SOME PROBLEMS IN APPROXIMATION FOR CERTAIN DISCRETE AND INTEGRAL OPERATORS

by

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Submitted
in fulfilment of the requirements of the degree of
Doctor of Philosophy
to the

DELHI TECHNOLOGICAL UNIVERSITY
BAWANA ROAD, DELHI-110 042
INDIA

DECEMBER 2014

CERTIFICATE

This is to certify that the thesis entitled **"Some Problems in Approximation for Certain Discrete and Integral Operators"** submitted to the Delhi technological University, Delhi for the award of **Doctor of Philosophy** is based on the original research work carried out by me under the supervision of Dr. Naokant Deo, Department of Applied Mathematics, Delhi Technological University, Delhi. It is further certified that the work embodied in this thesis has neither partially nor fully submitted to any other university or institution for the award of any degree or diploma.

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Acknowledgement

I humbly take the opportunity to thank all the people who have been associated directly or indirectly, throughout my research work. Firstly, I would like to thank the Almighty God for his blessings bestowed upon me which enabled me to achieve this place by providing me everything I needed.

I express my sincere gratitude to Dr. Naokant Deo my Ph.D. supervisor, for his valuable guidance and untiring encouragement. He had always been extremely generous with his precious time, knowledge and ideas and allowed me great freedom in this research work. Perhaps, I would not have been able to accomplish my work without his constant inspiring criticism, motivation and encouragement. His unflinching courage and conviction will always inspire me.

I also express my regards to Prof. H. C. Taneja, Head of the Department, for his support. My special thanks to my SRC members, Prof. Vijay Gupta and Prof. P.N. Agarwal for spending their invaluable time during the discussions over seminars. I owe a great deal of appreciation and gratitude to Prof. Vijay Gupta who willingly devoted time in giving guidance to me.

I would also like to extend huge warm thanks to my batchmates Vijay and Virendra for their wonderful companionship, help and support which buttressed me to perform work comfortably.

My special and heartfelt thanks to my friend Karunesh for encouraging and supporting me in difficult situations through the research period.

Last but not the least, I would like to pay high regards to my parents and entire family for their sincere encouragement and inspiration throughout my research work and lifting me uphill this phase of life. Besides this several people have knowingly and unknowingly helped me in the successful completion of my research endeavor.

Above all my heartfelt thanks to my husband Prakash Gaurav for his cooperation and support and to my little princess Kaashvi for her patience by giving me maximum time in completing this research work.

Delhi (Neha Bhardwaj)

December 10, 2014

Abstract

In this thesis, the problems that we study are with respect to the approximation and error estimation of the linear positive operators. The techniques of simultaneous approximation and King type modification have been applied successfully to improve the order of approximation for various operators.

Firstly, some theorems on approximation of the r-th derivative of a given function f by corresponding r-th derivative of the Durrmeyer variant of generalized Bernstein operator have been studied by contracting the interval of the definition of integrability of function from class $[0, 1]$ to $[0, 1 - \frac{1}{n+1}]$.

The basic properties and Voronoskaya type results for the ordinary approximation for modified Baskakov operators and Balázs operators have been studied and the results for better error estimation after considering King type modification of these operators have been obtained. Some results have been calculated for multidimensional Bernstein operators and its Durrmeyer variant. Quantitative global estimates for generalized double Baskakov operators have been studied. In the sequel, direct and inverse theorems for Beta Durrmeyer operators have been obtained.

In the end, some approximation properties of modified Beta operators and an operator introduced by Jain with the help of Poisson type distribution have been studied, which include rate of convergence and statistical convergence.

Contents

Introduction

0.1 General Introduction

The theory of approximation is an area of mathematical analysis, which at its core, is concerned with the approximations of functions by simpler and more easily calculated functions. The basis of the theory of approximation of functions of a real variable is the theorem discovered by K. Wierstrass in 1885, which asserts that for any continuous function f on the finite interval $[a, b]$, there exists a sequence of polynomials which converges uniformly to f on $[a, b]$. In 1912, S.N. Bernstein gave a simple and elegant proof of Weierstrass theorem, constructing by probabilistic methods, a sequence of polynomials that converges uniformly to the function to be approximated. Thus, introduced the Bernstein operators. These operators belong to the class of positive linear operators.

In the 50s, the theory of approximation of functions by positive linear operators developed a lot when T. Popoviciu(1951), H. Bohman(1952) and P.P. Krovokin(1953) [93] discovered independently, a simple and easily applicable criterion to check if a sequence of positive linear operators converges uniformly to the function to be approximated. This criterion says that the necessary and sufficient condition for the uniform convergence of the sequence P_n of positive linear operators to the continuous function f on the compact interval $[a, b]$, is the uniform convergence of the sequence $P_n f$ to f for the only three functions $e_k(x) = x^k, k = 0, 1, 2$. If the domain of definition of function f is unbounded, then the result remains valid only for the continuous function having a finite limit at infinity.

