \end{aligned} \quad (3.19)$$

Suppose (3.18) holds, then using (3.10) the inequality (3.18) reduces to $\lambda_{F_1}(t_0) < \lambda_{F_2}(t_0)$. If (3.19) holds, then using (3.10), it reduces to $\lambda_{F_1}(t_0) = \lambda_{F_2}(t_0)$. Combining these two we get $\lambda_{F_1}(t_0) \leq \lambda_{F_2}(t_0)$. This contradicts the assumption $\lambda_{F_1}(t_0) > \lambda_{F_2}(t_0)$ and, therefore, the set A is empty and this concludes the proof.

3.3.1 Properties of the Residual Inaccuracy Measure

Before working for the properties of the residual measure of inaccuracy we give following definitions.

Definition 3.1 *A distribution function $F(.)$ is said to be decreasing (increasing) mean residual life DMRL (IMRL), if its mean residual life function $\delta_F(t)$ is decreasing (increasing) in $t \geq 0$.*

Definition 3.2 *A survival function $\overline{F}(.)$ has decreasing (increasing) inaccuracy in residual life DIRL (IIRL), if $H'(f,g;t)$ is decreasing (increasing) in t, $t \geq 0$.*

Definition 3.3 *Let $\phi(x)$ be a monotone function. If $Y(\phi(X)) = Y(X)$ for all continuous random variable X, then $\phi(x)$ is affine transformation.*

We know that Shannon entropy is not invariant under affine transformation because it is shift invariant but not scale invariant, that is $H(aX + b) = H(X) + \log a$.

The dynamic measure of inaccuracy $H(f,g;t)$ satisfies the following important properties:

I *For a common increasing transformation ϕ of X and Y*

$$H(X,Y;\phi^{-1}(t)) = H(\phi(X),\phi(Y);t).$$

Proof Consider

$$H(\phi(X),\phi(Y),t) = -\int_t^{\infty} \frac{f(\phi^{-1}(x))}{\phi'(\phi^{-1}(x))\bar{F}(\phi^{-1}(t))} \log \frac{g(\phi^{-1}(x))}{\bar{G}(\phi^{-1}(t))} dx \qquad (3.20)$$

$$= -\int_{\phi^{-1}(t)}^{\infty} \frac{f(y)}{\bar{F}(\phi^{-1}(t))} \log \frac{g(y)}{\bar{G}(\phi^{-1}(t))} dy$$

$$= H(X,Y,\phi^{-1}(t)).$$

II *If $\bar{F}(.)$ and $\bar{G}(.)$ satisfy the proportional hazard model (3.8), and $\delta_F(t)$ is finite, then*

$$H(f,g,t) \leq \beta - \log\beta + \log\delta_F(t)\ , \qquad (3.21)$$

where $\delta_F(t)$ is the mean residual lifetime function.

Proof The dynamic measure of inaccuracy is

$$H(f,g;t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx\ .$$

Under proportional hazard model (3.9), we can express it as

$$H(f,g;t) = \beta - \log\beta - \int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log\lambda_F(x) dx$$
$$= (\beta - \log\beta - 1) + 1 - \int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log\lambda_F(x) dx. \qquad (3.22)$$

Also, the dynamic measure of entropy is

$$H(f;t) = 1 - \int_t^{\infty} \frac{f(x)}{\overline{F}(t)} \log\lambda_F(x) dx \leq 1 + \log\delta_F(t)\ ,$$

refer to [34]. Using this in (3.22), we get (3.21).

III *The maxima of dynamic inaccuracy measure under proportional hazard model exists when F is exponential .*

Proof From (3.22), under proportional hazard model, we have

$$H(f,g;t) = (\beta - \log\beta - 1) + H(f;t)\ . \tag{3.23}$$

Since the maxima of $H(f;t)$ exists, when $f(x) = \theta\exp(-\theta x),\ \theta > 0$ and $\max H(f;t) = 1 - \log\theta$, refer to [36], thus from (3.23) the maxima of $H(f,g;t)$ under proportional hazard model also exists only when $f(x) = \theta\exp(-\theta x)$, and it is given by

$$\max.H(f,g;t) = (\beta - \log\beta - 1) + (1 - \log\theta)$$

$$= \beta - \log\beta - \log\theta\ .$$

IV *If $\bar{F}(.)$ and $\bar{G}(.)$ satisfy the proportional hazard model with proportionality constant β and $\bar{F}(.)$ is decreasing mean residual life (DMRL), then it is decreasing inaccuracy in residual life (DIRL).*

Proof From (3.14), we have

$$H'(f,g;t) = \lambda_F(t)[-\beta + \log\beta + \log\lambda_F(t) + H(f,g;t)]\ .$$

Using (3.21), this gives

$$\begin{aligned} H'(f,g;t) &\leq \lambda_F(t)[\log\lambda_F(t) + \log\delta_F(t)] \\ &\leq \lambda_F(t)\log[\lambda_F(t)\delta_F(t)] \\ &\leq \lambda_F(t)\log[1 + \delta'_F(t)] \\ &\leq 0, \end{aligned}$$

for all $t \geq 0$. The last inequality comes from the assumption that $\delta_F(t)$ is decreasing. This proves the result.

3.4 Past Inaccuracy Measure

In many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. For instance if at time t , a system which is observed only at certain preassigned inspection times, is found to be down, then the uncertainty of the system's life relies on the past, that is, at which instant in (0, t) the system has failed. Based on this idea, Di Crescenzo and Longobardi [29, 30] have studied measures of entropy and discrimination based on the past entropy over $(0,\ t)$ given respectively as

$$H^*(f;t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx \ , \tag{3.24}$$

and

$$H^*(f/g;t) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)/F(t)}{g(x)/G(t)} dx \ . \tag{3.25}$$

In sequel to these measures of entropy and discrimination based on the past entropy over $(0,\ t)$, we propose

$$H^*(f,g;t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx \ , \tag{3.26}$$

as *a dynamic measure of past inaccuracy over the interval* $(0,\ t)$.

Here we observe that

$$H^*(f;t) + H^*(f/g;t) = H^*(f,g;t) \tag{3.27}$$

in confirmation with the result

$$H(f) + H(f/g) = H(f,g) \ ,$$

in the literature, refer to Kerridge [67]. Here $H(f),\ H(f/g)$ and $H(f,g)$ are given respectively by (3.1), (3.2) and (3.7).

In case we have a system with the baseline distribution function $F(.)$ and the reference distribution function $G(.)$, then the measure $H^*(f,g;t)$ can be interpreted as a measure of inaccuracy associated with the probability density functions f_t and g_t, where $f_t = \frac{f(x)}{F(t)}$ and $g_t = \frac{g(x)}{G(t)}$.

We observe that the measure of past inaccuracy defined by (3.26) can be considered analogous to the measure of residual inaccuracy defined by (3.6). When $t \to \infty$, then (3.26) reduces to (3.7), the Kerridge measure of inaccuracy [67], and further, when $g(x) = f(x)$, then (3.26) becomes (3.24), the dynamic measure of past entropy given by Di Crescenzo and Longobardi [29].

Next, consider the past dynamic inaccuracy measure (3.26) when the random variables satisfy the assumption of *proportional reversed hazard model* (PRHM). We recall that if X is a non- negative random variable with distribution function $F(.)$, denoting the lifetime of a component, then the *reversed hazard rate* of X, denoted by $\mu_X(x)$, is given by

$$\mu_X(x) = \frac{d}{dx} \log F(x) = \frac{f(x)}{F(x)} \ ,$$

where f is the probability density function (p.d.f.) of X.

Here $\mu_X(x)dx$ provides the probability of failing a component in the interval $(x - dx,\ x)$, when it has been found in failed state at time x. For example, if lifetime X of a component is uniformly distributed in the interval $[a,\ b]$, then the reversed hazard rate is, $\mu_X(x) = \frac{f(x)}{F(x)} = \frac{1}{x-a}$.

Next, the two random variables X and Y satisfy the *proportional reversed hazard model (PRHM)* with proportionality constant β (> 0), if

$$\mu_Y(x) = \beta\ \mu_X(x)\ ,\ \beta > 0, \tag{3.28}$$

which is equivalent to

$$G(x) = [F(x)]^{\beta}\ , \beta > 0\ , \tag{3.29}$$

where $F(x)$ is the baseline distribution function and $G(x)$ can be considered as some reference distribution function. This model was proposed by Gupta et al. [50] in contrast to the proportional hazard model (PHM) given by Cox [24] and Efron [42].

As an example, for some positive integral value of β, if $X_1, X_2, ...X_\beta$ are independent and identically distributed (i.i.d.) random variables each with distribution function $F(x)$ representing the lifetimes of components in a β-components parallel system, then the lifetime of the system is given by $Y = \max(X_1, X_2, ...X_\beta)$ with distribution function $G(x)$ given by (3.29).

Consider

$$H^*(f,g;t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx$$

$$= \log G(t) - \int_0^t \frac{f(x)}{F(t)} \log g(x) dx \tag{3.30}$$

$$= \log G(t) - \int_0^t \frac{f(x)}{F(t)} \log \mu_G(x) dx - \frac{1}{F(t)} \int_0^t f(x) \log G(x) dx. \tag{3.31}$$

Using (3.28) and (3.29) in (3.31) we obtain

$$H^*(f,g;t) = \beta - \log \beta - \int_0^t \frac{f(x)}{F(t)} \log \mu_F(x) dx \ . \tag{3.32}$$

When β =1, that is, $G(x) = F(x)$, then (3.32) becomes the past entropy given by Di Crescenzo and Longobardi [29].

Remark 3.1 We observe that the three inaccuracy measures viz. $H(f;g)$, $H(f,g;t)$ and $H^*(f,g;t)$ considered above, satisfy the relation

$$H(f;g) = \overline{F}(t)H(f,g;t) + F(t)H^*(f,g;t) + H\left[F(t), G(t)\right], \tag{3.33}$$

where

$$H\left[F(t), G(t)\right] = -F(t)\log G(t) - [1 - F(t)]\log[1 - G(t)],$$

corresponds to the Kerridge inaccuracy [67].

When $g = f$, then (3.33) reduces to

$$H(f) = H\left[F(t), \overline{F}(t)\right] + F(t)H^*(t) + \overline{F}(t)H(t),$$

a result obtained by Di Crescenzo and Longobardi [29], where $H(p, 1-p) = -p \log p - (1-p)\log(1-p)$ is the entropy of a Bernoulli random variable.

3.5 Characterization Based on Past Inaccuracy Measure

The characterization of specific distributions using relations between reliability measures has become of increasing interest. Several characterizations of probability models have been obtained based on the failure rate or mean residual life(MRL) functions. Asadi and Ebrahimi [8] have studied the characterization based on Shannon residual entropy. Characterizations based on aging measures and dynamic information measures have also been given by Belzunce et al. [16], Ruiz and Navarro [103] and Nanda et al. [84, 85]. In the preceding chapter in Section 2.3 we have characterized some specific lifetime distributions based on the non-additive dynamic entropy measure (2.10). In this section we characterize uniform distribution in term of the past inaccuracy measure (3.26) under the assumption that the two random variables X and Y satisfy the proportional reversed hazard model (3.28). We give the following theorem.

Theorem 3.2 *If two random variables X and Y satisfy the proportional reversed hazard model (PRHM) with proportionality constant β (> 0), then random variable X over $(a,\ b)$, $\ a < b$, has uniform distribution if, and only if*

$$H^*(f, g; t) = \beta - \log \beta - 1 + \log(t-a),\ a < t < b. \tag{3.34}$$

Proof The 'only if'; part of the theorem is straight forward since in case of uniform distribution of X over $(a,\ b)$

$$F(x) = \frac{x-a}{b-a} \text{ and } f(x) = \frac{1}{b-a}\ .$$

Hence, under PRHM, $G(x) = \left[\frac{x-a}{b-a}\right]^{\beta}$. This gives $g(x) = \frac{\beta(x-a)^{\beta-1}}{(b-a)^{\beta}}$. Substituting these in (3.30) and simplifying, we obtain

$$H^*(f, g; t) = \beta - \log\beta - 1 + \log(t-a) \ .$$

To prove the 'if part' let (3.34) be valid. Differentiating (3.30) w.r.t. t and using $\mu_G(x) = \beta\mu_F(x)$, we obtain

$$\frac{d}{dt}H^*(f, g, t) = \mu_G(t) - \mu_F(t)\log g(t) + \mu_F(t)\int_0^t \frac{f(x)}{F(t)}\log g(x)dx \qquad (3.35)$$

$$= \mu_F(t)[\beta - \log g(t) - \int_0^t \frac{f(x)}{F(t)}\log g(x)dx]$$

$$= \mu_F(t)[\beta - \log g(t) + \log G(t) + \int_0^t \frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}dx]$$

$$= \mu_F(t)[\beta - \log\mu_G(t) - H^*(f, g; t)]$$

$$= \mu_F(t)[\beta - \log\beta - \log\mu_F(t) - H^*(f, g; t)] \ . \qquad (3.36)$$

This gives

$$\frac{d}{dt}H^*(f, g; t) - \mu_F(t)[\beta - \log\beta - \log\mu_F(t) - H^*(f, g; t)] = 0.$$

Hence for a fixed $t > 0$, $\mu_F(t)$ is a solution of $g_1(x) = 0$, where

$$g_1(x) = \frac{d}{dt}H^*(f, g; t) - x[\beta - \log\beta - \log x - H^*(f, g; t)]. \qquad (3.37)$$

Differentiating (3.37) with respect to x , we obtain

$$g_1^{'}(x) = [1 - \beta + \log\beta + \log x + H^*(f, g; t)] \ ,$$

and $g_1^{'}(x) = 0$ gives

$$x = \exp[\beta - 1 - \log\beta - H^*(f, g; t)] = x_0, (say).$$

Then from (3.37), we have

$$g_1(0) = \frac{d}{dt} H^*(f, g; t) > 0.$$

Also we can show that $g_1(x)$ is a convex function with minima at $x = x_0$. So $g_1(x) = 0$ has a unique solution and if $g_1(x_0) = 0$, then we have

$$x_0 = \exp[\beta - 1 - \log \beta - H^*(f, g; t)] \ .$$

Using (3.34), we get $x_0 = \frac{1}{t-a}$, $\ t > a$ and

$$g_1(x_0) = \frac{d}{dt} H^*(f, g; t) - x_0[\beta - \log \beta - \log x_0 - H^*(f, g; t)] = 0 \ .$$

Thus $g_1(x) = 0$ has a unique solution given by $x = x_0$. But $\mu_F(t)$ is a solution to (3.37). Hence $\mu_F(t) = x_0 = (t - a)^{-1}$, $t > a$ is the unique solution to $g_1(x) = 0$. Thus the distribution is uniform, and this proves the result.

Example 3.1 Consider an $n-$components parallel system with components having independent and identically distributed (i.i.d) lifetimes $X_i's$, $i = 1, 2, ...n$, where $X_i's$ are exponentially distributed random variables with the same parameter θ, and let $Y = max\{X_1, X_2, ...X_n\}$ be the lifetime of the system. Further, let $f(x)$ and $F(x)$ be respectively the p.d.f. and c.d.f. of X_i. If G is the distribution function for Y, then under PRHM, the c.d.f. of Y is $G(x) = [F(x)]^n$ and its p.d.f. is $g(x) = n[F(x)]^{n-1} f(x)$.

Here

$$\begin{aligned} f(x) &= \theta \ e^{-\theta x}, \\ F(x) &= 1 - e^{(-\theta x)}, \\ G(x) &= \left(1 - e^{(-\theta x)}\right)^n, \\ \text{and,} \qquad g(x) &= n\theta \ e^{-\theta x}[1 - e^{(-\theta x)}]^{n-1}. \end{aligned}$$

Also,

$$H^*(f, g; t) = \log G(t) - \int_0^t \frac{f(x)}{F(t)} \log g(x) dx \ .$$

Substituting for G, F , f and g , this gives

$$H^*(f,g;t) = n - \log n\theta + \log(1-e^{-\theta t}) - \frac{\theta t e^{-\theta t}}{1-e^{-\theta t}} \ . \tag{3.38}$$

Taking limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty} H^*(f,g;t) = n - \log n\theta \ , \tag{3.39}$$

a result in confirmation with the inaccuracy measure $H(f,g)$ under PRHM for

$$f(x) = \theta e^{-\theta x} \ .$$

The graph of $H^*(f,g;t)$ versus t for $t \in [0\ ,2]$ is shown below in Fig. 3.1. It suggests that when n, the number of components increases in a parallel system then the past inaccuracy measure $H^*(f,g;t)$ also increases. Otherwise, for fix n, $H^*(f,g;t)$ is an increasing function of t.

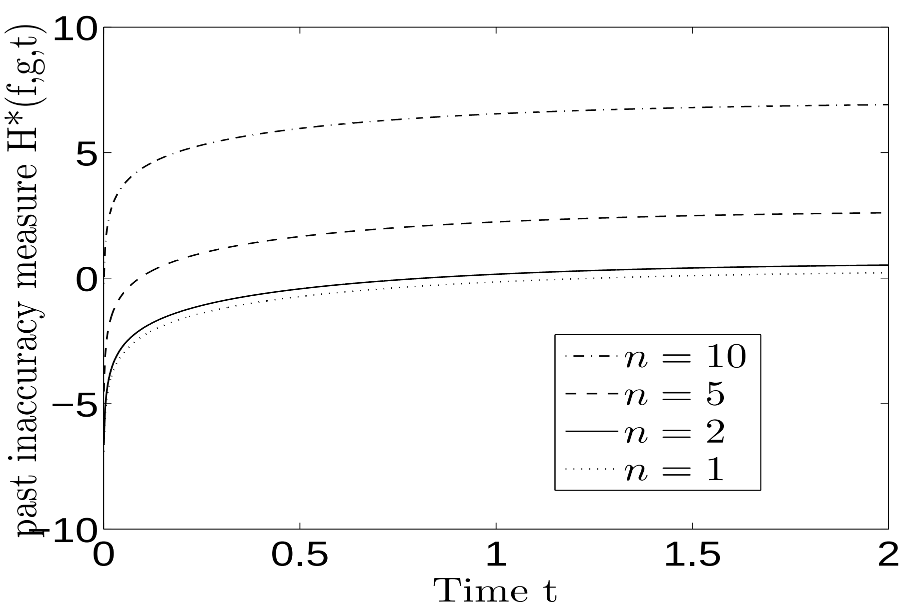

Fig. 3.1: Plot of $H^*(f,g;t)$ versus t for different values of n.

Next, we consider another example where $F(x)$ and $G(x)$ does not satisfy proportional reversed hazard model.

Example 3.2 Let X and Y be two nonnegative random variables having distribution functions respectively

$$F(x) = \begin{cases} \frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\ \frac{x^2+2}{6}, & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

and

$$G(x) = \begin{cases} \frac{x^2+x}{4}, & \text{for } 0 \leq x < 1 \\ \frac{x}{2}, & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2. \end{cases}$$

The past inaccuracy measure (3.26) is given by

$$H^*(f,g;t) = \begin{cases} \frac{1}{2} - \frac{1}{2t} + \frac{1}{4t^2}\log(2t+1) + \log\frac{t^2+t}{2t+1}, & \text{for } 0 < t < 1 \\ \log\frac{t}{2} + \left(\frac{t^2-1}{t^2+2}\right)\log 2 + \frac{6}{t^2+2}\log 2 - \frac{9}{4(t^2+2)}\log 3, & \text{for } 1 \leq t < 2 \\ \frac{3}{2}\log 2 - \frac{3}{8}\log 3, & \text{for } t \geq 2. \end{cases}$$

The graph of the past inaccuracy measure for $t \in [0,\ 1)$, is shown in Fig. 3.2 on the next page.