To the theorem of Popoviciu-Bohman-Krovokin to continuous and unbounded functions defined on $[0, \infty)$, some bounds on the functions must be required, which was first noted by Z. Ditzian. In 1974, A.D. Gadjiev introduced the weighted space $C_\rho(I)$, which is the set of all continuous function f on the interval $I \subset \mathbb{R}$ for which there exists a constant $M > 0$ such that $|f(x)| \leq M\rho(x)$, for every $x \in I$, where ρ is a positive continuous function called weight. This space is a Banach space, endowed with the norm

$$||f||_\rho = \sup_{x \in I} \frac{|f(x)|}{\rho(x)}.$$

The Krovokin type theorem found by Gadjiev is the following:

Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing, continuous and unbounded function and set $\rho(x) = 1 + \varphi^2(x)$; the sequence of positive linear operators $P_n : C_\rho [0, \infty) \to C_\rho [0, \infty)$ verifies

$$\lim_{n \to \infty} \left\| P_n \varphi^i - \varphi^i \right\|_\rho = 0, i = 0, 1, 2.$$

Then, $\lim\limits_{n \to \infty} \|P_n f - f\|_\rho = 0$, for every function $f \in C_\rho[0, \infty)$ for which the limit $\lim\limits_{x \to \infty} \frac{f(x)}{\rho(x)}$ exists and finite.

Some of the general problems arising in approximation of operators are: (i) the basic convergence in approximation (ii) various estimates of error for classes of functions (iii) a precise rate of convergence for smooth functions (iv) the possibility of simultaneously approximating the derivatives of operators. The recognition of the basic role of linear positive operators triggered a virtual chain reaction in approximation theory.

0.2 Preliminary and Auxiliary Results

Given a non-empty set X, we denote by $B(X)$ the space of all real valued bounded functions defined on X, endowed with the norm of the uniform convergence (or the sup-norm) defined by

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f \in B(X).$$

The set $B(X)$ is a linear subspace of $\mathbb{R}^X$.

If X is a topological space, then $C(X)$ denotes the space of all real-valued continuous functions on X. Furthermore, we set

$$C_B(X) = C(X) \cap B(X).$$

If X is a topological space, then $B(X)$ and $C_B(X)$ endowed with the sup-norm, are Banach spaces.

If X is a topological compact space, then $C(X) = C_B(X)$.

Let X, Y be two linear spaces of real functions. The mapping $L : X \to Y$ is called a linear operator if and only if $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \forall f, g \in X$ and $\alpha, \beta \in \mathbb{R}$.

The operator L is called positive if and only if $Lf \geq 0, \forall f \geq 0, f \in X$.

The set $L(X, Y) = \{L : X \to Y | \ L \ \textit{is a linear operator}\}$ is a real vector space.

Let $L : X \to Y$ be a linear positive operator, it has following properties:

1. If $f, g \in X$ with $f \leq g$, then $Lf \leq Lg$.
2. $\forall f \in X$ we have $|Lf| \leq L|f|$.

The main tools to measure the degree of approximation by linear positive operators are moduli of smoothness and K-functionals. Let f belongs to $C_B(I), I \subset \mathbb{R}$. The modulus of continuity $\omega(f; .) \in \mathbb{R}^{[0,\infty)}$ of the function f is defined by

$$\omega(f; \delta) \in \{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\} \tag{0.2.1}$$

for every $\delta \geq 0$. We recall the forward differences of a function $f : M \to \mathbb{R}$, where $M = \{a_k | a_k = x + kh, k = (0, r), x, h \in \mathbb{R}, h \neq 0\}$, $\triangle_h^1 f(x) = f(x + h) - f(x)$ and $\triangle_h^r f(x) = \triangle_h^1(\triangle_h^{r-1} f)(x), \ \ r \geq 2$.