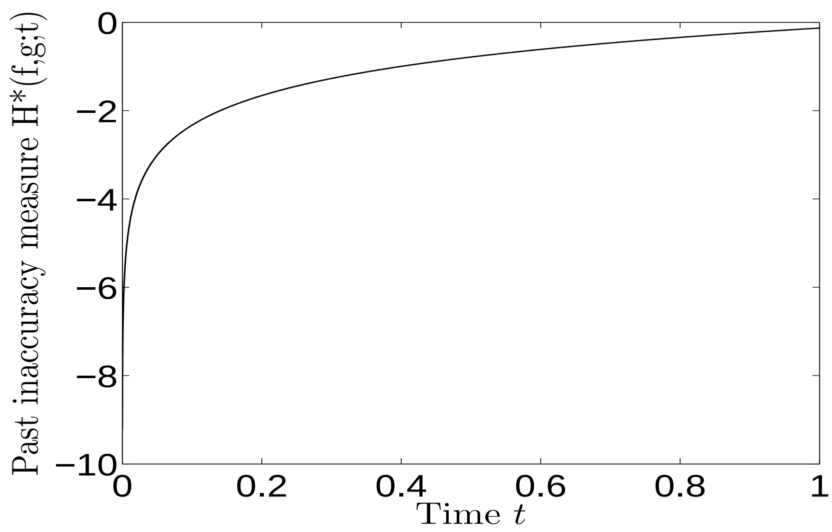

Fig. 3.2: Plot of $H^*(f,g;t)$ against $t \in [0\ ,1]$.

3.5.1 An Upper Bound to $H^*(f,g;t)$

To find an upper bound to $H^*(f,g;t)$, we prove the following result.

Theorem 3.3 *If $\bar{F}(.)$ and $\bar{G}(.)$ satisfying the proportional reversed hazard model (3.28) and $\mu_F(t)$ is decreasing in t, then*

$$H^*(f,g,t) \leq \beta - \log\beta - \log\mu_F(t)\ , \tag{3.40}$$

where $\mu_F(t)$ is the reversed failure rate function.

Proof The measure of past inaccuracy is

$$H^*(f,g;t) = -\int_0^t \frac{f(x)}{F(t)} \log\frac{g(x)}{G(t)} dx\ .$$

Using the proportional reversed hazard model (3.28), this gives

$$\begin{aligned} H^*(f,g;t) &= \beta - \log\beta - \int_0^t \frac{f(x)}{F(t)} \log\mu_F(x) dx \\ &= (\beta - \log\beta - 1) + 1 - \int_0^t \frac{f(x)}{\bar{F}(t)} \log\mu_F(x) dx. \end{aligned} \tag{3.41}$$

Also, in case of measure of past entropy

$$H^*(f;t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx \leq 1 - \log \mu_F(t), \tag{3.42}$$

refer to Di Crescenzo and Longobardi [29]. Using this in (3.41), we get (3.40).

3.6 Conclusion

The concept of inaccuracy given by Kerridge [67], measures the inaccuracy in the statement when the true distribution is not the same as the actual one. For a system which has survived up to time t, for the residual time $[X \mid X \geq t]$, the residual inaccuracy measure is $H(f,g;t)$. It characterizes the base line distribution $F(.)$ uniquely when $F(.)$ and $G(.)$ satisfy the proportional hazard model. For the past time $[X \mid X \leq t]$ distribution, the past inaccuracy measure is given by $H^*(f,g;t)$. It characterizes a specific distribution (uniform) under proportional reversed hazard model. So far we have carried over the study when the process is observed by assigning equal weights to all the observations. In the next chapter we will study this concept for the weighted distributions.

Chapter 4

Length Biased Dynamic Inaccuracy Measures

4.1 Introduction

The concept of weighted distribution introduced by Rao [96] is widely used in statistics and other applications. Jain et al. [61], Gupta and Kirmani [48] and Nanda and Jain [83] have used the weighted distribution in many practical problems to model unequal sampling probabilities. Such distributions arise when the observations generated from a stochastic process are recorded with some weight function . Let X be a non-negative continuous random variable with probability density function $f(x)$, and let X^w be a weighted random variable corresponding to X with weight function $w(x)$ which is positive for all value of $x \geq 0$. Then the corresponding p.d.f. $f^w(x)$ of the random variable X^w is given by

$$f^w(x) = \frac{w(x)f(x)}{E[w(X)]}, \quad 0 \leq x < \infty \tag{4.1}$$

with $0 < E[w(X)] < \infty$.

When $w(x) = x$, X^w is said to be a *length biased (or a size biased) random variable* and the p.d.f. (4.1) in this case becomes

$$f^L(x) = \frac{xf(x)}{E[X]} \ . \tag{4.2}$$

The length biased distribution function and the length biased survival function are defined respectively as

$$F^L(t) = \frac{1}{E[X]} \int_0^t xf(x)dx, \ \ and \ \ \bar{F}^L(t) = \frac{1}{E[X]} \int_t^\infty xf(x)dx \tag{4.3}$$

respectively. These functions characterize weighted distributions that arise in sampling procedures where the sampling probabilities are proportional to the sample values, refer to Patil et al. [95], Furman and Zitikis [45].

In literature Belis and Guiasu [15] raised the important issue of integrating the quantitative concept of information with the qualitative concept, called utility and characterized weighted information measure, called the quantitative-qualitative measure of information, refer to (1.13). Information theoretic measures of weighted relative information and of weighted inaccuracy have been given by Taneja and Tuteja [118] and Taneja [120] respectively. However in these studies the weights attached to the outcomes of a random variables were independent of their probabilities of occurrence.

In the preceding chapter we have proposed the dynamic (both, residual and past) inaccuracy measures. In the present chapter we extend the concept of dynamic inaccuracy measure to the length biased dynamic inaccuracy measures and study the characterization results pertaining to the measures proposed. The chapter is organized as follows. In Section 4.2, we propose a measure of length biased residual inaccuracy and express it in terms of residual inaccuracy measure studied in Chapter 3. Section 4.3 considers a characterization result that under proportional hazard model the measure proposed characterizes the distribution function uniquely; and also we have derived an upper bound to it. In Section 4.4 we consider a length biased measure of past inaccuracy and in Section 4.5 we prove a characterization result for this measure under proportional reversed hazard model. Some further results concerning the length biased past inaccuracy measure have been considered in Section 4.6. The chapter ends with the conclusion.

4.2 Length Biased Residual Inaccuracy Measure

We have observed in Chapter 3 that if a system has survived up to time t, the corresponding dynamic measures of uncertainty, refer to Ebrahimi [34], of discrimination, refer to Ebrahimi and Kirmani [38], and of inaccuracy, refer to Taneja et al. [117], are given as

$$H(f;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \ , \tag{4.4}$$

$$H(f/g;t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx \ , \tag{4.5}$$

and

$$H(f,g;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx, \tag{4.6}$$

respectively.

When $t = 0$, then (4.4), (4.5) and (4.6) reduce to measures of Shannon entropy [109], Kullback discrimination [70] and Kerridge inaccuracy [67] respectively. These information measures do not take into account the weightage of the random variable but only its probability density function.

Di Crescenzo and Longobardi [31] considered a length-biased shift dependent information measure related to the differential entropy in which higher weights are assigned to the larger values of the observed random variables. The residual measure of entropy (4.4) has been extended to the *length biased weighted residual entropy* given as

$$H^L(f,t) = -\int_t^\infty x\frac{f(x)}{\bar{F}(t)}\log\frac{f(x)}{\bar{F}(t)}dx\ . \tag{4.7}$$

The factor x in the integral on right-hand-side yields a "length-biased" shift dependent information measure assigning greater importance to the larger values of the random variable X.

In agreement with Taneja and Tuteja [118], we refer to the measure

$$H^L(f,g) = -\int_0^\infty xf(x)\log g(x)dx\ , \tag{4.8}$$

as the *length biased weighted inaccuracy*, and propose the measure

$$H^L(f,g;t) = -\int_t^\infty x\frac{f(x)}{\bar{F}(t)}\log\frac{g(x)}{\bar{G}(t)}dx\ , \tag{4.9}$$

as the *length biased residual inaccuracy measure.*

When $g(x) = f(x)$, the measure (4.9) reduces to (4.7), the length biased weighted residual entropy given by Di Crescenzo and Longobardi [31]. In case the weights are independent of x then (4.9) reduces to (4.4), the dynamic measure of uncertainty proposed by Ebrahimi [34], and also when $t = 0$, the measure (4.9) reduces to the measure (4.8), the length biased weighted inaccuracy.

4.2.1 Weighted Residual Inaccuracy in Terms of Residual Inaccuracy

Rewriting $H^L(f,g;t)$ as

$$H^L(f,g;t) = -\int_t^\infty dx\int_0^x \frac{f(x)}{\bar{F}(t)}\log\frac{g(x)}{\bar{G}(t)}dy$$

$$= -\int_0^t dy \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx - \int_t^\infty dy \int_y^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx. \tag{4.10}$$

Using (4.6) in (4.10), we obtain

$$H^L(f,g;t) = tH(f,g;t) - \left(\int_t^\infty dy\right)\left(\int_y^\infty \alpha(x;t)dx\right),$$

where $\alpha(x;t) = \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)}$.

Writing $\int_y^\infty \alpha(x;t)dx$ as $\beta(y;t)$, this can be written as

$$H^L(f,g;t) = tH(f,g;t) - \left(\int_t^\infty \beta(y;t)dy\right). \tag{4.11}$$

Differentiating (4.11) w.r.t. t both sides using Leibnitz rule for differentiation under integration, we obtain

$$\frac{d}{dt}H^L(f,g;t) = t\frac{d}{dt}H(f,g;t) + H(f,g;t) + \beta(t;t)$$

$$= t\frac{d}{dt}H(f,g;t) + H(f,g;t) + \int_t^\infty \alpha(x;t)dx.$$

Thus

$$\frac{d}{dt}H^L(f,g;t) = t\frac{d}{dt}H(f,g;t), \tag{4.12}$$

a relation giving the rate of change of weighted residual inaccuracy (4.9) in terms of rate of change of residual inaccuracy (4.6) . This relation is used in the characterization problem considered in the next section.

4.3 Characterization Problem

The general characterization problem is to determine whether the residual measure characterizes the distribution function uniquely. In this section, we study characterization problem for the weighted residual inaccuracy measure (4.9) under the proportional hazard model (PHM) as already introduced in Chapter 3. Under this model, refer to Cox [24] and Efron [42], the survival functions of two random lifetime variables are related by

$$\bar{G}(x) = [\bar{F}(x)]^{\beta},\ \beta > 0\ , \tag{4.13}$$

where β is the proportionality constant. We note that based on the proportional hazard model (4.13), the hazard rate functions $\lambda_F(.)$ and $\lambda_G(.)$ satisfy the relation $\lambda_G(x) = \beta\lambda_F(x)$. We consider the following characterization theorem.

Theorem 4.1 *If the two random variables X and Y satisfy the proportional hazard model (4.13) with proportionality constant β (> 0), and if $H^L(f, g; t)$ is increasing in t with $H^L(f, g; t) < \infty$, then $H^L(f, g; t)$ uniquely determines $\bar{F}(.)$, the survival function of X.*

Proof Consider the weighted residual inaccuracy

$$H^L(f, g; t) = -\int_t^{\infty} x\frac{f(x)}{\bar{F}(t)} \log\frac{g(x)}{\bar{G}(t)} dx\ .$$

Rewriting this as

$$\int_t^{\infty} xf(x)\log g(x)dx = \log\bar{G}(t)\left[\int_t^{\infty} xf(x)dx\right] - \bar{F}(t)H^L(f, g; t)\ ,$$

or,

$$\int_t^{\infty} xf(x)\log g(x)dx = \log\bar{G}(t)\left[t\bar{F}(t) + \int_t^{\infty} \bar{F}(y)dy\right] - \bar{F}(t)H^L(f, g; t)\ .$$

Differentiating this both sides w.r.t. t, and then using (4.12) and substituting $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$, and $\lambda_G(t) = \frac{g(t)}{\bar{G}(t)}$, we obtain

$$-t\lambda_F(t)\log\lambda_G(t) + t\lambda_G(t) = -\lambda_G(t)\left[\frac{\int_t^{\infty}\bar{F}(y)dy}{\bar{F}(t)}\right] + \lambda_F(t)H^L(f, g; t) - t\frac{d}{dt}H(f, g; t). \tag{4.14}$$

Now under proportional hazard model $\lambda_G(t) = \beta\lambda_F(t)$. Using this, Eq. (4.14) becomes

$$-t\lambda_F(t)\log\beta\lambda_F(t)+\beta t\lambda_F(t) = -\beta\lambda_F(t)\frac{\int_t^\infty \bar{F}(y)dy}{\bar{F}(t)}+\lambda_F(t)H^L(f,g;t)-t\frac{d}{dt}H(f,g;t).$$

Thus for any fixed t, $\lambda_F(t)$ is a positive solution of the equation $h(x) = 0$, where

$$h(x) = x\left\{\beta t - t\log\beta x + \frac{\beta\int_t^\infty \bar{F}(y)dy}{\bar{F}(t)} - H^L(f,g;t)\right\} + t\frac{d}{dt}H(f,g;t). \quad (4.15)$$

Here $h(0) = t\frac{d}{dt}H(f,g;t) \geq 0$, since we have assumed that $H(f,g;t)$ is increasing in t, and also as $x \to \infty$, $h(x) \to -\infty$. Further differentiating (4.15) with respect to x, we get

$$\frac{d}{dx}h(x) = \beta t - t\log\beta x + \frac{\beta\int_t^\infty \bar{F}(y)dy}{\bar{F}(t)} - H^L(f,g;t) - t.$$

Now, $\frac{d}{dx}h(x) = 0$ if, and only if,

$$x = \frac{1}{\beta}\exp\left[-\frac{1}{t}\left\{t - \beta t - \frac{\beta\int_t^\infty \bar{F}(y)dy}{\bar{F}(t)} + H^L(f,g;t)\right\}\right] = x_0, say.$$

In view of the above, $h(x) = 0$ has a unique positive solution. Thus $\lambda_F(t)$, and hence $\bar{F}(t)$ is uniquely determined by the weighted residual inaccuracy measure $H^L(f,g;t)$ under the assumption that $\frac{d}{dt}H(f,g;t) \geq 0$. This concludes the proof.

4.3.1 A Lower Bound to $H^L(f,g;t)$

To derive a lower bound for the weighted residual inaccuracy measure (4.9), we consider the following conditional mean value of a random variable X as

$$\delta_t = E(X \mid X > t) = \frac{1}{\bar{F}(t)}\int_t^\infty xf(x)dx\ , \quad (4.16)$$

a result which finds applications in insurance and economics, refer to Furman and Zitikis [45]. We have the following result:

Theorem 4.2 *If the hazard rate function $\lambda_G(t)$ is decreasing in t, then*

$$H^L(f,g;t) \geq -\delta_t \log \lambda_G(t)\ . \tag{4.17}$$

Proof From (4.9), we have

$$H^L(f,g;t) = -\frac{1}{\bar{F}(t)}\int_t^{\infty} xf(x)\log\lambda_G(x)dx - \frac{1}{\bar{F}(t)}\int_t^{\infty} xf(x)\log\frac{\bar{G}(x)}{\bar{G}(t)}dx.$$

Since $\log\frac{\overline{G}(x)}{\overline{G}(t)} \leq 0$, for $x \geq t$, and by assumption that hazard rate is decreasing in t, we have $\log\lambda_G(x) \leq \log\lambda_G(t)$, thus

$$H^L(f,g;t) \geq -\frac{1}{\bar{F}(t)}\int_t^{\infty} xf(x)\log\lambda_G(x)dx$$

$$\geq -\frac{\log\lambda_G(t)}{\bar{F}(t)}\int_t^{\infty} xf(x)dx,$$

which gives

$$H^L(f,g;t) \geq -\delta_t \log\lambda_G(t).$$

Example 4.1 If the true distribution function $F(x)$ and the reference distribution function $G(x)$ are exponentially distributed with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively, then

$$f(x) = \lambda_1\, e^{-\lambda_1 x},\ g(x) = \lambda_2\, e^{-\lambda_2 x},$$

$$\overline{F}(x) = 1 - F(x) = e^{-\lambda_1 x},$$

$$\text{and,}\qquad \overline{G}(x) = 1 - G(x) = e^{-\lambda_2 x}.$$

Substituting for $\overline{G}$, $\overline{F}$, f, and g in (4.9) , we obtain the length biased weighted residual inaccuracy measure as

$$H^L(f,g;t) = -\int_t^\infty x\frac{\lambda_1\, e^{-\lambda_1 x}}{e^{-\lambda_1 t}}\log\frac{\lambda_2\, e^{-\lambda_2 x}}{e^{-\lambda_2 t}}dx$$

$$= \frac{\lambda_2}{\lambda_1}\left(t+\frac{2}{\lambda_1}\right) - \left(t+\frac{1}{\lambda_1}\right)\log\lambda_2.$$

Further, we note that hazard rate is constant for an exponential distribution, that is, $\lambda(t) = \lambda$, and the conditional mean value is $\delta_t = t + \frac{1}{\lambda}$. Thus (4.17) holds .

4.4 Length Biased Past Inaccuracy Measure

In many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. For instance if at time t, a system which is observed only at certain preassigned inspection times is found to be down, then the uncertainty of the system's life relies on the past, that is, at which instant in the interval (0, t) the system has failed.

Based on this idea, measures of past entropy [29], discrimination [30] and of inaccuracy [72] over $(0,\ t)$ are given respectively as

$$H^*(f;t) = -\int_0^t \frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx\ , \tag{4.18}$$

$$H^*(f/g;t) = \int_0^t \frac{f(x)}{F(t)}\log\frac{f(x)/F(t)}{g(x)/G(t)}dx\ , \tag{4.19}$$

and

$$H^*(f,g;t) = -\int_0^t \frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}dx\ . \tag{4.20}$$

Further, the concept of past entropy given by (4.18) has been extended to the length biased past entropy given as

$$H^{*L}(f,t) = -\int_0^t x\frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx, \tag{4.21}$$

refer to Di Crescenzo and Longobardi [31].

In sequel to this, we extend the past inaccuracy measure (4.20) to the length biased past inaccuracy given by

$$H^{*L}(f,g;t) = -\int_0^t x\frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}dx\ . \tag{4.22}$$

This may be considered as the differential weighted inaccuracy of the random variable $[X \mid X \leq t]$. When $g(x) = f(x)$, then (4.22) is the weighted past entropy (4.21), and when the weights are independent of x, then (4.22) reduces to the past inaccuracy measure studied in Chapter 3.

4.4.1 Weighted Past Inaccuracy Measure in Term of Past Inaccuracy

Rewriting $H^{*L}(f,g;t)$ as

$$H^{*L}(f,g;t) = -\int_0^t dx\int_0^x \frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}dy$$

$$= t\int_0^t \frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}dx - \int_0^t dy\int_0^y \frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}dx. \tag{4.23}$$

Using (4.20) in (4.23) , we obtain

$$H^{*L}(f,g;t) = tH^{*}(f,g;t) - \int_0^t H^{*}(f,g;y)dy\ . \tag{4.24}$$

Differentiating (4.24) w.r.t. t both sides, we obtain

$$\frac{d}{dt}H^{*L}(f,g;t) = t\frac{d}{dt}H^{*}(f,g;t)\ , \tag{4.25}$$

a result analogous to the result (4.12) studied in context with weighted residual inaccuracy. We shall use this result in the characterization problem studied next in Section 4.5.

When $g(x) = f(x)$, then (4.25) reduces to

$$\frac{d}{dt}H^{*L}(f;t) = t\frac{d}{dt}H^{*}(f;t), \tag{4.26}$$

a result given by Di Crescenzo and Longobardi [31].