Let $f \in C_B(I)$, the following properties of modulus of continuity take place:

1. $\omega(f;.) \geq 0$,
2. $\omega(f;0) = 0$,
3. $\omega(f;.)$ is non-decreasing,
4. $\omega(f;.)$ is sub-additive,
5. $\omega(f;.)$ is uniform continuous,
6. $\forall \delta \geq 0, \forall n \in \mathbb{N}, \omega(f;n\delta) \leq n\omega(f;\delta)$,
7. $\forall \delta \geq 0, \forall \lambda > 0, \omega(f;\lambda\delta) \leq (\lambda+1)\omega(f;\delta)$,
8. $\forall \delta > 0, |f(y) - f(x)| \leq \left(1 + \delta^{-2}(y-x)^2\right)\omega(f;\lambda\delta) \leq (\lambda+1)\omega(f;\delta)$,
9. $\forall f_1, f_2 \in C_B(I), \omega(f_1 f_2;\delta) \leq \|f_1\|\,\omega(f_2;\delta) + \|f_2\|\,\omega(f_1;\delta)$,
10. If $\alpha \in (0,1]$ then $f \in Lip_{M^\alpha} \Leftrightarrow \omega(f;\delta) \leq M\delta^\alpha$.

For $k \in \mathbb{N}$ and $f \in B(I), I \subset \mathbb{R}$, the modulus of smoothness of order k of f is defined by

$$\omega_k(f;\delta) = \sup\left\{\left|\triangle_h^k f(x)\right|, 0 \leq h \leq \delta, x, x+kh \in I\right\}, \quad \delta \geq 0.$$

In case, $k = 1, \omega_1(f;\delta) = \omega(f;\delta)$.

In 1963, J. Peetre [112] introduced an expression called Peetre's K-functional, which represents another important instrument to measure the smoothness of a function. For any $f \in C([a,b]),\ \ \delta \geq 0$ and integer $s \geq 1$, we call

$$\begin{aligned} K_s(f;\delta)_{[a,b]} &= K(f;\delta;C([a,b]),C^s([a,b])) \\ &= \inf\left\{\|f-g\| + \delta\left\|g^{(s)}\right\| : g \in C^s([a,b])\right\}, \end{aligned} \tag{0.2.2}$$

Peetre's K-functional of order s. Whenever there is no doubt about the interval of definition of f we shall use the notation $K_s(f;\delta)$ instead of $K_s(f;\delta)_{[a,b]}$.

Let us assume that $0 < a < a_1 < b_1 < b < \infty$ and $f \in C_\alpha[0,\infty)$, then for $m \in \mathbb{N}$ the Steklov mean $f_{\eta,m}$ of $m-th$ order corresponding to f, for sufficiently small values of $\eta > 0$ is defined by

$$f_{\eta,m}(x) = \eta^{-m}\left(\int_{-\eta/2}^{\eta/2}\right)^m \left\{f(x) + (-1)^{m-1}\triangle^m_{\sum\limits_{i=1}^{m} x_i} f(x)\right\}\prod_{i=1}^{m} dx_i, \tag{0.2.3}$$

where $x \in [a,b]$ and $\triangle^m_\eta f(x)$ is the $m-th$ order forward difference with step length η. It is easily checked (see e.g. [58], [77]) that

(i) $f_{\eta,m} \in C[a,b]$;
(ii) $\|f^{(r)}_{\eta,m}\|_{C[a_1,b_1]} \leq M_1\eta^{-r}\omega_r(f,\eta,a,b),\ r = 1,2,...,m$;
(iii) $\|f - f_{\eta,m}\|_{C[a_1,b_1]} \leq M_2\omega_m(f,\eta,a,b)$;
(iv) $\|f_{\eta,m}\|_{C[a_1,b_1]} \leq M_3\|f\|_{C[a,b]} \leq M_4\|f\|_{C_\alpha}$,

where $M_i, i = 1,2,3,4$ are certain constants independent of f and η.

Lipschitz-type space is defined as

$$Lip^*_M(\alpha) = \left\{f \in C[0,\infty) : |f(y) - f(x)| \leq M\frac{|y-x|^\alpha}{(y+x)^{\alpha/2}}; x,y \in (0,\infty)\right\} \tag{0.2.4}$$

where M is any $+ve$ constant and $0 < \alpha \leq 1$.

0.3 Historical Background

In 1885, Karl Weierstrass [134], has given two starting results in the Theory of Approximation, which are the milestone for the development in this area. Result for algebraic polynomials is as follows:

If a function $f(x)$ is continuous in the interval $a \leq x \leq b$ and $\varepsilon > 0$, then we can find a polynomial $p(x)$ such that the inequality

$$|f(x) - p(x)| < \varepsilon$$

would hold for all values of x in this interval.