4.5 Characterization Problem

The general characterization problem is to determine when the dynamic information-theoretic measure determines the distribution function uniquely. In this section, we study the characterization problem for the weighted past inaccuracy measure (4.22) under the proportional reversed hazard model as already stated in Chapter 3 and restated as follows.

Two random variables X and Y satisfy the proportional reversed hazard model (PRHM) with proportionality constant β (> 0), if

$$\mu_G(x) = \beta\ \mu_F(x)\ ,\quad \beta > 0, \tag{4.27}$$

where $\mu_F(x) = \frac{f(x)}{F(x)}$.

We know that the PRHM is equivalent to the model

$$G(x) = [F(x)]^{\beta}\ , \tag{4.28}$$

where $F(x)$ can be considered as the baseline distribution function and $G(x)$ as some reference distribution function, refer to Gupta et al. [50].

Next, we consider the following characterization theorem.

Theorem 4.3 *If two random variables X and Y satisfy the proportional reversed hazard model (4.27) with proportionality constant β (>0) and $H^{*L}(f,g;t)$ is decreasing for all $t>0$, then $H^{*L}(f,g;t)$ uniquely determines $\bar{F}(.)$, the survival function of X.*

Proof Consider the weighted past inaccuracy measure (4.22) given by

$$H^{*L}(f,g;t) = -\int_0^t x\frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}dx\ .$$

Rewriting this as

$$\int_0^t xf(x)\log g(x)dx = \log G(t)\left[\int_0^t xf(x)dx\right] - F(t)H^{*L}(f,g;t) \qquad (4.29)$$

or,

$$\int_0^t xf(x)\log g(x)dx = \log G(t)\left[tF(t) - \int_0^t F(y)dy\right] - F(t)H^{*L}(f,g;t)\ .$$

Differentiating both sides w.r.t. t, and using $\mu_F(t) = \frac{f(t)}{F(t)}$ *and* $\mu_G(t) = \frac{g(t)}{G(t)}$, we obtain

$$t\mu_F(t)\log\mu_G(t) - t\mu_G(t) = -\mu_G(t)\left[\frac{\int_0^t F(y)dy}{F(t)}\right] + \mu_F(t)H^{*L}(f,g;t) - \frac{d}{dt}H^{*L}(f,g;t)\ . \qquad (4.30)$$

Under proportional reversed hazard model (4.27), this gives

$$t\mu_F(t)\log\beta\mu_F(t) - \beta t\mu_F(t) = -\beta\mu_F(t)\frac{\int_0^t F(y)dy}{F(t)} - \mu_F(t)H^{*L}(f,g;t) - \frac{d}{dt}H^{*L}(f,g;t)\ .$$

Thus for any fixed t, $\mu_F(t)$ is a positive solution of the equation $h_1(x)=0$, where

$$h_1(x) = \frac{d}{dt}H^{*L}(f,g;t) + x\left\{-\beta t + t\log\beta x + \frac{\beta\int_0^t F(y)dy}{F(t)} + H^{*L}(f,g;t)\right\} \qquad (4.31)$$

Here $h_1(0) = \frac{d}{dt}H^{*L}(f,g;t) \le 0$; since we have assumed that $H^{*L}(f,g;t)$ is decreasing in t, and also, when $x\to\infty,\ h_1(x)\to\infty$.

Differentiating (4.31) with respect to x, we get

$$\frac{d}{dx}h_1(x) = -\beta t + t\log\beta x + \frac{\beta\int_0^t F(y)dy}{F(t)} + \overline{H}^w(f,g;t) + t\ .$$

So that $\frac{d}{dx}h_1(x) = 0$ if, and only if

$$x = \frac{1}{\beta}exp\left[-\frac{1}{t}\left\{t - \beta t + \frac{\beta\int_0^t F(y)dy}{F(t)} + H^{*L}(f,g;t)\right\}\right] = \overline{x}_0,\ \text{say.} \qquad (4.32)$$

Therefore $h_1(x) = 0$ has a unique positive solution. Thus $\mu_F(t)$, and hence $\bar{F}(t)$, is uniquely determined by the weighted past inaccuracy measure $H^{*L}(f,g;t)$. This concludes the proof.

Example 4.2 If a random variable X is uniformly distributed over $(a\ ,b)$, $\ a < b$, then its density and distribution functions are given respectively by

$$f(x) = \frac{1}{b-a} \ \text{ and } \ F(x) = \frac{x-a}{b-a}\ ,\quad a < x < b.$$

Further if X and Y satisfy the PRHM with proportionality constant $\beta > 0$, then distribution function of the variable Y is

$$G(x) = \left[\frac{x-a}{b-a}\right]^\beta,\ \text{ which gives } \ g(x) = \frac{\beta(x-a)^{\beta-1}}{(b-a)^\beta}\ ,\quad a < x < b.$$

Substituting these in (4.22) and simplifying, we obtain the weighted past inaccuracy measure as

$$H^{*L}(f,g;t) = \left(\frac{t+a}{2}\right)\log\left(\frac{t-a}{\beta}\right) + (\beta-1)\left(\frac{t+3a}{4}\right).\ \ a < t < b. \qquad (4.33)$$

For $\beta = 1$, this reduces to

$$H^{*L}(f;t) = \left(\frac{t+a}{2}\right)\log(t-a),$$

a result obtained in case of weighted past entropy, refer to Di Crescenzo and Longobardi [31].

Example 4.3 If true distribution function $F(x)$ and reference distribution function $G(x)$ are exponentially distributed with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively, then

$$f(x) = \lambda_1\, e^{-\lambda_1 x},\;\; \overline{F}(x) = 1 - F(x) = e^{-\lambda_1 x},\; x > 0$$

and,

$$g(x) = \lambda_2\, e^{-\lambda_2 x},\;\; \overline{G}(x) = 1 - G(x) = e^{-\lambda_2 x},\; x > 0. \tag{4.34}$$

Substituting for $\overline{G}$, $\overline{F}$, f and g in (4.22) , we obtain the weighted past inaccuracy measure as

$$H^{*L}(f,g;t) = \frac{1}{[1-e^{-\lambda_1 t}]}\left\{\log\frac{1-e^{-\lambda_2 t}}{\lambda_2}\left[\frac{1}{\lambda_1} - \frac{e^{-\lambda_1 t}}{\lambda_1} - te^{-\lambda_1 t}\right]\right\}$$

$$+\frac{1}{[1-e^{-\lambda_1 t}]}\left\{\frac{2\lambda_2}{\lambda_1^2} - \frac{2\lambda_2 e^{-\lambda_1 t}}{\lambda_1^2} - \frac{2\lambda_2 t e^{-\lambda_1 t}}{\lambda_1} - \lambda_2 t^2 e^{-\lambda_1 t}\right\}. \tag{4.35}$$

In addition to the general case as given by (4.35), the following two particular cases are of specific interest. The case I is of PRHM and case II is of PHM. But we must note that $H^{*L}(f,g;t)$ characterizes the distribution uniquely only under PRHM.

Case I: $\lambda_1 = \lambda_2$, that is, $G(x) = F(x)$. In this case the weighted past inaccuracy $H^{*L}(f,g;t)$ reduces to the weighted past entropy $H^{*L}(f;t)$, given by

$$H^{*L}(f;t) = \frac{1}{[1-e^{-\lambda_1 t}]}\left\{\log\frac{1-e^{-\lambda_1 t}}{\lambda_1}\left[\frac{1}{\lambda_1} - \frac{e^{-\lambda_1 t}}{\lambda_1} - te^{-\lambda_1 t}\right]\right\}$$

$$+\frac{1}{[1-e^{-\lambda_1 t}]}\left\{\frac{2}{\lambda_1} - \frac{2e^{-\lambda_1 t}}{\lambda_1} - 2te^{-\lambda_1 t} - \lambda_1 t^2 e^{-\lambda_1 t}\right\}. \tag{4.36}$$

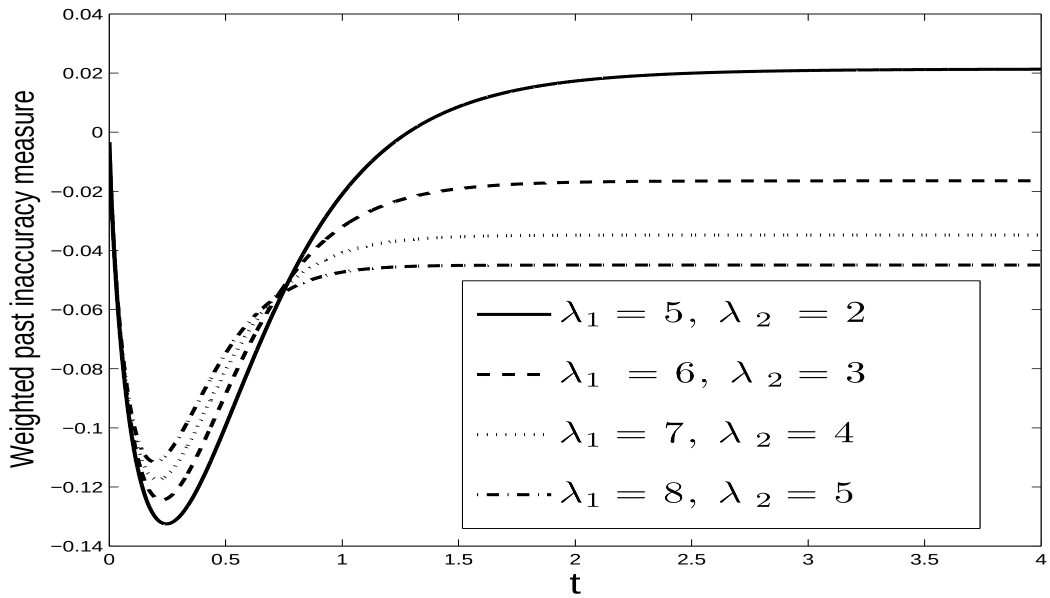

Fig.4.1: Plot of $H^{*L}(f, g; t)$ when $\lambda_1 \neq \lambda_2$ for $t \in [0\ ,4]$.

Case II: $\lambda_2 = n\lambda_1$, that is, $\overline{G}(x) = [\overline{F}(x)]^n$. This corresponds to n-components series system where each component is having independent and identically distributed lifetime X_i, $i = 1, 2, ...n$, with distribution function $F(x)$, and here $G(x)$ is the distribution function of the lifetime $Y = min\{X_1, X_2, ...X_n\}$ of the system. The weighted past inaccuracy (4.35) in this case becomes

$$H^{*L}(f, g; t) = \frac{1}{[1 - e^{-\lambda_1 t}]}\left\{\log\frac{1 - e^{-n\lambda_1 t}}{n\lambda_1}\left[\frac{1}{\lambda_1} - \frac{e^{-\lambda_1 t}}{\lambda_1} - te^{-\lambda_1 t}\right]\right\}$$

$$+ \frac{1}{[1 - e^{-\lambda_1 t}]}\left\{\frac{2n}{\lambda_1} - \frac{2ne^{-\lambda_1 t}}{\lambda_1} - 2nte^{-\lambda_1 t} - n\lambda_1 t^2 e^{-\lambda_1 t}\right\}. \qquad (4.37)$$

The plots in case of the measures (4.35), (4.36) and (4.37) for various values of λ_1 and λ_2 are given respectively in Figs. 4.1, 4.2 and 4.3 .

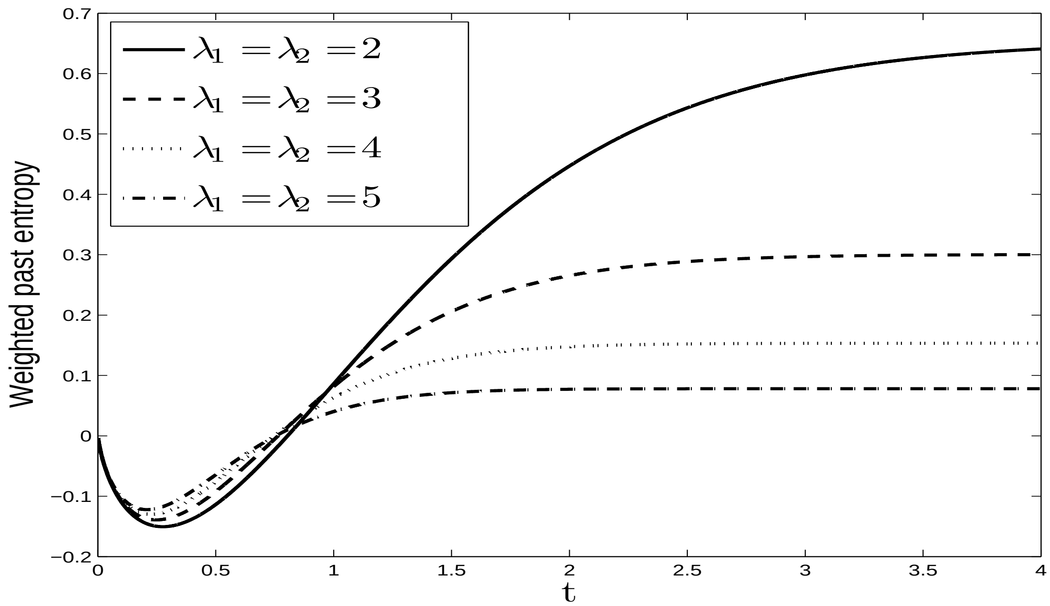

Fig. 4.2: Plot of $H^{*L}(f,g;t)$ when $\lambda_1 = \lambda_2$ for $t \in [0\ ,4]$.

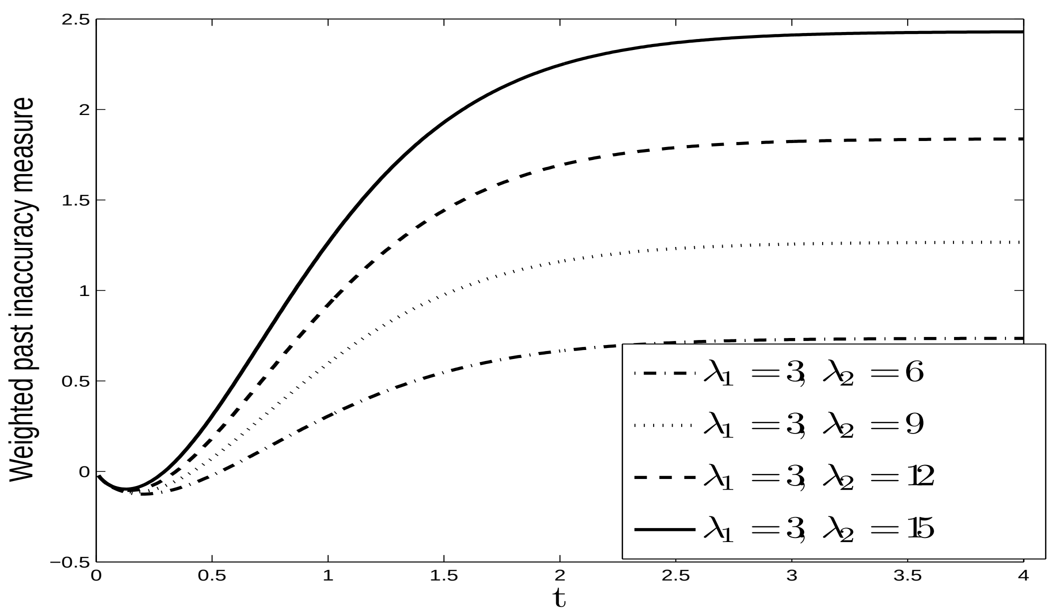

Fig. 4.3: Plot of $H^{*L}(f,g;t)$ when $\lambda_2 = n\lambda_1$ for $t \in [0\ ,4]$,where $\lambda_1 = 3$ fixed and $n = 2, 3, 4, 5$.

4.6 Some Further Results on Past Inaccuracy

4.6.1 An Upper Bound to $H^{*L}(f, g; t)$

To obtain an upper bound to the weighted past inaccuracy measure (4.22), we define the mean past lifetime of the system as

$$\tau(t) = E(X \mid X \leq t) = \int_0^t x \frac{f(x)}{F(t)} dx = t - \frac{1}{F(t)} \int_0^t F(y) dy \ . \qquad (4.38)$$

Next, we consider the following result.

Theorem 4.4 *If $\mu_G(t) = \frac{g(t)}{G(t)}$, the reversed hazard rate is decreasing in t, then*

$$H^{*L}(f, g; t) \leq -\tau_F(t) \left[\log \mu_G(t) + 1\right] + \frac{G(t)}{F(t)} \int_0^t \frac{x f(x)}{G(x)} dx \ . \qquad (4.39)$$

Proof From (4.22), we have

$$H^{*L}(f, g; t) = -\frac{1}{F(t)} \int_0^t x f(x) \log \mu_G(x) dx + \frac{1}{F(t)} \int_0^t x f(x) \log \frac{G(t)}{G(x)} dx \ .$$

Since $\mu_G(t) = \frac{g(t)}{G(t)}$ is decreasing in t, we have $\log \mu_G(x) \geq \log \mu_G(t)$ for $0 < x < t$. Moreover, $\log x \leq x - 1$ for $x > 0$. We obtain

$$H^{*L}(f, g; t) \leq -\frac{\log \mu_G(t)}{F(t)} \int_0^t x f(x) dx + \frac{1}{F(t)} \int_0^t x f(x) [\frac{G(t)}{G(x)} - 1] dx$$

and, after simplification, we get (4.39).

Example 4.4 Let X be a non-negative random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x & ; \quad \text{if } \ 0 \leq x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and let random variable Y be uniformly distributed over $(0,\ a)$ with density and distribution functions given respectively by

$$g_Y(x) = \frac{1}{a} \text{ and } G_Y(x) = \frac{x}{a}\ , \quad 0 < x < a.$$

Substituting these values in (4.22), we obtain the length biased past inaccuracy measure as

$$H^{*L}(f,g;t) = \frac{2t\log t}{3}\ , \quad 0 < t < 1. \tag{4.40}$$

Also the right hand side of (4.39) gives

$$-\tau_F(t)\left[\log \mu_G(t) + 1\right] + \frac{G(t)}{F(t)}\int_0^t \frac{xf(x)}{G(x)}dx = \frac{2t\log t}{3} + \frac{t}{3}\ .$$

Comparing this with (4.40), it is easily seen that (4.39) is fullfilled.

4.6.2 Weighted Inaccuracy in Terms of Residual and Past Inaccuracy

We express weighted inaccuracy measure (4.8) in terms of length biased past inaccuracy measure (4.22) and length biased residual inaccuracy measure (4.9). We prove the following result.

Theorem 4.5 *For a random variable X having finite mean $E(X)$ for all $t > 0$, the weighted inaccuracy measure $H^L(f,g)$ as given by (4.8) can be expressed as*

$$\begin{aligned} H^L(f,g) &= F(t)H^{*L}(f,g;t) + \bar{F}(t)H^L(f,g;t) \\ &\quad - E(X)\left\{F^L(t)\log G(t) + \bar{F}^L(t)\log \bar{G}(t)\right\}. \end{aligned} \tag{4.41}$$

Proof We have

$$H^L(f,g) = -\int_0^\infty xf(x)\log g(x)dx.$$

This can be rewritten as

$$H^L(f,g) = -F(t)\int_0^t x\frac{f(x)}{F(t)}\log g(x)dx - \bar{F}(t)\int_t^\infty x\frac{f(x)}{\bar{F}(t)}\log g(x)dx\ .$$

Using (4.22) and (4.9) , we obtain

$$\begin{aligned} H^L(f,g) &= F(t)H^{*L}(f,g;t) + \bar{F}(t)H^L(f,g;t) \\ &\quad - \log G(t)\left\{\int_0^t xf(x)dx\right\} - \log\bar{G}(t)\left\{\int_t^\infty xf(x)dx\right\}. \end{aligned}$$

Using (4.3), we obtain (4.41), the desired result.

4.7 Conclusion

The concept of weighted distributions and hence that of weighted information measures is of wide interest when a stochastic process is recorded with some weight function. We have seen here that the dynamic inaccuracy measures (both residual and past) studied in Chapter 3 find a natural extension to the corresponding length biased (weighted) residual and past inaccuracy measures. These measures also characterize the underlying distribution uniquely. So far we have concentrated only on p.d.f. based information-theoretic measures which have their own inherent limitations. In the subsequent chapters we consider distribution function based information theoretic measures which overcome the limitations of density based measures.

Chapter 5

Generalized Cumulative Entropy Measures

5.1 Introduction

The measure of differential entropy [109] given by

$$H(f) = -\int_0^\infty f(x) \log f(x) dx \ , \tag{5.1}$$

where $f(.)$ is the p.d.f. of a continuous random variable X, was considered to be the straight forward extension of the entropy measure $H(P)$ given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \ , \ \ 0 \leq p_i \leq 1, \ \ \sum_{i=1}^{n} p_i = 1 \ , \tag{5.2}$$

studied in the discrete case, where $P = (p_1, p_2, \cdots, p_n)$ is the probability distribution of the discrete random variable $X = (X_1, X_2, \cdots, X_n)$. We have already observed that the measure (5.1) and its extension to residual and past entropies, and further to inaccuracy measures have been studied widely.

Rao et al. [98] have pointed out some basic shortcomings of the differential entropy measure, like that it is defined only for distributions with densities; it is inconsistent in the sense that it may also take negative values over the range of some specific random variables and a few more, for details refer to Rao [97].

Considering the fact that the main objective was to extend the Shannon entropy (5.1) to random variables with continuous distribution. Rao et al. [98] proposed an alternative measure of entropy of a random variable X with distribution function F, given by

$$\xi(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx \ , \tag{5.3}$$

where $\overline{F}(x) = 1 - F(x)$ is the survival function. The measure (5.3) is called the *cumulative residual entropy (CRE).*

The CRE measure defined by (5.3) has consistent definition in both the continuous and discrete domains; it is always non-negative and can be easily computed from the sample data. The measure (5.3) is more consistent since it is based on distribution function rather than the density function which is a derivative of the distribution function. The detailed properties of the CRE measures and its relation with Shannon entropy along with its applications in reliability engineering and computer vision have been studied by Rao et al. [98]. Some general results regarding this measure have been studied by Drissi et al. [33] and Navarro et al. [86]. Applications of CRE related measures to image alignment and to measurement of similarity between images can be found in Wang and Vemuri [124].

The CRE measure (5.3) is not applicable to a system which has survived for some unit of time, say t. Asadi and Zohrevand [11] have considered the corresponding dynamic measure, *the dynamic cumulative residual entropy (DCRE),* defined as the cumulative residual entropy of the random variable $X_t = [X - t|X > t]$. It is given by

$$\xi(X;t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} dx \ . \tag{5.4}$$

Considering the importance of the CRE measures, in this chapter we extend the concept of cumulative residual entropy to one parameter and two parameters entropies and also their dynamic versions. The chapter is organized as follows. In Section 5.2 we consider cumulative residual entropy of order α based on Renyi's entropy [101] and propose a dynamic version of it. Some specific lifetime distributions based on the dynamic cumulative entropy measure have been characterized in Section 5.3. In Section 5.4 we consider a generalized cumulative entropy and also generalized dynamic cumulative residual entropy of type α and order β. Section 5.5 deals with the characterization results based on the generalized dynamic cumulative entropy of type α and order β. Section 5.6 includes the conclusion.

5.2 Cumulative Residual Entropy of Order α

Analogous to the definition of cumulative residual entropy (5.3) by Rao et al. [98], we propose the cumulative residual entropy of order α of the random variable X as

$$\xi_\alpha(X) = \frac{1}{1-\alpha} \log \left[\int_0^\infty \bar{F}^\alpha(x) dx \right] , \alpha \neq 1, \ \alpha > 0. \tag{5.5}$$

When $\alpha \to 1$, then (5.5) reduces to (5.3, under the assumption that $E(X) = 1$.

For some specific univariate continuous distributions, the measure (5.5) is evaluated as given below.

(i) If X is uniformly distributed on the interval $[a, b]$, then

$$\xi_\alpha(X) = \frac{1}{1-\alpha} \log \left(\frac{b-a}{\alpha+1} \right) \ . \tag{5.6}$$

(ii) If X is exponentially distributed with parameter θ, then

$$\xi_\alpha(X) = \frac{1}{1-\alpha}\log\left(\frac{1}{\alpha}\right) + \frac{1}{1-\alpha}\log\left(\frac{1}{\theta}\right) .$$

(iii) If X has Pareto distribution with parameters a and b so that

$$\bar{F}(x) = 1 - F(x) = \left(1 + \frac{x}{b}\right)^{-a} = \frac{b^a}{(x+b)^a} ,$$

then

$$\xi_\alpha(X) = \frac{1}{1-\alpha}\log\left(\frac{b}{a\alpha - 1}\right).$$

(iv) If X has folded Cramer distribution with parameter θ, so that the probability density function is

$$f(x) = \frac{\theta}{(1+\theta x)^2} \quad , \quad \theta > 0$$

then its survival function is given by

$$\bar{F}(x) = \frac{1}{1+\theta x} ,$$

and thus

$$\xi_\alpha(X) = \frac{1}{1-\alpha}\log\frac{1}{\theta(\alpha-1)} . \tag{5.7}$$

In life-testing situations, the remaining lifetime given that the component has survived up to time t called the *residual lifetime* of the component, is of specific interest. For computing the uncertainty of the component, it is easy to see that the survival function of the residual lifetime distribution is given by the function

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)} & ; \quad \text{if } \ x > t, \\ 1 & ; \quad \text{otherwise.} \end{cases}$$

Thus dynamic cumulative residual entropy of order α of the random variable $X_t = [X|X > t]$ is defined as

$$\xi_\alpha(X;t) = \frac{1}{1-\alpha}\log\left[\int_t^\infty \bar{F}_t^{\,\alpha}(x)dx\right] , \alpha \neq 1, \ \alpha > 0. \tag{5.8}$$

This can be rewritten as

$$\xi_{\alpha}(X;t) = \frac{1}{1-\alpha} \log \left[\frac{\int_t^{\infty} \bar{F}^{\alpha}(x)dx}{\bar{F}^{\alpha}(t)} \right] , \alpha \neq 1, \ \alpha > 0. \tag{5.9}$$

Obviously at $t = 0$, the dynamic version (5.9) reduces to (5.5), the cumulative residual entropy of order α.

Recently, Sunoj and Linu [112] have proposed the uniqueness property of the dynamic cumulative residual entropy of order α defined in equation (5.9) and have studied its properties. Also they have extended this measure to the bivariate set-up and have proved certain characterizing relationships to identify different bivariate lifetime models.

5.3 Specific Lifetime Distributions Based on Entropy of Order α

In the previous chapters we have characterized some specific probability distributions using relations between different reliability measures. In this section we characterize some well known distributions in term of $\xi_{\alpha}(X;t)$ using the mean residual life (MRL) function $\delta_F(t)$ of the random variable X, which is defined as follows

$$\delta_F(t) = E[X - t | X > t] = \frac{\int_t^{\infty} \overline{F}(x)dx}{\overline{F}(t)},$$

where $F(x)$ is the distribution function of the random variable X. This represents the expected value of the residual lifetime of a system which has survived up to a certain point of time t. It is well known that $\delta_F(t)$ uniquely determines the distribution function $F(x)$ and the relation between this and the hazard rate $\lambda_F(t) = \frac{f(t)}{\overline{F}(t)}$ is given as

$$\lambda_F(t) = \frac{\delta'_F(t) + 1}{\delta_F(t)} . \tag{5.10}$$

Next, we prove the following theorem.

Theorem 5.1 *Let X be a non-negative continuous random variable with survival function $\bar{F}(t)$, mean residual life $\delta_F(t)$ and cumulative residual entropy of order α given by*

$$\xi_\alpha(X;t) = \frac{1}{1-\alpha}\log k + \frac{1}{1-\alpha}\log \delta_F(t)\ , \quad \alpha \neq 1, \alpha > 0. \tag{5.11}$$

Then X has

(i) an exponential distribution iff $k = \frac{1}{\alpha}$,

(ii) a Pareto distribution iff $k < \frac{1}{\alpha}$, and

(iii) a finite range distribution iff $k > \frac{1}{\alpha}$.

Proof (i) Let X be an exponential random variable with parameter $\theta > 0$, then its p.d.f. is given by

$$f(x) = \theta e^{-\theta x}\ . \tag{5.12}$$

This gives the survival function as $\bar{F}(x) = e^{-\theta x}$, and the mean residual life as $\delta_F(t) = \frac{1}{\theta}$. The generalized cumulative residual entropy $\xi_\alpha(X;t)$ is

$$\xi_\alpha(X;t) = \frac{1}{1-\alpha}\log\left[\frac{\int_t^\infty \bar{F}^\alpha(x)dx}{\bar{F}^\alpha(t)}\right]$$

$$= \frac{1}{1-\alpha}\log\left[\frac{\int_t^\infty (e^{-\theta x})^\alpha dx}{e^{-\theta\alpha t}}\right]$$

$$= \frac{1}{1-\alpha}\log(\frac{1}{\alpha}) + \frac{1}{1-\alpha}\log\frac{1}{\theta}$$

$$= \frac{1}{1-\alpha}\log k + \frac{1}{1-\alpha}\log \delta_F(t),$$

where $k = \frac{1}{\alpha}$, and $\delta_F(t) = \frac{1}{\theta}$. Thus (5.11) holds.

(ii) The p.d.f. of the Pareto distribution with parameters a and b is given by

$$f(x) = \frac{ab^a}{(x+b)^{a+1}}, \quad a > 1, \;\; b > 0.$$

The survival function is

$$\bar{F}(x) = 1 - F(x) = \left(1 + \frac{x}{b}\right)^{-a} = \frac{b^a}{(x+b)^a}$$

and the mean residual life is

$$\delta_F(t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} = \frac{t+b}{a-1} \; . \tag{5.13}$$

Substituting these values in (5.9) and simplifying, we obtain

$$\xi_\alpha(X;t) = \frac{1}{1-\alpha}\log k + \frac{1}{1-\alpha}\log \delta_F(t) \; ,$$

where $k = \frac{a-1}{a\alpha-1} < \frac{1}{\alpha}$, for $\alpha > 1$ and $\delta_F(t) = \frac{t+b}{a-1}$. Thus (5.11) holds.

(iii) The p.d.f. of the finite range distribution is given by

$$f(x) = a(1-x)^{a-1}, \quad a > 1, \;\; 0 \le x \le 1.$$

The survival function is

$$\bar{F}(x) = 1 - F(x) = (1-x)^a \; ,$$

and the mean residual life is

$$\delta_F(t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} = \frac{1-t}{1+a} \; .$$

Substituting these values in (5.9), after simplification we obtain

$$\xi_\alpha(X;t) = \frac{1}{1-\alpha}\log k + \frac{1}{1-\alpha}\log \delta_F(t) \; ,$$

where $k = \frac{a+1}{a\alpha+1} > \frac{1}{\alpha}$, for $\alpha > 1$ and $\delta_F(t) = \frac{1-t}{1+a}$. Thus (5.11) holds.

To prove the converse part, from (5.9), we have

$$(1-\alpha)\xi_\alpha(X;t) = \log \int_t^\infty \bar{F}^\alpha(x)dx - \alpha \log \bar{F}(t) \ . \tag{5.14}$$

Differentiating this with respect to t, we obtain

$$(1-\alpha)\xi'_\alpha(X;t) = \alpha \frac{f(t)}{\bar{F}(t)} - \frac{\bar{F}^\alpha(t)}{\int_t^\infty \bar{F}^\alpha(x)dx} \ ,$$

or,

$$(1-\alpha)\xi'_\alpha(X;t) = \alpha\lambda_F(t) - \exp[-(1-\alpha)\xi_\alpha(X;t)]. \tag{5.15}$$

Using (5.10), we obtain

$$\delta_F(t)(1-\alpha)\xi'_\alpha(X;t) = \alpha\delta'_F(t) + \alpha - \delta_F(t)\exp\left[-(1-\alpha)\xi_\alpha(X;t)\right] \ . \tag{5.16}$$

From (5.11), we have

$$\xi'_\alpha(X;t) = \frac{\delta'_F(t)}{(1-\alpha)\delta_F(t)}.$$

Thus (5.16) becomes

$$\delta'_F(t) = \alpha\delta'_F(t) + \alpha - \frac{1}{k},$$

or

$$\delta'_F(t) = \frac{1-k\alpha}{k(\alpha-1)}. \tag{5.17}$$

This gives

$$\delta_F(t) = \left(\frac{1-k\alpha}{k(\alpha-1)}\right) t + \delta_F(0) \ . \tag{5.18}$$

The equation (5.18) represents the mean residual life (MRL) function $\delta_F(t)$ of the continuous random variable X as a linear function in t. Thus the underlying distribution is exponential if $k = \frac{1}{\alpha}$, Pareto distribution if $k < \frac{1}{\alpha}$, and finite range distribution if $k > \frac{1}{\alpha}$, refer to Hall and Waller [54]. This proves the converse part.

We note that the expression (5.11) can be viewed as

$$\xi_\alpha(X;t) = c + \frac{1}{1-\alpha}\log\delta_F(t)\ ,$$

where $c = \frac{1}{1-\alpha}\log k$ is a constant. Next, we extend the result to a more general case taking c as a function of t. We state the following theorem.

Theorem 5.2 *Let X be a non-negative continuous random variable such that*

$$\xi_\alpha(X;t) = c(t) + \frac{1}{1-\alpha}\log\delta_F(t)\ ,\ for\, t \geq 0, \tag{5.19}$$

then

$$\delta_F(t) = \left(\mu e^{c(0)} + \int_0^t \left[\frac{\alpha - e^{-(1-\alpha)c(x)}}{1-\alpha}\right] e^{c(x)}dx\right) e^{-c(t)}\ , \tag{5.20}$$

where $\mu = E(X)$.

Proof From (5.9), we have

$$(1-\alpha)\xi_\alpha(X;t) = \log\int_t^\infty \bar{F}^\alpha(x)dx - \alpha\log\bar{F}(t)\ .$$

Substituting for $\xi_\alpha(X;t)$ from (5.19) , we get

$$(1-\alpha)c(t) + \log\delta_F(t) = \log\int_t^\infty \bar{F}^\alpha(x)dx - \alpha\log\bar{F}(t)\ . \tag{5.21}$$

Differentiating w.r.t. t, we obtain

$$(1-\alpha)c'(t) + \frac{\delta'_F(t)}{\delta_F(t)} = \alpha\lambda_F(t) - \exp[-(1-\alpha)\xi_\alpha(X;t)]\ . \tag{5.22}$$

Substituting from (5.19) and simplifying, it gives

$$\delta'_F(t) + c'(t)\delta_F(t) = \frac{\alpha - \exp[-(1-\alpha)c(t)]}{1-\alpha}\ , \tag{5.23}$$

a linear differential equation in $\delta_F(t)$. Solving this for $\delta_F(t)$, we obtain

$$\delta_F(t) = \left(\mu e^{c(0)} + \int_0^t \left[\frac{\alpha - e^{-(1-\alpha)c(x)}}{1-\alpha} \right] e^{c(x)} dx \right) e^{-c(t)} .$$

This proves the result.

In particular if $c(t) = at + b$, and $a > 0$, then (5.20) gives

$$\delta_F(t) = d\, e^{-at-b} + \frac{1}{1-\alpha} \left[\frac{\alpha(1-e^{-at})}{a} + \frac{e^{-(at+b)+b\alpha} - e^{(at+b)(\alpha-1)}}{a\alpha} \right]. \tag{5.24}$$

Further we note that the expression (5.24) for $a = 0$, gives the characterization result given by Theorem 5.1.

Next, we characterize the lifetime models when dynamic cumulative residual entropy of order α is expressed in terms of hazard rate function. we give the following theorem.

Theorem 5.3 *Let X be a non-negative continuous random variable with survival function $\bar{F}(t)$, hazard rate function $\lambda_F(t)$ and cumulative residual entropy of order α given by*

$$\xi_\alpha(X;t) = \frac{1}{1-\alpha} \log k - \frac{1}{1-\alpha} \log \lambda_F(t). \tag{5.25}$$

Then X has

(i) an exponential distribution iff $k = \frac{1}{\alpha}$,

(ii) a Pareto distribution iff $k > \frac{1}{\alpha}$, and

(iii) a finite range distribution iff $k < \frac{1}{\alpha}$.

Proof Let (5.25) be valid, then

$$\frac{1}{1-\alpha}\log\left[\frac{\int_t^\infty \bar{F}^\alpha(x)dx}{\bar{F}^\alpha(t)}\right] = \frac{1}{1-\alpha}\log k - \frac{1}{1-\alpha}\log \lambda_F(t)\ .$$

This gives

$$f(t)\int_t^\infty \bar{F}^\alpha(x)dx = k\bar{F}^{\alpha+1}(t)\ .$$

Differentiating it with respect to t both sides, we get

$$\lambda_F(t) = \left[\frac{k\alpha-1}{k}t + \frac{1}{\lambda_F(0)}\right]^{-1} = (at+b)^{-1}\ , \tag{5.26}$$

where $a = \frac{k\alpha-1}{k}$, and $b = \frac{1}{\lambda_F(0)}$. We observe the following.

(i) If $k = \frac{1}{\alpha}$, then hazard rate function is constant, that is, X follows the exponential distribution.

(ii) If $k\alpha > 1$, then $a > 0$ and (5.26) is the hazard rate function of the Pareto distribution.

(iii) If $k\alpha < 1$, then $a < 0$ and (5.26) is the hazard rate function of the finite range distribution.

The only if part of the theorem is easy to prove.

5.4 Cumulative Residual Entropy of Order α and Type β

In Section 5.2 we have considered one parameter cumulative residual entropy. A two parameters generalization of order α and type β of the entropy (5.1) is the

Verma entropy [123] defined as

$$H_\alpha^\beta(X) = \frac{1}{\beta-\alpha}\log\left[\int_0^\infty f^{\alpha+\beta-1}(x)dx\right];\ \ \beta-1<\alpha<\beta,\ \beta\geq 1\ . \tag{5.27}$$

When $\beta = 1$, (5.27) reduces to the Renyi entropy [101] of order α; and in case of $\beta = 1$ and $\alpha \to 1$ this reduces to Shannon differential entropy [109]. As already mentioned in Chapter 2, Verma entropy plays a vital role as a measure of complexity and uncertainty in different areas such as physics and electronics to describe many chaotic systems.

Analogous to the generalized entropy (5.27), the cumulative residual entropy of order α and type β of the random variable X is proposed as

$$\xi_\alpha^\beta(X) = \frac{1}{\beta-\alpha}\log\left[\int_0^\infty \bar{F}^{\alpha+\beta-1}(x)dx\right];\ \ \beta-1<\alpha<\beta,\ \beta\geq 1. \tag{5.28}$$

When β=1, $\alpha \to 1$ then under the assumption that $E(X) = 1$, (5.28) reduces to

$$\lim_{\beta=1,\alpha\to 1}\xi_\alpha^\beta(X) = -\int_0^\infty \bar{F}(x)\log\bar{F}(x)dx,$$

the cumulative residual entropy (5.3) as suggested by Rao et al. [98].

For some specific probability distributions the generalized cumulative residual entropy (5.28) is given as below.