For trigonometric polynomials, he gave the following result:

If a function $f(x)$ has period 2π and is continuous on the real axis, then we can find a trigonometric polynomial $T(x)$ for $\varepsilon > 0$ such that there holds the inequality

$$|T(x) - f(x)| < \varepsilon, -\pi \leq x \leq \pi.$$

In the year 1912, Russian mathematician, **Sergei Bernstein (1880-1968)** [22], one of the “fathers”of approximation theory, defined, well known Bernstein operators as: For $f \in C[0,1]$, the Bernstein polynomials of f are

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]. \tag{0.3.1}$$

Later **O. Szász and G. Mirakian** [124] extended the result of S. Bernstein and gave the following results:

Let f be a continuous function defined on the interval $[0, \infty)$, the Szász-Mirakian operators $(S_n f)$ is defined as follows:

$$(S_n f)(x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0, \infty). \tag{0.3.2}$$

Similarly, **Baskakov** [18] introduced the following operators and studied some approximation results

$$(V_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \in [0, \infty). \tag{0.3.3}$$

At the same time, Lupaş also introduced these type of operators, therefore these operators are also known as **Lupaş** operators.

In 1960, Meyer-König and Zeller [102] introduced a sequence of positive linear operators which were studied, modified and generalized by several authors. The classical **Meyer-König and Zeller operators** $Z_n : C[0,1] \to C[0,1], n \in \mathbb{N}$ are defined as:

$$(Z_nf)(x) = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k(1-x)^{n+1} f\left(\frac{k}{k+n}\right), \quad x \in [0,1].$$

To approximate integrable functions, **Kantorovich** [87] was the first to introduce the integral variant of Bernstein polynomials. The Bernstein-Kantorovich operators are defined as:

$$(\tilde{B}_nf)(x) = (n+1) \sum_{k=0}^{\infty} b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,$$

where $b_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$.

Totik [126] introduced integral variant of Szász-Mirakian operators known as Szász Kantorovich operators:

$$(\tilde{S}_nf)(x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\,(t)\,dt,$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$.

For Baskakov operators appeared in literature as following:

$$(\tilde{V}_nf)(x) = (n-1) \sum_{k=0}^{\infty} v_{n,k}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt,$$

where $v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$.

In 1967, Durrmeyer [54] introduced a more generalized integral modification of the Bernstein polynomials so as to approximate integrable function on the interval [0,1]. The integral modification given by Durrmeyer is independent on the value of k in integration. Bernstein-Durrmeyer operators were first studied by Derrienic [41],

$$(\bar{B}_nf)(x) = (n+1) \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^1 b_{n,k}(t)f(t)dt \tag{0.3.4}$$

who gave the interesting results.

On the similar lines, the integral modification of the Szász operators [100] is as follows:

$$(\bar{S}_n f)(x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t) dt. \tag{0.3.5}$$

The Baskakov-Durrmeyer operators [117] on $[0, \infty)$ are defined as

$$(\bar{V}_n f)(x) = (n-1) \sum_{k=0}^{\infty} v_{n,k}(x) \int_0^{\infty} v_{n,k}(t) f(t) dt. \tag{0.3.6}$$

In the recent years, several modifications have been done for these operators to make the convergence rate faster. Hybrid form of different operators have been obtained by several mathematicians for example, Szász-Beta operators, Baskakov-Szász operators, Baskakov-Beta operators, their Durrmeyer and Kantorovich forms etc.. Approximation properties, Voronskaya type results, error estimation, rate of convergence, central theorems etc. results have already been calculated, see([1, 2, 12, 38, 130, 132, 133]).

0.4 Approximation of Functions, Direct and Inverse theorem

For a given approximation method, the study of the central theorems of approximation eg. direct, inverse and saturation are the most important aspects of consideration. A direct theorem provides the order of approximation for functions of specified smoothness. On the other hand, an inverse theorem infers the nature of smoothness of a function when the order of approximation is specified. A saturation theorem refers to an inherent limitation of the approximation method. The order of approximation beyond a certain limit is possible only for a trivial (finite dimensional) subspace. The functions for which the order is attained, form the saturation (Favard) class and those with the approximation order $o(\phi(n))$ come in the trivial class. Thus, a saturation theorem consists of a determination of the saturation order $\phi(n)$, the saturation class and the trivial class.

The study of direct theorems in approximation theory was initiated by the classical work of Jackson [81] on algebraic and trigonometric polynomials of best approximation. The corresponding inverse theorems for Bernstein polynomials was obtained by Bernstein by an ingenious application of his famous inequality. But the result for $\alpha = 1$, in trigonometric case was proved by Zygmund. In the trigonometric case, the results of Jackson-Bernstein had an essential gap for the case $\alpha = 1$. This was filled, much later by Zygmund through the introduction of the class $Z(Lip^{*}1)$. The generalisations of Zygmund class have been found very useful in approximation theory.

Direct, inverse and saturation theorems in approximation by semigroup of operators had been extensively developed by Butzer and Berens [25]. Some studied the convergence of iterates of certain sequences of linear positive operators to semigroup of operators.