(i) If X is uniformly distributed on (a, b) with $a < b$, then

$$\xi_\alpha^\beta(X) = \frac{1}{\beta-\alpha}\log\left(\frac{b-a}{\alpha+\beta}\right)\ .$$

(ii) If X is folded Cramer distribution with probability density function $f(x) = \frac{\theta}{(1+\theta x)^2}$, $\theta > 0$, and survival function $\bar{F}(x) = \frac{1}{1+\theta x}$, then cumulative residual entropy of order α and type β is

$$\xi_{\alpha}^{\beta}(X) = \frac{1}{\beta-\alpha}\log\frac{1}{\theta(\alpha-\beta)}\ . \tag{5.29}$$

In case the system has already survived up to time t, then using the survival function $\overline{F}_t(x)$ of the residual lifetime distribution *the dynamic cumulative residual entropy of order α and type β of the random variable X* is defined as

$$\xi_{\alpha}^{\beta}(X;t) = \frac{1}{\beta-\alpha}\log\left[\int_0^{\infty}\bar{F}_t^{\alpha+\beta-1}(x)dx\right]\ , \tag{5.30}$$

which, as noted in Section 5.2, can be rewritten as

$$\xi_{\alpha}^{\beta}(X;t) = \frac{1}{\beta-\alpha}\log\left[\frac{\int_t^{\infty}\bar{F}^{\alpha+\beta-1}(x)dx}{\bar{F}^{\alpha+\beta-1}(t)}\right]\ . \tag{5.31}$$

When $\beta = 1$, then (5.31) reduce to *the dynamic cumulative entropy of order α*, given by (5.9).

Next, we prove that $\xi_{\alpha}^{\beta}(X;t)$ characterizes the lifetime distribution uniquely. In this context we prove the following result.

Theorem 5.4 *Let X be a non-negative random variable having continuous density function $f(.)$ and survival function $\overline{F}(.)$, and if, $\xi_{\alpha}^{\beta}(X;t) < \infty$, $t \geq 0$, $\forall\ \beta-1 < \alpha < \beta$, $\beta \geq 1$, then $\xi_{\alpha}^{\beta}(X;t)$ determines the survival function $\bar{F}(.)$ uniquely.*

Proof Rewriting (5.31) as

$$(\beta-\alpha)\xi_{\alpha}^{\beta}(X;t) = \log\left(\int_t^{\infty}\bar{F}^{\alpha+\beta-1}(x)dx\right) - (\alpha+\beta-1)\log\bar{F}(t)\ . \tag{5.32}$$

Differentiating (5.32) with respect to t, we obtain

$$(\beta-\alpha)\xi_{\alpha}'^{\beta}(X;t) = (\alpha+\beta-1)[\lambda_F(t)] - \frac{\bar{F}^{\alpha+\beta-1}(t)}{\int_t^{\infty}\bar{F}^{\alpha+\beta-1}(x)dx}\ , \tag{5.33}$$

where $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ is the failure rate of X. Using (5.31), we can rewrite (5.33) as

$$(\beta - \alpha)\xi_\alpha'^\beta(X;t) = (\alpha + \beta - 1)[\lambda_F(t)] - \exp[-(\beta - \alpha)\xi_\alpha^\beta(X;t)]. \tag{5.34}$$

Let $\bar{F}_1(t)$ and $\bar{F}_2(t)$ be two survival functions with dynamic entropies as $\xi_\alpha^\beta(X_1;t)$ and $\xi_\alpha^\beta(X_2;t)$ and hazard rates $\lambda_{F_1}(t)$ and $\lambda_{F_2}(t)$ respectively. Now $\xi_\alpha^\beta(X_1;t) = \xi_\alpha^\beta(X_2;t)$ implies that

$$\xi_\alpha'^\beta(X_1;t) = \xi_\alpha'^\beta(X_2;t) \ ,$$

which is equivalent to

$$(\beta - \alpha)\xi_\alpha'^\beta(X_1;t) = (\beta - \alpha)\xi_\alpha'^\beta(X_2;t) \ . \tag{5.35}$$

Using (5.34), this becomes

$$\begin{aligned}(\alpha + \beta - 1)[\lambda_{F_1}(t)] - \exp[-(\beta - \alpha)\xi_\alpha^\beta(X_1;t)] = (\alpha + \beta - 1)[\lambda_{F_2}(t)] \\ - \exp[-(\beta - \alpha)\xi_\alpha^\beta(X_2;t)].\end{aligned} \tag{5.36}$$

Since $\xi_\alpha^\beta(X_1;t) = \xi_\alpha^\beta(X_2;t)$, (5.36) reduces to

$$(\alpha + \beta - 1)[\lambda_{F_1}(t)] = (\alpha + \beta - 1)[\lambda_{F_2}(t)] \tag{5.37}$$

which implies that $\lambda_{F_1}(t) = \lambda_{F_2}(t)$, or equivalently $F_1 = F_2$. This completes the proof for the characterization theorem.

5.5 Lifetime Distributions Based on Entropy of Order α and Type β

In this section we characterize some specific lifetime distribution functions based on $\xi_\alpha^\beta(X;t)$, the generalized cumulative entropy of order α and type β. We will achieve this by considering a relation between $\xi_{\alpha}^{\beta}(X;t)$ and $\delta_F(t)$, the mean residual life function. We give the following result.

Theorem 5.5 *Let X be a non-negative continuous random variable with survival function $\bar{F}(t)$, mean residual life $\delta_F(t)$ and generalized cumulative residual entropy $\xi_{\alpha}^{\beta}(X;t)$ given by*

$$(\beta-\alpha)\xi_{\alpha}^{\beta}(X;t) = \log k + \log \delta_F(t). \tag{5.38}$$

Then X has

(i) an exponential distribution iff $k = \frac{1}{\alpha+\beta-1}$,

(ii) a Pareto distribution iff $k < \frac{1}{\alpha+\beta-1}$, and

(iii) a finite range distribution iff $k > \frac{1}{\alpha+\beta-1}$.

Proof (i) The p.d.f. and survival function of an exponential variable X with parameter $\theta > 0$, are given respectively by

$$f(x) = \theta e^{-\theta x}$$
$$\text{and} \;\; \bar{F}(x) = e^{-\theta x}.$$

The mean residual life is

$$\delta_F(t) = \frac{\int_t^{\infty} \bar{F}(x)dx}{\bar{F}(t)} = \frac{1}{\theta}. \tag{5.39}$$

The generalized residual entropy $\xi_{\alpha}^{\beta}(X;t)$ is given by

$$\begin{aligned}
\xi_{\alpha}^{\beta}(X;t) &= \frac{1}{\beta-\alpha}\log\left[\frac{\int_t^{\infty} \bar{F}^{\alpha+\beta-1}(x)dx}{\bar{F}^{\alpha+\beta-1}(t)}\right] \\
&= \frac{1}{\beta-\alpha}\log\left[\frac{\int_t^{\infty} (e^{-\theta x})^{\alpha+\beta-1}dx}{e^{-(\alpha+\beta-1)\theta t}}\right]
\end{aligned} \tag{5.40}$$

$$= \frac{1}{\beta-\alpha}\log\left(\frac{1}{\alpha+\beta-1}\right)+\frac{1}{\beta-\alpha}\log\left(\frac{1}{\theta}\right),$$

$$= \frac{1}{\beta-\alpha}\log k+\frac{1}{\beta-\alpha}\log\delta_F(t),$$

where $k = \frac{1}{\alpha+\beta-1}$, and $\delta_F(t) = \frac{1}{\theta}$, from (5.39). Thus (5.38) holds.

(ii) The p.d.f. of the pareto distribution is given by

$$f(x) = \frac{ab^a}{(x+b)^{a+1}}, \quad a\ > 1, \;\; b >\ 0,$$

and the survival function is

$$\bar{F}(x) = 1 - F(x) = \left(1+\frac{x}{b}\right)^{-a} = \frac{b^a}{(x+b)^a}.$$

The mean residual life is

$$\delta_F(t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} = \frac{t+b}{a-1}\ . \tag{5.41}$$

Substituting in (5.31) and simplifying, we obtain

$$\xi_\alpha^\beta(X;t) = \frac{1}{\beta-\alpha}\log k+\frac{1}{\beta-\alpha}\log\delta_F(t)\ ,$$

where $k = \frac{a-1}{a(\alpha+\beta-1)-1} < \frac{1}{\alpha+\beta-1}$, for $\alpha > 1$ and $\delta_F(t) = \frac{t+b}{a-1}$ from (5.40). Thus (5.38) holds.

(iii) The p.d.f. and survival function of the finite range distribution are given respectively by

$$f(x) = a(1-x)^{a-1}, \quad a\ > 1, \;\; 0 \le x \le\ 1,$$

and

$$\bar{F}(x) = 1 - F(x) = (1-x)^a.$$

The mean residual life is

$$\delta_F(t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} = \frac{1-t}{1+a} \ .$$

Substituting in (5.31) and simplifying, we get

$$\xi_\alpha^\beta(X;t) = \frac{1}{\beta-\alpha}\log k + \frac{1}{\beta-\alpha}\log \delta_F(t) \ ,$$

where $k = \frac{a+1}{a(\alpha+\beta-1)+1} > \frac{1}{\alpha+\beta-1}$, for $\alpha > 1$ and $\delta_F(t) = \frac{1-t}{1+a}$. Thus (5.38) holds.

Conversely, assume that (5.38) is valid. Differentiating it w.r.t. t, we have

$$(\beta-\alpha)\xi_\alpha^{\prime\beta}(X;t) = \frac{\delta_F'(t)}{\delta_F(t)}. \tag{5.42}$$

From (5.34) we have

$$(\beta-\alpha)\xi_\alpha^{\prime\beta}(X;t) = (\alpha+\beta-1)[\lambda_F(t)] - \exp[-(\beta-\alpha)\xi_\alpha^\beta(X;t)]. \tag{5.43}$$

Using (5.38) and (5.41), Eq. (5.42) gives

$$\frac{\delta_F'(t)}{\delta_F(t)} = (\alpha+\beta-1)[\lambda_F(t)] - \frac{1}{k\delta_F(t)}. \tag{5.44}$$

Using $\lambda_F(t) = \frac{\delta_F'(t)+1}{\delta_F(t)}$, as given by (5.10), we obtain

$$\delta_F'(t) = (\alpha+\beta-1)\delta_F'(t) + (\alpha+\beta-1) - \frac{1}{k},$$

or

$$\delta_F'(t) = \frac{1-k(\alpha+\beta-1)}{k(\alpha+\beta-2)}. \tag{5.45}$$

This gives

$$\delta_F(t) = \left(\frac{1-k(\alpha+\beta-1)}{k\{(\alpha+\beta-1)-1\}}\right)t + \delta_F(0) \ . \tag{5.46}$$

The Eq. (5.45) represents the mean residual life (MRL) function $\delta_F(t)$, of the continuous random variable X, as a linear function in t, thus the underlying distribution is exponential if $k = \frac{1}{\alpha+\beta-1}$, Pareto distribution for $k < \frac{1}{\alpha+\beta-1}$, and finite range distribution for $k > \frac{1}{\alpha+\beta-1}$, refer to Hall and Waller [54]. This proves the theorem.

Next, we extend the result (5.38) given by Theorem 5.5 to a more general case.

Theorem 5.6 *If X is a non-negative continuous random variable with generalized entropy measure $\xi_\alpha^\beta(X;t)$ given by*

$$\xi_\alpha^\beta(X;t) = c(t) + \frac{1}{\beta-\alpha}\log\delta_F(t)\ , for\ \ t \geq 0, \tag{5.47}$$

then

$$\delta_F(t) = \left(d + \int_0^t \left[\frac{(\alpha+\beta-1) - e^{-(\beta-\alpha)c(x)}}{(2-\alpha-\beta)}\right] e^{\frac{(\beta-\alpha)c(x)}{(2-\alpha-\beta)}}dx\right) e^{-(\frac{\beta-\alpha}{2-\alpha-\beta})c(t)}\ , \tag{5.48}$$

where $d = \mu e^{-(\frac{\beta-\alpha}{2-\alpha-\beta})c(0)}$ and $\mu = E(X)$.

Proof From (5.31), we have

$$(\beta-\alpha)\xi_\alpha^\beta(X;t) = \log\int_t^\infty \bar{F}^{\alpha+\beta-1}(x)dx - (\alpha+\beta-1)\log\bar{F}(t).$$

Substituting for $\xi_\alpha^\beta(X;t)$, we get

$$(\beta-\alpha)c(t) + \log\delta_F(t) = \log\int_t^\infty \bar{F}^{\alpha+\beta-1}(x)dx - (\alpha+\beta-1)\log\bar{F}(t). \tag{5.49}$$

Differentiating (5.48) w.r.t. t,

$$(\beta-\alpha)c'(t) + \frac{\delta_F'(t)}{\delta_F(t)} = (\alpha+\beta-1)[\lambda_F(t)] - \exp[-(\beta-\alpha)\xi_\alpha^\beta(X;t)]\ . \tag{5.50}$$

Substituting from (5.10) and (5.46) and simplifying, this gives

$$(2-\alpha-\beta)\delta'_F(t)+(\beta-\alpha)c'(t)\delta_F(t)=(\alpha+\beta-1)-\exp[-(\beta-\alpha)c(t)]\ , \quad (5.51)$$

or

$$\delta'_F(t)+\left(\frac{\beta-\alpha}{2-\alpha-\beta}\right)c'(t)\delta_F(t)=\frac{(\alpha+\beta-1)-\exp[-(\beta-\alpha)c(t)]}{2-\alpha-\beta}\ , \qquad (5.52)$$

which is a linear differential equation in $\delta_F(t)$. Solving this we obtain

$$\delta_F(t)=\left(d+\int_0^t\left[\frac{(\alpha+\beta-1)-e^{-(\beta-\alpha)c(x)}}{(2-\alpha-\beta)}\right]e^{\frac{(\beta-\alpha)c(x)}{(2-\alpha-\beta)}}dx\right)e^{-(\frac{\beta-\alpha}{2-\alpha-\beta})c(t)}\ .$$

This proves the result.

In particular if $c(t)=at+b$ for $t>0$ and $a>0$, (5.47) gives

$$\begin{aligned}\delta_F(t)&=d\ e^{-Ac(t)}\\&+\frac{1}{2-\alpha-\beta}\left[\frac{(\alpha+\beta-1)(1-e^{-at})}{Aa}+\frac{e^{-Ac(t)+(A-\beta+\alpha)b}-e^{c(t)(\alpha-\beta)}}{a(A-\beta+\alpha)}\right]\ , \quad (5.53)\end{aligned}$$

where $A=\left(\frac{\beta-\alpha}{2-\alpha-\beta}\right)$. Further for if $a=0$, (5.52) gives the characterization results given by Theorem 5.5 .

Remark 5.1 For $\beta=1$, (5.52) reduces to

$$\delta_F(t)=d\ e^{-at-b}+\frac{1}{1-\alpha}\left[\frac{\alpha(1-e^{-at})}{a}+\frac{e^{-(at+b)+b\alpha}-e^{(at+b)(\alpha-1)}}{a\alpha}\right], \qquad (5.54)$$

a result given by (5.24).

Remark 5.2 For $\beta=1$, $\alpha\to1$, (5.52) reduces to

$$\delta_F(t)=d\ e^{-at-b}+\frac{b-2+at}{a}-\frac{(b-2)e^{-at}}{a}, \qquad (5.55)$$

a result given by Navarro et al. [86].

Similarly lifetime models may be characterized when the dynamic cumulative residual entropy of order α and type β is expressed in terms of hazard rate function. We give the following result.

Theorem 5.7 *Let X be a non-negative continuous random variable with survival function $\bar{F}(t)$, hazard rate function $\lambda_F(t)$ and dynamic cumulative residual entropy $\xi_\alpha^\beta(X;t)$, and let*

$$\xi_\alpha^\beta(X;t) = \frac{1}{\beta-\alpha}\log k - \frac{1}{\beta-\alpha}\log \lambda_F(t). \tag{5.56}$$

Then X has

(i) an exponential distribution iff $k = \frac{1}{\alpha+\beta-1}$,

(ii) a Pareto distribution iff $k > \frac{1}{\alpha+\beta-1}$ and

(iii) a finite range distribution iff $k < \frac{1}{\alpha+\beta-1}$.

The proof is similar to that of Theorem 5.5 and hence omitted.

In sequel to Theorem 5.6 and Theorem 5.7, another result of interest is given as follows.

Theorem 5.8 *The relationship*

$$(\beta-\alpha)\xi_\alpha'^\beta(X;t) = c\lambda_F(t) \tag{5.57}$$

characterizes

(i) exponential distribution with survival function

$$\overline{F}(t) = e^{-\theta t}\ , \quad \lambda > 0 \tag{5.58}$$

for $c = 0$,

(ii) Pareto II distribution with survival function

$$\overline{F}(t) = \left(1+\frac{t}{a}\right)^{-a}\ , \ a > 0, \quad t > 0 \tag{5.59}$$

for $c > 0$*, and*

(iii) beta distribution with survival function

$$\overline{F}(t) = (1-t)^a, \;\; a > 0, \;\; 0 < t < 1 \, , \tag{5.60}$$

for $c < 0$.

Proof Assume that (5.56) holds. Using (5.34), (5.56) becomes

$$(\alpha+\beta-1)[\lambda_F(t)] - \exp[-(\beta-\alpha)\xi_\alpha^\beta(X;t)] = c\lambda_F(t) \, . \tag{5.61}$$

From (5.31) and (5.60), we get

$$(\alpha+\beta-c-1)f(t)\int_t^\infty \bar{F}^{\alpha+\beta-1}(x)dx = \bar{F}^{\alpha+\beta}(t). \tag{5.62}$$

Differentiating (5.61) with respect to t, we obtain

$$(\alpha+\beta-c-1)\left\{f'(t)\int_t^\infty \bar{F}^{\alpha+\beta-1}(x)dx\right\} - \left\{(\alpha+\beta-c-1)f(t)\bar{F}^{\alpha+\beta-1}(t)\right\}$$
$$= -(\alpha+\beta)\bar{F}^{\alpha+\beta-1}(t)f(t).$$

Substituting (5.61), this gives

$$\frac{f'(t)\bar{F}^{\alpha+\beta}(t)}{f(t)} - \left\{(\alpha+\beta-c-1)f(t)\bar{F}^{\alpha+\beta-1}(t)\right\} = -(\alpha+\beta)\bar{F}^{\alpha+\beta-1}(t)f(t).$$

After some simplifications the above expression reduces to

$$\frac{f'(t)}{f(t)} = -\frac{(c+1)f(t)}{\bar{F}(t)} \, . \tag{5.63}$$

The solution of the differential equation (5.62) is obtained as

$$\log\lambda_F(t) = c\log\bar{F}(t) + c_1 \, , \tag{5.64}$$

where c_1 is a constant of integration.

Differentiating (5.63) with respect to t, we have

$$\frac{d}{dt}\left\{\frac{1}{\lambda_F(t)}\right\} = c \ ,$$

which leads to

$$\lambda_F(t) = \frac{1}{ct+d} \ , \tag{5.65}$$

where $d > 0$ is a constant.

Since the hazard rate uniquely determines the survival function using the relationship $\bar{F}(t) = \exp\left\{-\int_0^t \lambda_F(x)dx\right\}$, the results (5.57) to (5.59) follows according as $c = 0, c > 0$ and $c < 0$.

5.6 Conclusion

The cumulative distribution function based measures of entropy $\xi(X)$ are in general more stable in comparison to probability density function based measure $H(f)$ given by Shannon [109]. The concept of *cumulative residual entropy(CRE)* given by Rao [98] has been extended to one parameter and two parameters cumulative residual entropies and further to their dynamic versions viz. $\xi_\alpha(X;t)$ and $\xi_\alpha^\beta(X;t)$. The exponential, the Pareto and the finite range distributions which are commonly used in the reliability modeling have been characterized in terms of the proposed generalized dynamic cumulative entropy measures. The proposed dynamic entropy functions uniquely determine the survival functions. The results obtained are the generalized one in conformity with the result already existing in the literature. In the subsequent chapter we extend the concept of cumulative distribution function based entropy measures to the inaccuracy measures.

Chapter 6

Dynamic Cumulative Inaccuracy Measures

6.1 Introduction

The average amount of uncertainty associated with the random variable X with p.d.f. $f(x)$, as given by Shannon differential entropy [109], is

$$H(f) = -\int_0^\infty f(x) \log f(x) dx \ . \tag{6.1}$$

The concept of entropy has been generalized in a number of different ways. An extension of Shannon's idea has been given by Kerridge [67], as Kerridge's inaccuracy. If $f(x)$ is the actual probability density function (p.d.f.) and $g(x)$ is the reference p.d.f. of a random variable X associated with a system, then Kerridge's measure of inaccuracy [67] is

$$H(f; g) = -\int_0^\infty f(x) \log g(x) dx \ . \tag{6.2}$$

The measure of inaccuracy suggested by Kerridge has many useful applications in statistics and has been studied by many researchers from various aspects. We have also studied the dynamic and length biased dynamic measures of inaccuracy in Chapter 3 and Chapter 4 respectively. In the preceding chapter we have considered the concept of cumulative residual entropy (CRE)

$$\xi(X) = \xi(F) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx \ , \tag{6.3}$$

as given by Rao et al. [98] and have studied its one parameter and two parameters generalizations and also their dynamic versions.

Taking into considerations the advantage of distribution function based information theoretic measures over probability density function based measure as discussed in Chapter 5, in this chapter we study the distribution functions based inaccuracy measures analogous to the Kerridge inaccuracy measure (6.2). The distribution function based inaccuracy measure can also be viewed as a natural extension of the cumulative residual entropy measure suggested by [98]. The chapter has been organized as follows. In Section 6.2 we propose a measure of *cumulative residual inaccuracy (CRI)* and derive an upper bound to it. Section 6.3 considers *dynamic cumulative residual inaccuracy (DCRI)* measure in context with residual lifetime distribution $[X|X \geq t]$. A characterization theorem for the dynamic cumulative residual inaccuracy under proportional hazard rate model has been proved in Section 6.4 and some specific lifetime distributions have been characterized. Section 6.5 introduces the *dynamic cumulative past inaccuracy (DCPI)* in context with past lifetime distribution $[X|X \leq t]$, and characterization result for this has been studied in Section 6.6 which also includes a few other results on this measure. Section 6.7 concludes the chapter.

6.2 Cumulative Residual Inaccuracy

Let X and Y be two random variables with the same support, and let $\bar{F}(x)$ and $\bar{G}(x)$ be their survival functions then the *cumulative residual inaccuracy (CRI)* analogous to the inaccuracy measure (6.2) is defined as

$$\xi(F;G) = -\int_0^\infty \bar{F}(x)\log\bar{G}(x)dx\ . \tag{6.4}$$

When these two distributions coincide the measure (6.4) reduces to the cumulative residual entropy (6.3).

If the two random variables X and Y satisfy the proportional hazard model (PHM), refer to Cox [24] and Efron [42], that is, if $\lambda_G(x) = \beta\lambda_F(x)$, or equivalently

$$\bar{G}(x) = [\bar{F}(x)]^\beta, \tag{6.5}$$

for some constant $\beta > 0$, then obviously the cumulative residual inaccuracy (6.4) reduces to a constant multiple of the cumulative residual entropy (6.3).

Example 6.1 Let a non-negative random variable X be uniformly distributed over (a,b), $a < b$, with density and distribution functions respectively given by

$$f(x) = \frac{1}{b-a} \ \text{ and } \ F(x) = \frac{x-a}{b-a}, \quad a < x < b.$$

If the random variables X and Y satisfy the proportional hazard model (PHM), then the distribution function of the random variable Y is

$$\bar{G}(x) = [\bar{F}(x)]^\beta = \left[\frac{b-x}{b-a}\right]^\beta \quad a < x < b,\ \beta > 0.$$

Substituting these in (6.4) and simplifying we obtain the cumulative inaccuracy measure as

$$\xi(F;G) = \frac{\beta(b-a)}{4}\ .$$

6.2.1 A Lower Bound to $\xi(F; G)$

Before deriving the lower bound to $\xi(F; G)$, we state the *log-sum inequality* given as follows;

Let m be a sigma finite measure. If f and g are positive and m-integrable then

$$\int \log\left(\frac{f}{g}\right) dm \geq \left[\int f dm\right] \log \frac{\int f dm}{\int g dm} \ . \tag{6.6}$$

Also another result of interest which we will use is the inequality given by

$$x \log \frac{x}{y} \geq x - y, \tag{6.7}$$

for all non-negative x and y. We prove the following result.

Theorem 6.1 *If X and Y are two non-negative random variables with finite means $E(X)$ and $E(Y)$ respectively and if CRE measure $\xi(X)$ given by (6.3) is finite, then*

$$\xi(F; G) \geq \int_0^\infty F(x)\bar{F}(x)dx + E(X) - E(Y). \tag{6.8}$$

Proof We have

$$\begin{aligned} \xi(F; G) &= -\int_0^\infty \bar{F}(x) \log \bar{G}(x)dx \\ &= -\int_0^\infty \bar{F}(x) \log \bar{F}(x)dx + \int_0^\infty \bar{F}(x) \log \frac{\bar{F}(x)}{\bar{G}(x)}dx \ . \end{aligned}$$

Using the log-sum inequality (6.6), we have

$$\begin{aligned} \xi(F; G) &\geq \xi(X) + \int_0^\infty \bar{F}(x)dx \log \frac{\int_0^\infty \bar{F}(x)dx}{\int_0^\infty \bar{G}(x)dx} \\ &\geq \xi(X) + E(X) \log \frac{E(X)}{E(Y)} \\ &\geq \int_0^\infty \bar{F}(x)F(x)dx + E(X) - E(Y) \ . \end{aligned}$$

The last inequality has been obtained using (6.7).This proves the result.

In the next section we extend the concept of cumulative residual inaccuracy to dynamic cumulative residual inaccuracy.

6.3 Dynamic Cumulative Residual Inaccuracy

In life-testing experiments normally the experimenter has information about the current age of the system under consideration. Obviously the CRI (6.4) is not suitable in such a situation and needs to be modified to take into account the current age also. Further, if X is the lifetime of a component which has already survived up to time t, then the random variable $X_t = [X - t|X > t]$, called the residual lifetime random variable, has the survival function

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)} & ; \quad \text{if } \ x > t \\ 1 & ; \quad \text{otherwise} \end{cases}$$

and similarly for $\bar{G}_t(x)$.

The cumulative inaccuracy measure (6.4), for the residual lifetime random variable X_t, is

$$\xi(F, G; t) = -\int_t^\infty \bar{F}_t(x) \log \bar{G}_t(x) dx \tag{6.9}$$

$$= -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{G}(x)}{\bar{G}(t)} dx \ . \tag{6.10}$$

The measure (6.10) is defined as the *dynamic cumulative residual inaccuracy measure (DCRI)*. Obviously when $t = 0$, then (6.10) becomes (6.4).

We observe that (6.10) is analogous to the residual inaccuracy measure

$$H(f, g; t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx \tag{6.11}$$

as discussed in Chapter 3.

Example 6.2 Let X be a non-negative random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x & ; \quad \text{if } \ 0 \leq x < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and suvival function $\bar{F}(x) = 1 - F(x) = (1 - x^2)$, and let the random variable Y be uniformly distributed over $(0, 1)$ with density and survival functions given respectively by $g_Y(x) = 1$ and $\bar{G}_Y(x) = 1 - x$, $0 < x < 1$.

Substituting these values in (6.10), the dynamic cumulative residual inaccuracy measure is

$$\xi(F, G; t) = \begin{cases} \frac{9(1-t)-2(1-t)^2}{18(1+t)} & ; \quad \text{if } \ 0 \leq t < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The plot of the dynamic cumulative residual inaccuracy measure $\xi(F, G; t)$ for $t \in [0,\ 1]$ is shown in Fig. 6.1

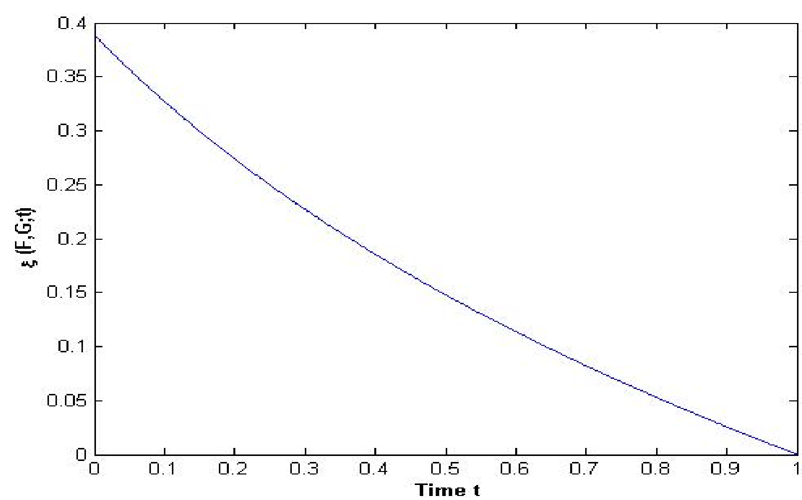

Fig. 6.1: Plot of $\xi(F, G; t)$ against $t \in [0\ , 1]$.

6.4 Characterization Result of Dynamic Cumulative Inaccuracy

The general characterization problem is to determine when the proposed dynamic inaccuracy measure (6.10) characterizes the distribution function uniquely. We study the characterization problem under the proportional hazard model (6.5). Also we know that the hazard rate function $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$ and the mean residual life function $\delta_F(t) = \frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)}$, characterize the distribution function of a random variable X and the relation between the two is given by

$$\lambda_F(t) = \frac{1 + \delta'_F(t)}{\delta_F(t)} \; . \tag{6.12}$$

We shall use (6.12) in establishing the characterization result stated next.

Theorem 6.2 *Let X and Y be two non-negative random variables with survival functions $\bar{F}(.)$ and $\bar{G}(.)$ satisfying the proportional hazard model (6.5), and let $\xi(F, G; t) < \infty, \forall\, t \geq 0$ be an increasing function of t, then $\xi(F, G; t)$ determines the survival function $\bar{F}(.)$ of the variable X uniquely.*

Proof The dynamic cumulative residual inaccuracy measure (6.10) can be expressed is

$$\xi(F, G; t) = -\frac{1}{\overline{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{G}(x) dx + \delta_F(t) \log \bar{G}(t) \; , \tag{6.13}$$

where $\delta_F(t)$ is the mean residual life function. Substituting (6.5) into (6.13) gives

$$\xi(F, G; t) = -\frac{\beta}{\overline{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx + \beta \delta_F(t) \log \bar{F}(t) \; .$$

Differentiating this w.r.t. t both sides, we obtain

$$\xi'(F,G;t) = \beta \log \bar{F}(t)[1+\delta'_F(t)]$$

$$-\beta\lambda_F(t)\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}\log \bar{F}(x)dx - \beta\lambda_F(t)\delta_F(t)\ , \tag{6.14}$$

where $\lambda_F(t)$ is hazard rate function. Substituting (6.12) and (6.13) in (6.14), we obtain

$$\xi'(F,G;t) = \lambda_F(t)\{\xi(F,G;t) - \beta\delta_F(t)\}\ . \tag{6.15}$$

Let F_1, G_1 and F_2, G_2 be two sets of the probability distribution functions satisfying the proportional hazard model, that is, $\lambda_{G_1}(x) = \beta\lambda_{F_1}(x)$, and $\lambda_{G_2}(x) = \beta\lambda_{F_2}(x)$, and let

$$\xi(F_1,G_1;t) = \xi(F_2,G_2;t)\ , \forall\ \ t \geq 0\ . \tag{6.16}$$

Differentiating it both sides w.r.t. t and using (6.15), we obtain

$$\lambda_{F_1}(t)\{\xi(F_1,G_1;t) - \beta\delta_{F_1}(t)\} = \lambda_{F_2}(t)\{\xi(F_2,G_2;t) - \beta\delta_{F_2}(t)\}. \tag{6.17}$$

If for all $t \geq 0,\ \lambda_{F_1}(t) = \lambda_{F_2}(t)$, then $\bar{F}_1(t) = \bar{F}_2(t)$ and the proof is over, otherwise, let

$$A = \{t : t \geq 0, \text{and}\ \ \lambda_{F_1}(t) \neq \lambda_{F_2}(t)\} \tag{6.18}$$

and assume the set A to be non empty . Thus for at least one $t_0 \in A$, $\lambda_{F_1}(t_0) \neq \lambda_{F_2}(t_0)$.

Without loss of generality suppose that $\lambda_{F_2}(t_0) > \lambda_{F_1}(t_0)$. Using this, (6.17) for $t = t_0$ gives

$$\xi(F_1,G_1;t_0) - \beta\delta_{F_1}(t_0) > \xi(F_2,G_2;t_0) - \beta\delta_{F_2}(t_0),$$

which implies that

$$\delta_{F_1}(t_0) < \delta_{F_2}(t_0),$$

a contradiction. Thus the set A is empty set and this concludes the proof.

Next, we characterize some specific lifetime distributions using the dynamic cumulative inaccuracy measure (6.10). We give the following theorem.

Theorem 6.3 *Let X and Y be two non-negative continuous random variables satisfying the proportional hazard model (6.5). If X is with mean residual life $\delta_F(t)$, then the dynamic cumulative residual inaccuracy measure*

$$\xi(F,G;t) = c\,\delta_F(t), \quad c > 0 \tag{6.19}$$

if, and only if X follows the

(i) exponential distribution for $c = \beta$,

(ii) Pareto distribution for $c > \beta$,

(iii) finite range distribution for $0 < c < \beta$.

Proof First we prove the 'if' part.

(i) If X has an exponential distribution with survival function $\bar{F}(x) = \exp(-\theta x),\ \theta > 0$, then the mean residual life function $\delta_F(t) = \frac{1}{\theta}$. The dynamic cumulative residual inaccuracy measure (6.10) under PHM is given as

$$\xi(F,G;t) = \frac{\beta}{\theta} = c\delta_F(t),$$

for $c = \beta$.

(ii) If X follows Pareto distribution with p.d.f.

$$f(x) = \frac{ab^a}{(x+b)^{a+1}}, \quad a > 1, \ \ b > 0,$$

then the survival function is

$$\bar{F}(x) = 1 - F(x) = \left(1 + \frac{x}{b}\right)^{-a} = \frac{b^a}{(x+b)^a} ,$$

and the mean residual life is

$$\delta_F(t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} = \frac{t+b}{a-1} . \tag{6.20}$$

The dynamic cumulative inaccuracy measure (6.10), under PHM is given by

$$\xi(F, G; t) = \frac{\beta a(t+b)}{(a-1)^2} = c\delta_F(t) ,$$

for $c = \frac{\beta a}{a-1} > \beta$.

(iii) In case X follows finite range distribution with p.d.f.

$$f(x) = a(1-x)^{a-1}, \quad a > 1, \ \ 0 \le x \le 1,$$

then the survival function is

$$\bar{F}(x) = 1 - F(x) = (1-x)^a,$$

and the mean residual life is

$$\delta_F(t) = \frac{1-t}{a+1}.$$

The inaccuracy measure (6.10) under PHM is given by

$$\xi(F, G; t) = \frac{\beta a(1-t)}{(a+1)^2} = c\delta_F(t) ,$$

for $c = \frac{\beta a}{a+1} < \beta$.

This proves the 'if' part.

To prove the 'only if' part, consider (6.19) to be valid.

Eq. (6.13) under PHM, using (6.19) gives

$$-\frac{\beta}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x)dx + \beta\delta_F(t) \log \bar{F}(t) = c\delta_F(t) .$$

Differentiating both sides w.r.t. t, we obtain

$$\frac{c}{\beta}\delta'_F(t) = \delta'_F(t)\log\bar{F}(t) - \lambda_F(t)\delta_F(t) + \log\bar{F}(t)$$

$$-\lambda_F(t)\frac{1}{\bar{F}(t)}\int_t^\infty \bar{F}(x)\log\bar{F}(x)dx,$$

$$= \delta'_F(t)\log\bar{F}(t) - \lambda_F(t)\delta_F(t) + \log\bar{F}(t) + \lambda_F(t)[\frac{c}{\beta}\delta_F(t) - \delta_F(t)\log\bar{F}(t)]\ .$$

From (6.12), put $\delta'_F(t) = \lambda_F(t)\delta_F(t) - 1$ and simplify, we obtain

$$\lambda_F(t)\delta_F(t) = \frac{c}{\beta}\ ,$$

which implies

$$\delta'_F(t) = \frac{c}{\beta} - 1\ .$$

Integrating both sides of this w.r.t. t over $(0,\ x)$ yields

$$\delta_F(x) = (\frac{c}{\beta} - 1)x + \delta_F(0). \tag{6.21}$$

The mean residual life function $\delta_F(x)$ of a continuous non-negative random variable X is linear of the form (6.21) if, and only if the underlying distribution is exponential for $c = \beta$, Pareto for $c > \beta$, or finite range for $0 < c < \beta$, refer to Hall and Wellner [54]. This completes the theorem.

Next, we extend the result (6.19) to a more general case taking c as a function of t. We state the following result:

Theorem 6.4 *Let X and Y be two non-negative continuous random variables and satisfying the proportional hazard model (PHM) (6.5) and if*

$$\xi(F,G;t) = c(t)\delta_F(t) \text{ , for } t \geq 0, \tag{6.22}$$

then

$$\delta_F(t) = \left[k + \left(\int_0^t \left\{\frac{c(x)-\beta}{\beta}\right\} e^{\frac{c(x)}{\beta}} dx\right)\right] e^{-\frac{c(t)}{\beta}}, \tag{6.23}$$

where $k = \delta_F(0)e^{\frac{c(0)}{\beta}}$.

Proof Substituting (6.22) in (6.15), we obtain

$$\xi'(F,G;t) = \lambda_F(t)\delta_F(t)\{c(t)-\beta\} \ . \tag{6.24}$$

Differentiating (6.22) w.r.t. t and substituting for $\xi'(F,G;t)$, from (6.24) we obtain

$$c'(t)\delta_F(t) + c(t)\delta'_F(t) = \lambda_F(t)\delta_F(t)\{c(t)-\beta\} \ .$$

Substituting $\lambda_F(t)\delta_F(t) = 1 + \delta'_F(t)$ in above expression and simplifying, we obtain

$$\delta'_F(t) + \frac{c'(t)}{\beta}\delta_F(t) = \frac{c(t)-\beta}{\beta}, \tag{6.25}$$

a linear differential equation in $\delta_F(t)$. Solving this we obtain (6.23).

Example 6.3 Let $c(t) = at + b$, $t > 0$ and $a > 0$. From (6.23), we obtain the general model with mean residual life function

$$\delta_F(t) = ke^{\frac{-(at+b)}{\beta}} + \frac{at - 2\beta + b}{a} - \frac{(b-2\beta)e^{\frac{-at}{\beta}}}{a}. \tag{6.26}$$

If $a = 0$, we obtain the characterization results given by Theorem 6.3 .

Remark 6.1 For $\beta = 1$, (6.26) reduces to

$$\delta_F(t) = k\ e^{-at-b} + \frac{b-2+at}{a} - \frac{(b-2)e^{-at}}{a},$$

a result given by Navarro et al. [86] in context with the cumulative residual entropy (6.3).

6.5 Dynamic Cumulative Past Inaccuracy Measure

Measures of uncertainty in context with past lifetime distributions have been studied extensively in the literature, refer to, Di Crescenzo and Longobardi [29, 30] Nanda and Paul [85]. We have also studied such measures in the proceeding chapters. For instance if at time t a system, which is observed only at certain preassigned inspection times, is found to be down, then the uncertainty of the system's life relies on the past, that is, at which instant in $(0,\ t)$ the system has failed. In this situation, the random variable ${}_tX = [X|X \leq t]$ is suitable to describe the time elapsed between the failure of a system and the time when it is found to be 'down'. The past lifetime random variable ${}_tX$ is related with two relevant aging functions, *the reversed hazard rate* defined by $\mu_F(x) = \frac{f(x)}{F(x)}$, and *the mean past lifetime* (MPT) defined by $\delta^*_F(t) = E(t - X|X < t) = \frac{1}{F(t)}\int_0^t F(x)dx$, which are further related as follows

$$\mu_F(t) = \frac{1 - \delta'^*_F(t)}{\delta^*_F(t)} \ , \tag{6.27}$$

where $\delta'^*_F(t) = \frac{d}{dt}\delta^*_F(t)$. For further results on reversed hazard rate function refer to Gupta and Nanda [51].

In analogy with the cumulative residual entropy (CRE) measure (6.3), based on the survival function $\bar{F}(x)$, Di Crescenzo and Longobardi [32] introduced and studied the cumulative entropy, defined as

$$\xi^*(F) = -\int_0^\infty F(x)\log F(x)dx, \tag{6.28}$$

based on the failure function $F(x)$.

A dynamic version of the cumulative entropy (6.28) given as

$$\xi^*(F;t) = -\int_0^t \frac{F(x)}{F(t)}\log\frac{F(x)}{F(t)}dx \ , \tag{6.29}$$

was also studied by Di Crescenzo and Longobardi [32].

Analogous to the Kerridge measure of inaccuracy (6.2), we propose a cumulative inaccuracy measure as

$$\xi^*(F;G) = -\int_0^\infty F(x)\log G(x)dx \ , \tag{6.30}$$

where $F(x)$ is the baseline distribution function and $G(x)$ can be considered as some reference distribution function. When these two distributions coincide, the measure (6.30) reduces to the measure (6.28) the cumulative entropy.

In case the two random variables X and Y satisfy the *proportional reversed hazard model* (PRHM), refer to Gupta et al. [50] , that is, if $\mu_G(x) = \beta\ \mu_F(x)$, or equivalently

$$G(x) = [F(x)]^\beta \ , \beta > 0 \ , \tag{6.31}$$

then obviously the cumulative inaccuracy measure (6.30) reduces to a constant multiple of the cumulative information measure (6.28).

The distribution function of the past lifetime random variable $[X|X \leq t]$ is given by

$$F_{{}_tX}(x) = \begin{cases} \frac{F(x)}{F(t)} & ; \quad \text{if } \ x < t \\ 1 & ; \quad \text{otherwise} \end{cases}$$

and similarly for $\bar{G}_t(x)$. Thus the cumulative inaccuracy measure analogous to the inaccuracy measure (6.30), for the past lifetime distribution is given by

$$\xi^*(F,G;t) = -\int_0^t F_{{}_tX}(x)\log G_{{}_tX}(x)dx,$$

$$= -\int_0^t \frac{F(x)}{F(t)}\log\frac{G(x)}{G(t)}dx \ . \tag{6.32}$$

We define the measure (6.32) as the *dynamic cumulative past inaccuracy* measure. When $t \to \infty$, the measure (6.32) reduces to (6.30).

Example 6.4 Let X be a non-negative random variable with distribution function $F_X(x) = x^2,\ 0 < x < 1$, and let the random variable Y be uniformly distributed over (0, 1) with distribution function given by $G_Y(x) = x$. Substituting these values in (6.32), we obtain the cumulative past inaccuracy measure as

$$\xi^*(F, G; t) = \frac{t}{9}\ .$$

Example 6.5 Let X and Y be two non-negative random variables having distribution functions respectively

$$F(x) = \begin{cases} \frac{x^2}{2}, & \text{for } \ 0 \le x < 1 \\ \frac{x^2+2}{6}, & \text{for } \ 1 \le x < 2 \\ 1 & \text{for } \ x \ge 2 \end{cases}$$

and

$$G(x) = \begin{cases} \frac{x^2+x}{4}, & \text{for } \ 0 \le x < 1 \\ \frac{x}{2}, & \text{for } \ 1 \le x < 2 \\ 1 & \text{for } \ x \ge 2. \end{cases}$$

The cumulative past inaccuracy measure is given by

$$\xi^*(F, G; t) = \begin{cases} \frac{2t}{9} - \frac{(t-2)}{6t} - \frac{1}{3t^2}\log(t+1), & \text{for } \ 0 < t < 1 \\ \frac{t}{9} + \frac{16t}{9(t^2+2)} - \frac{17}{18(t^2+2)} - \frac{18\log 2 + 24\log t}{18(t^2+2)}, & \text{for } \ 1 \le t < 2 \\ \log 2 + \frac{1}{6}\log 5 - \frac{41}{54} - \frac{8}{3}\tan^{-1}(\frac{1}{2}) & \text{for } \ t \ge 2. \end{cases}$$

Next, we study the characterization problem in case of the dynamic measure (6.32) under the proportional reversed hazard rate model (6.31). This is analogous to Theorem 6.2 proved in case of dynamic cumulative residual inaccuracy.

6.6 Characterization Results of Dynamic Cumulative Past Inaccuracy

We consider the characterization problem for the dynamic cumulative past inaccuracy measure under the proportional reversed hazard model (6.31). We state the following theorem.

Theorem 6.5 *Let X and Y be two non-negative random variables with distribution functions $F(.)$ and $G(.)$ satisfying the proportional reversed hazard rate model (6.31), and let, $\xi^*(F,G;t) < \infty, \forall\, t \geq 0$ be an decreasing function of t, then $\xi^*(F,G;t)$ determines the distribution function $F(.)$ uniquely.*

The proof is similar to that of Theorem 6.2, hence omitted.

Next, we characterize a specific distribution by using the dynamic cumulative past inaccuracy measure (6.32). The result is stated as follows.

Theorem 6.6 *Let $F(.)$ and $G(.)$ be two distribution functions satisfying the proportional reversed hazard model (6.31). The dynamic cumulative past inaccuracy measure*

$$\xi^*(F,G;t) = c\delta_F^*(t)\ ,\ 0 < c < \beta\ , \tag{6.33}$$

if, and only if $F(x) = \left(\frac{x}{b}\right)^{\frac{c}{\beta - c}}\ ,\ b > 0$.

Proof Rewriting (6.32) as

$$\xi^*(F,G;t) = -\frac{1}{F(t)}\int_0^t F(x)\log G(x)dx + \delta_F^*(t)\log G(t). \tag{6.34}$$

Substituting (6.31), this gives

$$\xi^*(F,G;t) = -\frac{\beta}{F(t)}\int_0^t F(x)\log F(x)dx + \beta\delta_F^*(t)\log F(t)\ . \tag{6.35}$$

Differentiating this w.r.t. t both sides, we obtain

$$\begin{aligned}\xi'^*(F,G;t) &= \beta\log F(t)[\delta_F'^*(t)-1] \\ &\quad + \beta\mu_F(t)\int_0^t \frac{F(x)}{F(t)}\log F(x)dx + \beta\mu_F(t)\delta_F^*(t).\end{aligned} \tag{6.36}$$

Substituting (6.27) and (6.35) in Eq. (6.36), we obtain

$$\xi'^*(F,G;t) = \mu_F(t)\{\beta\delta_F^*(t) - \xi^*(F,G;t)\}\ . \tag{6.37}$$

Let (6.33) be valid. Differentiating both sides w.r.t. t, we get

$$\xi'^*(F,G;t) = c\delta_F'^*(t)\ . \tag{6.38}$$

Substituting this in (6.37) , we get

$$c\delta_F'^*(t) = (\beta - c)\mu_F(t)\delta_F^*(t)\ . \tag{6.39}$$

Using (6.27) and simplifying, we obtain

$$\delta_F'^*(t) = \left(\frac{\beta - c}{\beta}\right) = 1 - \frac{c}{\beta}. \tag{6.40}$$

This gives

$$\delta_F^*(t) = \left(\frac{\beta - c}{\beta}\right)t\ . \tag{6.41}$$

Dividing (6.40) by (6.41), we obtain

$$\frac{1-\delta_F'^*(t)}{\delta_F^*(t)} = \mu_F(t) = \left(\frac{c}{\beta - c}\right)\frac{1}{t}\ . \tag{6.42}$$

Using the relationship between reversed hazard rate and distribution function is given by

$$F(x) = \exp\left[\int_0^x \mu_F(t)dt\right]\ ,$$

we obtain

$$F(x) = \left(\frac{x}{b}\right)^{\frac{c}{\beta - c}}, \ b > 0. \tag{6.43}$$

Conversely, when the distribution of X is specified by (6.43), using (6.35), we get

$$\xi^*(F, G; t) = -\frac{\beta}{F(t)} \int_0^t F(x) \log F(x) dx + \beta \delta_F^*(t) \log F(t)$$

$$= -\frac{\beta}{F(t)} \int_0^t \left(\frac{x}{b}\right)^k \log \left(\frac{x}{b}\right)^k dx + \left(\frac{\beta k t}{k+1}\right) \log \left(\frac{t}{b}\right).$$

After simplification, we obtain

$$\begin{aligned} \xi^*(F, G; t) &= \frac{\beta k t}{(k+1)^2} \\ &= \frac{\beta k \ \delta_F^*(t)}{k+1} = c \ \delta_F^*(t), \end{aligned}$$

where $\delta_F^*(t) = \frac{1}{F(t)} \int_0^t F(x) dx = \frac{t}{k+1}$. This prove the result.

Example 6.6 Let X and Y be two non-negative random variables satisfying the proportional reversed hazard model (PRHM) and let

$$f_X(x) = \begin{cases} a x^{a-1} & ; \quad \text{if } \ 0 \le x < 1, \ a > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The distribution function $F(x) = x^a$, and $G(x) = [F(x)]^{\beta}, \ \beta > 0$.
Substituting these values in (6.32), after simplification we get

$$\xi^*(F, G; t) = \frac{t}{(a+1)^2} = c \delta_F^*(t) \ ,$$

where $c = \frac{1}{a+1}$ and mean past lifetime is $\delta_F^*(t) = \frac{t}{a+1}$.

Next, we extend the result (6.33) to a more general case taking c as a function of t. We state the following result:

Theorem 6.7 *If X and Y satisfy the PRHM (6.31), and*

$$\xi^*(F,G;t) = c(t)\delta_F^*(t), \quad for \ t \geq 0, \tag{6.44}$$

then

$$\delta_F^*(t) = \left(\int_0^t \left\{\frac{\beta - c(x)}{\beta}\right\} e^{\frac{c(x)}{\beta}} dx\right) e^{-\frac{c(t)}{\beta}}. \tag{6.45}$$

The proof is similar to that of Theorem 6.4, hence omitted.

Example 6.7 Let $c(t) = at + b$, $t > 0$ and $a > 0$. From (6.45), we obtain the general model with mean inactivity time function

$$\delta_F^*(t) = \frac{2\beta - at - b}{a} + \frac{(b - 2\beta)e^{\frac{-at}{\beta}}}{a}. \tag{6.46}$$

For $\beta = 1$, (6.46) reduces to

$$\delta_F^*(t) = \frac{2 - at - b}{a} - \frac{(b-2)e^{-at}}{a},$$

a result in context with the cumulative entropy, refer to Di Crescenzo and Longobardi [32].

6.7 Conclusion

The distribution functions based measure of cumulative residual inaccuracy and cumulative past inaccuracy have been considered as natural extension of the distribution functions based dynamic entropy measures. The proposed cumulative inaccuracy measures determine the underlying distribution uniquely under PHM (for residual) and PRHM (for past) models; and also characterize certain specific probability distributions using relation between different reliability measure. It is expected that dynamic cumulative inaccuracy measures introduced in this chapter will further, extend the scope of study of information theoretic measures.

Chapter 7

Conclusion and Further Scope of Work

In this chapter we conclude the investigations carried through out this thesis and also give scope for further study which may be undertaken on the basis of the results reported.

7.1 Conclusion of the Work Reported

The concept of entropy $H(f)$ introduced by Shannon (1948) in the literature measures the average uncertainty associated with a random variable X with probability density function $f(.)$. For a component, which has survived up to time t, $H(f;t)$ measures the uncertainty about the remaining lifetime $[X|X \geq t]$. Observing that highly uncertain components are inherently not reliable, Ebrahimi and Pellery [40] have used the Shannon's residual entropy, as a measure of the stability of a component or a system. This approach seemed more realistic and has opened the applications of information- theoretic measures in the area of reliability.

Considering the importance of non-additive entropy measure we have proposed one parameter generalized residual entropy measure $H^{\alpha}(f;t)$ and have observed that the proposed measure determines the distribution function uniquely. Further we have seen that it characterizes three specific lifetime distributions.

Next, we have extended the scope of dynamic entropy measures to the concept of inaccuracy measure given by Kerridge (1961). The dynamic inaccuracy measures, both residual and past, can be employed respectively under proportional hazard model (PHM) and proportional reversed hazard model (PRHM) to characterize specific lifetime distributions.

The concept of weighted distributions and hence that of weighted information measures is of wide interest when a stochastic process is recorded with some weight function. The dynamic inaccuracy measures, both residual and past, find a natural extension to the corresponding length biased residual and past inaccuracy measures. These measures also characterize the underlying distribution uniquely.

The cumulative distribution function based measures of entropy $\xi(X)$ are in general more stable in comparison to probability density function based measure $H(f)$ given by Shannon in (1948). The concept of *cumulative residual entropy(CRE)* given by Rao et al. in (2004) has been extended to cumulative residual entropies with one parameter and two parameters and further to their dynamic versions viz. $\xi_{\alpha}(X;t)$ and $\xi_{\alpha}^{\beta}(X;t)$. The dynamic cumulative entropy functions determine the distribution function uniquely. The exponential, the Pareto and the finite range distributions which are commonly used in the reliability modeling have been characterized in terms of the proposed generalized dynamic cumulative entropy measures.

The distribution function based dynamic measures of cumulative residual inaccuracy and cumulative past inaccuracy have been considered as natural extension of distribution function based dynamic entropy measures. The proposed cumulative inaccuracy measures determine the underlying distribution uniquely under PHM (for residual) and PRHM (for past) models; and also characterize certain specific probability distributions using relation between different reliability measure.

7.2 Scope For Future Study

During the present investigation several ideas have originated which have the potential to extend the study further. The work reported in this thesis can be extended to bivariate and multivariate domains. The problem of extending the concept of the cumulative residual entropy (CRE) function to higher dimensions is yet to be examined. Characterizations of some bivariate distributions based on the functional form of the bivariate cumulative residual entropy function can be obtained analogous to that of bivariate failure rate.

In comparison to the quantum of work done on cumulative residual entropy in the continuous case, a little work seems to have been done in discrete domain. We can consider the dynamic measure proposed further for discrete cases, since practically discrete cases are suitable from applications point of view. Further the discrete measures of the dynamic version proposed can possibly find applications in image processing and information retrieval etc.

Another domain which can be explored in this context is that of order statistics. A number of researchers like Wong and Chen [128], Ebrahimi et al. [41], Baratpour et al. [14], Agrahimi et al. [6] and Zarezadeh and Asadi [131] are working in the area of information theoretic measures in order statistics. We can study the dynamic information measures and dynamic inaccuracy measures in the context of order statistics; also we can study the scope of measures of cumulative residual entropy in order statistics.

Bibliography

[1] Abbasnejada, M. Arghamia, N. R. Morgenthalerb, S. Borzadarana, G. R. M. (2010). On the dynamic survival entropy, Statistics and Probability Letters, **80** (23), 1962-1971.

[2] Abraham, B. and Sankaran, P. G. (2005). Renyi's entropy for residual life distribution, Statistical Papers, **46**, 17-30.

[3] Aczel, J. and Daroczy, Z. (1975). On Measures of Information and Their Characterizations, Academic Press, New York.

[4] Aggarwal, N. L. and Picard C. F. (1976). Functional equations and information measures with preferences, Kybernetika, **3**, 174-181.

[5] Arimoto, S. (1971). Information-theoretical considerations on estimation problems, Information and Control, **19**, 181-194.

[6] Arghami, N. R. and Abbasnejad, M. (2011). Renyi entropy properties of order statistics. Communications in Statistics-Theory and Methods, **40**, 40-52.

[7] Asadi, M. and Bayramoglu, I. (2005). A note on the MRL function of a parallel system. Communications in Statistics-Theory and Methods **34** (2), 475-484.

[8] Asadi, M. and Ebrahimi, N. (2000). Residual entropy and its characterizations in terms of hazard function and mean residual life function, Statistics and Probability Letters, **49**, 263-269.

[9] Asadi, M. Ebrahimi, N. and Soofi, E. (2005). Dynamic generalized information measures, Statistics and Probability Letters, **71**, 89-98.

[10] Asadi, M. Ebrahimi, N. Hamedani, G. G. and Soofi, E. S. (2004). Maximum dynamic entropy models, Journal of Applied Probability, **41** (2), 379-390.

[11] Asadi, M. and Zohrevand, Y. (2007). On the dynamic cumulative residual entropy, Journal of Statistical Planning and Inference, **137** (6), 1931-1941.

[12] Baig, M. A. K. and Dar, J. G. (2008). Generalized residual entropy function and its applications, European Journal of Pure and Applied Mathematics, **4**, 30-40.

[13] Barlow, R. E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing, Holt, Rinehart and Winston, New York.

[14] Baratpour, S. Ahmadi, J. Arghami, N. R. (2007). Some characterizations based on entropy of order statistics and record values. Communications in Statistics-Theory and Methods, 36, 47-57.

[15] Belis, M. and Guiasu, S. (1968). A qualitative measure of information in cybernatic systems, IEEE Transactions of Information Theory, **4**, 593-94.

[16] Belzunce, F. Navarro, J. Ruiz, J. M. and Aguila, Y. (2004). Some results on residual entropy function, Metrika , **59**, 147-161.

[17] Bhatia, P. K. and Taneja, H. C. (1990). On characterization of quantitative qualitative measure in inaccuracy, Information Sciences, **51**, 135-141.

[18] Block, H. W. Savits, T. H. Singh, H. (1998). The reversed hazard rate function, Probability in the Engineering and Informational Sciences, **12**, 69-90.

[19] Boghosian, B. M. (1996). Thermodynamic description of the relaxation of two dimensional turbulence using Tsallis statistics, Physical Review E, **53**, 4754-4763.

[20] Chandra, N. K. and Roy, D. (2001). Some results on reversed hazard rate, Probability in the Engineering and Informational Sciences, **15**, 95-102.

[21] Chandra, N. K. and Roy, D. (2005). Classification of distributions based on reversed hazard rate. Calcutta Statistical Association Bulletin, **56**, 221-224.

[22] Compte, A. and Jou, D. (1996). Non-equilibrium thermodynamics and anomalous diffusion, Journal of Physics A: Mathematical and General, **29**, 4321-4329.

[23] Cover, T. M. and Thomas, J. (1991). The Elements of Information Theory, John Wiley and Sons.

[24] Cox, D. R. (1959). The analysis of exponentially distributed lifetimes with two type of failure, Journal of the Royal Statistical Society, Series B, **21**, 411-421.

[25] Cox, D. R. (1962). Renewal Theory, Methuen, London.

[26] Cox, D. R. (1972). Regression models and life tables, Journal of the Royal Statistical Society, Series B, **34**, 187-220.

[27] Daniels, H. E. (1942). A new technique for the analysis of fiber length distribution in wool, Journal of Textile Institute, **33**, 137-150.

[28] Di Crescenzo, A. (2000). Some results on the proportional reversed hazard model, Statistics and Probability Letters, **50**, 313-321.

[29] Di Crescenzo, A. and Longobardi, M. (2002). Entropy-based measure of uncertainty in past lifetime distributions, Journal of Applied Probability, **39**, 434-440.

[30] Di Crescenzo, A. and Longobardi, M. (2004). A measure of discrimination between past lifetime distributions, Statistics and Probability Letters, **67**, 173-182.

[31] Di Crescenzo, A. and Longobardi, M. (2006). On weighted residual and past entropies, Scientiae Mathematicae Japonicae, **64** (3), 255-266.

[32] Di Crescenzo, A. and Longobardi, M. (2009). On cumulative entropies, Journal of Statistical Planning and Inference, **139** (12), 4072-4087.

[33] Drissi, N. Chonavel, T. and Boucher J. M., Generalized cumulative residual entropy for distributions with unrestricted supports, Research Letters in Signal Processing, doi:10.1155/2008/790607.

[34] Ebrahimi, N . (1996). How to measure uncertainty in the residual life time distribution, Sankhya A, **58**, 48-56.

[35] Ebrahimi, N. (1997). Testing whether lifetime distribution is decreasing uncertainty, Journal of Statistical Planning and Inference, **64**, 9-19.

[36] Ebrahimi, N. (2000). The maximum entropy method for lifetime distributions, Sankhya A, **62**, 236-243.

[37] Ebrahimi, N. and Kirmani, S.N.U.A. (1996 a). A characterization of the proportional hazards model through a measure of discrimination between two residual life distributions, Biomertika, **83** (1), 233-235.

[38] Ebrahimi, N. and Kirmani, S.N.U.A. (1996 b). A measure of discrimination between two residual life time distributions and its applications, The Annals of Mathematical Statistics, **48**, 257-265.

[39] Ebrahimi, N. and Kirmani, S.N.U.A. (1996). Some result on ordering of survival function through uncertainty, Statistics and Probability Letters, **29**, 167-176.

[40] Ebrahimi, N. and Pellerey, F. (1995). New partial ordering of survival functions based on the notion of uncertainty, Journal of Applied Probability, **32**, 202-211.

[41] Ebrahimi, N. Soofi, E. S. Zahedi, H. (2004). Information properties of order statistics and spacings. IEEE Transactions of Information Theory, 50, 177-183.

[42] Efron, B. (1981). Censored data and the bootstrap, Journal of the American Statistical Association, **76**, 312-319.

[43] Ferreri, C. (1980). Hypoentropy and related heterogeneity divergence measures, Statistica, **40**, 55-118.

[44] Finkelstein, M. S. (2002). On the reversed hazard rate, Reliability Engineering and System Safety, **78**, 71-75.

[45] Furman, E. and Zitikis, R. (2008). Weighted premium calculations principles, Insurance; Mathematics and Economics **42**, 459-465.

[46] Gupta, R. C. and Gupta, R. D. (1998). Proportional reversed hazard rate model and its applications, Journal of Statistical Planning and Inference, **137** (11), 3525-3536.

[47] Gupta, R. C. and Keating, J. P (1986). Relations for reliability measures under length biased sampling, Scandinavian Journal of Statistics, **13**, 49-56.

[48] Gupta, R. C. and Kirmani , S.N.U.A. (1990). The role of weighted distributions on stochastic modeling, Communications in Statistics-Theory and Methods, **19**, 3147- 3162.

[49] Gupta, R. C. and Kirmani, S.N.U.A. (2004). Some characterization of distributions by functions of failure rates and mean residual life, Communications in Statistics- Theory and Methods, **33**, 3115-3131.

[50] Gupta, R. C. Gupta, P. L. and Gupta, R. D. (1998). Modeling failure time data with lehman alternative, Communications in Statistics-Theory and methods, **27**, 887-904.

[51] Gupta, R. D. and Nanda, A. K. (2001). Some results on reversed hazard rate ordering, Communications in Statistics -Theory and Methods, **30**, 2447-2457.

[52] Gupta, R. C. and Wu, H. (2001). Analyzing survival data by proportional reversed hazard model, International Journal of Reliability and Applications, **2** (1), 126.

[53] Gurdial, F. Pessoa (1977). On "useful" information of order α, Journal of Combinatorics Information and System Sciences, 151-155.

[54] Hall, W. J. and Wellner, J. A. (1981). Mean residual life. In Statistics and Related Topics (Csrg, et al. eds), North-Holland, Amsterdam, 169-184.

[55] Hamity, V. H. and Barraco, D. E. (1996). Generalized non-exitensive thermodynamics applied to the cosmical background radiation in Robertson-Walker Universe, Physical Review Letters, **76**, 466-64.

[56] Hartly, R. T. V. (1928). Transmission of information, Bell System Technical Journal, **7**, 535-563.

[57] Havrda, J. and Charvat, F. (1967). Quantification method of classification process: concept of structural $\alpha-$entropy, Kybernetika **3**, 30-35.

[58] Hooda, D. S and Saxena, S. (2011). Generalized measures of discrimination between past life time distributions, Pakistan Journal of Statistics and Operation Research, **2**, 233-243.

[59] Jain, K. C. and Srivastava, A. (2009) Some new weighted information generating functions of discrete probability distributions. Journal of Applied Mathematics, Statistics and Informatics **5,** 15-22.

[60] Ion, D. B. and Ion, M. L. D. (1999). Optimal bounds for Tsallis-like entropies in quantum scattering, Physical Review Letters, **83**, 4634-4667.

[61] Jain, K. Singh, H. and Bagai, I. (1989). Relations for reliability measures of weighted distributions, Communications in Statistics-Theory and Methods, **18** (2), 4393-4412.

[62] Johnson, D. H. Glantz, R. M. (2004). When does interval coding occur?, Neurocomputing, **58** (60), 13-18.

[63] Kannappan, P. I. and Rathie, P. N. (1973). On characterization of directed divergence, Information and Control, **22**, 163-171.

[64] Kapur, J. N. (1968). On information of order α and β, Proceeding of the Indian Academic Sciences, **48** (A), 65-75.

[65] Kapur, J. N. (1994). Measure of Information and Their Applications, Wiley Eastern Ltd., New Delhi.

[66] Kayid, M. and Ahmad, I. A. (2004). On the mean inactivity time ordering with reliability applications, Probability in the Engineering and Informational Sciences, **18** (3), 395-409.

[67] Kerridge, D. F. (1961). Inaccuracy and inference, Journal of the Royal Statistical Society, Series B, **23**, 184-194.

[68] Keilson, J. and Sumita, V. (1982). Uniform stochastic ordering and related inequalities, The Canadian Journal of Statistics, **10** (3), 181-191.

[69] Khinchin, A. J. (1957). Mathematical Foundation of Information Theory, Dover, New York .

[70] Kullback, S. and Leibler, R. A. (1951). On information and sufficiency, Annals of Mathematics Statistics, **22**, 79-86.

[71] Kullback, S. (1959). Information Theory and Statistics, Wiley, New York.

[72] Kumar, V. Taneja, H. C. and Srivastava, R. (2010). A dynamic measure of inaccuracy between two past lifetime distributions, Metrika, **74** (1), 1-10.

[73] Kundu, C. Nanda, A. K. and Maiti, S. S. (2010). Some distributional results through past entropy, Journal of Statistical Planning and Inference, **140** (5), 1280-1291.

[74] Lehmann, E. L. (1953). The power of rank test, Annals of Mathematical Statistics, **24**, 23-43.

[75] Leemis, L. M. and Beneke, M. (1990). Burn-In models and methods: A Review, IIE Transactions, **22**, 172-180.

[76] McEliece, R. J. (1977). The theory of information and coding, A Mathematical Framework For Communication. Addison-Wesley, Reading, Mass.

[77] Mukherjee, S. P. and Roy, D. (1986). Some characterizations of the exponential and related life distributions, Calcutta Statistical Association Bulletin, **36**, 189-197.

[78] Muth, E. J. (1977). The null distribution for a test of constant versus bathtub failure rate, Scandinavian Journal of Statistics, **12**, 55-61.

[79] Nath, P. (1968). Inaccuracy and coding theory, Metrika, **13**, 123-135.

[80] Nair, N. U. and Gupta, R. P. (2007). Characterization of proportional hazard models by properties of information measures, International Journal of Statistical Sciences, **6**, 223-231.

[81] Nair, N. U. Sankaran, P. G. and Asha, G. (2005). Characterizations of distributions using reliability concepts, Journal of Applied Statistical Science, **14**, 237-242.

[82] Nanda, A. K. and Das, S. (2011). Dynamic proportional hazard rate and reversed hazard rate models, Journal of Statistical Planning and Inference, **141** (6), 2120-2127.

[83] Nanda, A. K. and Jain, K. (1999). Some weighted distribution result on univariate and bivariate cases, Journal of Statistical Planning and Inference, **77**, 169-180.

[84] Nanda, A. K. and Paul, P. (2006). Some properties of past entropy and their applications, Metrika, **64** (1), 47-61.

[85] Nanda, A. K. and Paul, P. (2006). Some results on generalized residual entropy, Information Sciences, **176**, 24-47.

[86] Navarro, J. Aguila, Y. and Asadi, M. (2010). Some new results on the cumulative residual entropy, Journal of Statistical Planning and Inference, **140** (1), 310-322.

[87] Navarro, J. Sunoj, S. M. and Linu, M. N. (2011). Characterizations of bivariate models using dynamic Kullback-Leibler discrimination measures, Statistics and Probability Letters, **81**, 1594-1598.

[88] Nyquist, H. (1924). Certain factors affecting telegraph speed, Bell System Technical Journal, **3**, 324-346.

[89] Nyquist, H. (1928). Certain topics in telegraph transmission theory, Trans. AIEE, **47**, 617-644.

[90] Oluyede, B. O. (1999). On inequalities and selection of experiments for length-biased distributions, Probability in the Engineering and Informational Sciences, **13**, 169-185.

[91] Oluyede, B. O. (2000). On some length-biased inequalities for reliability measure, Journal of Inequality and Applications, **5**, 447-466.

[92] Oluyede, B. O. (2002). Some inequalities and bounds for weighted reliability measures, Journal of Inequalities in Pure and Applied Mathematics, **3** (4), Article 60.

[93] Parkash, O. Sharma, P. K. and Mahajan, R. (2008). New measures of weighted fuzzy entropy and their applications for the study of maximum weighted fuzzy entropy principle, Information Sciences, **178**, 2389-2395.

[94] Patil, G. P. and Rao, C. R. (1978). Weighted distributions and size- biased sampling with applications to wildlife and human families, Biometrics, **34**, 179-189.

[95] Patil, G. P. and Ord, J. K. (1976). On size-biased sampling and related form invariant weighted distributions, Sankhya B, **38**, 48-61.

[96] Rao, C. R. (1965). On discrete distributions arising out of methods of ascertainment, In classical and contagious discrete distributions, G. P. Patil (ed.), Calcutta India Pergamon Press and Statistical Publishing Society, 320-332.

[97] Rao, M. (2005). More on a new concept of entropy and information, Journal of Theoretical Probability, **18**, 967-981.

[98] Rao, M. Chen, Y. Vemuri, B. C. and Wang, F. (2004). Cumulative residual entropy: a new measure of information, IEEE Transactions on Information Theory, **50**, 1220-1228.

[99] Rajesh, G. and Nair, K. R. M. (1998). Residual entropy function in discrete time, Far East Journal of Theoretical Statistic, **2** (l), 1-10.

[100] Rathie, P. N. (1971). On some measures of uncertainty, inaccuracy and information and their characterizations, Kybernetika, **7**, 394-403.

[101] Renyi, A. (1961). On measures of entropy and information, Proceeding of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, university of California Press, Berkeley, 547-561.

[102] Roy, D. (2002). A characterization of model approach for generating bivariate life distributions using reversed hazard rates, Journal of the Japan Statistical Society, **32** (2), 239-245.

[103] Ruiz, J. M. and Navarro, J. (1994). Characterization of distributions by relationships between failure rate and mean residual life, IEEE Transactions on Reliability, **43** (4), 640-644. 9

[104] Sankaran, P. G. and Gleeja, V. L. (2006). On bivariate reversed hazard rates, Journal of the Japan Statistical Society, **36** (2), 213-224.

[105] Sankaran, P. G. and Gupta, R. P. (1999). Characterization of lifetime distributions using measure of uncertainty, Calcutta Statistical Association Bullettin, 154-166.

[106] Sankaran, P. G. and Nair, N. U. (1993). On form-invariant length-biased models from pearson family, Journal of the Indian Statistical Association, **31**, 35-89.

[107] Schneider, T. D. (1997). Information content of individual genetic sequences, Journal of Theoretical Biology, 427-441.

[108] Sengupta, D. Singh, H. and Nanda, A. K. (1999). The proportional reversed hazards model, Technical Report, Applied Statistics Division, Indian statistical Institute, Kolkata.

[109] Shannon, C. E. (1948). A mathematical theory of communication, Bell System Technical Journal, 279-423 and 623-656.

[110] Sharma, B. D. and Taneja, I. J. (1975). Entropy of type (α, β) and other generalized measures in information theory, Metrika, **22**, 205-215.

[111] Sullo, P. and Rutherford, D. K. (1977). Characterizations of the power distribution by conditional excedence, The American Statistical Association Proceedings of the Business and Economics, 787-792.

[112] Sunoj, S. M. and Linu, M. N. (2012). Dynamic cumulative residual Renyi's entropy, Statistics, **46** (1), 41-56.

[113] Sunoj, S. M. Sankaran, P. G. and Maya, S. S. (2009). Characterizations of life distributions using conditional expectations of doubly (interval) truncated random variables, Communications in Statistics-Theory and Methods, **38**, 1441-1452.

[114] Swartz, G. B. (1973). The mean residual life time function, IEEE Transactions on Reliability, **22**, 108-109.

[115] Taneja, I. J. (1995). New Developments in Generalized Information Measures, Chapter in: Advances in Imaging and Electron Physics, Ed. P.W. Hawkes, **91**, 37-136.

[116] Taneja, I. J. (2001). Generalized Information Measures and Their Applications, ONLINE http://www.mtm.ufsc.br/ taneja/book/book.html.

[117] Taneja, H. C. Kumar, V. and Srivastava, R. (2009). A dynamic measure of inaccuracy between two residual lifetime distributions, International Mathematical Forum, **4** (25), 1213-1220.

[118] Taneja, H. C. and Tuteja, R. K. (1984). Characterization of a quantitative-qualitative of relative information, Information Sciences, **33**, 217-222.

[119] Taneja, H. C. and Tuteja, R. K. (1984). Characterization of a quantitative-qualitative measure of inaccuracy, Kybernetika, **22** (5), 393-402.

[120] Taneja, H. C. (1985). On measure of relative useful information, Kybernetika, **21** (2), 148-156.

[121] Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics, Journal Statistical Physics, **52**, 479-487.

[122] Tsallis, C. and Brigatti, E. (2004). Non-extensive statistical mechanics: a brief introduction, Continuum Mechanics and Thermodynamics, **16**, 223-235.

[123] Verma, R. S. (1966). Generalizations of Renyi's entropy of order α, Journal of Mathematical Sciences, **1**, 34-48.

[124] Wang, F. and Vemuri, B. C. (2007). Non-Rigid multi-modal image registration using cross-cumulative residual entropy, International Journal of Computer Vision, **74** (2), 201-215.

[125] Weiner, N. (1948). Cybernetics, Wiley, New York.

[126] Wells (2009). Applied Coding and Information Theory for Engineers. Pearson Education.

[127] Whittle, A. J. and Tennakoon, J. (2005). Predicting the residual life of PVC sewer pipes, Plastics Rubber and Composites, **34** (7), 311-317.

[128] Wong, K. M. Chen, S. (1990). The entropy of ordered sequences and order statistics. IEEE Transactions of Information Theory, **36**, 276-284.

[129] Xekalaki, E. (1983). Hazard function and life distributions in discrete time, Communications in Statistics-Theory and Methods, **12**, 2503-2509.

[130] Xie, M. Gaudoin, O. and Bracquemond, C. (2002). Redefining failure rate function for discrete distributions, International Journal of Reliability Quality Safety Engineering, **9**, 275-285.

[131] Zarezadeh, S. Asadi, M. (2010). Results on residual Renyi entropy of order statistics and record values. Information Sciences, **180**, 4195-4206.

[132] Zografos, K. and Nadarajah, S. (2005). Survival exponential entropies, IEEE Transactions on Information Theory, **51**, 1239-1246.

List of Publications

1. HC Taneja, Vikas Kumar and R. Srivastava, *A Dynamic Measure of Inaccuracy Between Two Residual Lifetime Distribution*, **International Mathematical Forum**, 2009, 4 (25), 1213-1220.

2. Vikas Kumar, HC Taneja and R. Srivastava, *A Dynamic Measure of Inaccuracy Between Two Past Lifetime Distribution*, **Metrika**, 2010, 74 (1), 1-10.

3. Vikas Kumar, HC Taneja and R. Srivastava, *Length Biased Weighted Residual Inaccuracy Measure*, **Metron**, 2010, LXIII (2), 153-160.

4. Vikas Kumar, HC Taneja and R. Srivastava, *Non-additive Entropy Measure Based Residual Lifetime Distributions*, **JMI International Journal of Mathematical Sciences**, 2010, 1 (2), 1 - 9.

5. Vikas Kumar and HC Taneja, *Some Characterization Results on Generalized Cumulative Residual Entropy Measure*, **Statistics Probability Letters**, 2011, 81 (8), 72-77.

6. Vikas Kumar, HC Taneja and R. Srivastava, *On Dynamic Renyi Cumulative Residual Entropy Measure*, **Journal of Statistical Theory and Applications**, 2011, 10 (3), 491-500.

7. Vikas Kumar and HC Taneja, *A Generalized Entropy Based Residual Lifetime Distributions*, **International Journal of Biomathematics**, 2011, 4 (2), 1-14.

8. Vikas Kumar and HC Taneja, *On Length Biased Dynamic Measure of Past Inaccuracy*, **Metrika**, 2012, 75 (1) 73-84.

9. HC Taneja and Vikas Kumar, *On Dynamic Cumulative Residual Inaccuracy Measure*, Proceeding at *World Congress of Engineering*,WCE- Vol. I, 2012, held at London, U.K., July 06-08.

10. Vikas Kumar and HC Taneja, *Dynamic Cumulative Residual and Past Inaccuracy Measure*, (Under Review) 2012.

Papers Presented in International Conferences

1. Vikas Kumar and HC Taneja, *On weighted past inaccuracy measure.* Presented at International Conference in Mathematics and Applications (ICMA-MU) held at Bangkok, Dec. 17-21, 2009.

2. HC Taneja and Vikas Kumar, *Length biased weighted residual inaccuracy measure.* Presented at International Congress of Mathematicians (ICM) held at Hyderabad, August 19-27, 2010.

3. Vikas Kumar, HC Taneja and R. Srivastava, *On dynamic Renyi cumulative residual entropy measure.* Presented at International Congress of Mathematicians (ICM) held at Hyderabad, August 19-27, 2010.

4. HC Taneja and Vikas Kumar, *On dynamic cumulative residual inaccuracy measure.* Presented at World Congress of Engineering (WCE) held at London, U.K., July 06-08, 2011.

5. Vikas Kumar and HC Taneja, *Generalized dynamic cumulative residual entropy.* Presented at International Conference on Statistics, Probability and Related Areas held at Cochin University of Science and Technology, Kerla, Dec. 19-22, 2011 .