Theorems and equalities establishing a connection between the difference- differential properties of the function to be approximated and the magnitude(and behaviour) of the error of approximation, by various methods. Direct theorems give an estimate of the error of approximation of a function in terms of its smoothness properties (the existence of derivatives of a given order, the modulus of continuity of f or of some of its derivatives, etc.). In the case of best approximation by polynomials, direct theorems are also known as Jackson-type theorems [80], together with their many generalizations and refinements. Inverse theorems characterize difference-differential properties of functions depending on the rapidity with which the errors of best, or any other, approximations tend to zero. The problem of obtaining inverse theorems in the approximation of functions was first stated, and in some cases solved, by S.N. Bernstein [23]. A comparison of direct and inverse theorems allows one sometimes to characterize completely the class of functions having specific smoothness properties using sequence of best approximation.

0.5 Jackson Inequality

An inequality estimating the rate of decrease of the best approximation error of a function by trigonometric or algebraic polynomials in dependence on its differentiability and finite-difference properties. Let f be a 2π-periodic continuous function on the real axis, let $E_n(f)$be the best uniform approximation error of f by trigonometric polynomials T_n of degree n, i.e.

$$E_n\left(f\right) = \inf_{T_n} \max_{x} \left|f(x) - T_n(x)\right|$$

and let

$$\omega\left(f;\delta\right) = \max_{|t_1 - t_2| \leq \delta} \left|f\left(t_1\right) - f\left(t_2\right)\right|$$

be the modulus of continuity of f. It was shown by D. Jackson [80] that

$$E_n\left(f\right) \leq C\omega\left(f;\frac{1}{n}\right) \tag{0.5.1}$$

(where C is an absolute constant), while if f has an rth order continuous derivative $f^{(r)} \geq 1$, then

$$E_n\left(f\right) \leq \frac{C_r}{n^r}\omega\left(f^{(r)};\frac{1}{n}\right),$$

where the constant C_r depends on r only. S.N. Bernstein [23] obtained inequality (0.5.1) in an independent manner for the case

$$\omega\left(f;t\right) \leq Kt^{\alpha}, 0 < \alpha < 1.$$

If f is continuous or r times continuously differentiable on a closed interval $[a,b], r = 1,2,...,$ and if $E_n(f;a,b)$ is the best approximation error of the function f on $[a,b]$ by algebraic polynomials of degree n, then, for $n > r$ one has the relation $f^0 = f$

$$E_n\left(f;a,b\right) \leq \frac{A_r(b-a)^r}{n^r}\omega\left(f^{(r)};\frac{b-a}{n}\right),$$

where the constant A_r depends on r only.

The Jackson inequalities are also known as the Jackson theorems or as direct theorems in the theory of approximation of functions. They may be generalized in various directions: to approximate using an integral metric, to approximate by entire functions of finite order, to an estimate concerning the approximation using a modulus of smoothness of order k, or to a function of several variables. The exact values of the constants in Jackson's inequalities have been determined in several cases.

Another topic of interest is the phenomena of simultaneous approximation (approximation of derivatives of functions by the corresponding order derivatives of operators). The study in this direction began with a remarkable result for Bernstein polynomials (B_n) owing to Lorentz [96], who proved that $B_n^{(k)}(f)(x) \to f^{(k)}(x), n \to \infty$, whenever the latter exists at the particular point $x \in [0,1], k = 1, 2, 3, ...$ being arbitrary. His method for this pointwise convergence in simultaneous approximation has since been applied by several mathematicians to other operators.

For the simultaneous approximation, a crucial property is the convexity of higher order of the operators. The main result in this direction is a theorem of Sendov and Popov [118]. A simplified version of it is the following:

Theorem 0.5.1. *If L_n is a sequence of linear positive operators. $L_n : C[a,b] \to C[a,b], p \geq 1$, such that*

(i) L_n are convex of order k, for any $k, 0 \leq k \leq p$, and

(ii) $\lim\limits_{n\to\infty} \|L_n(e_i) - e_i\|_{[a,b]} = 0$, for $i = 0, 1, 2$

then for any $f \in C^p[a,b]$ and any subinterval $[c,d] \subset [a,b]$ we have

$$\lim_{n\to\infty} \left\|L_n^{(p)}(f) - f^p\right\|_{[c,d]} = 0.$$

0.6 Improvement in Order of Approximation

Though the linear positive operators are conceptually simpler, easier to construct and study, they lack in the rapidity of convergence for sufficiently smooth functions. In the same context a well known theorem of Korovkin (1960) states that the optimal rate of convergence for any sequence of linear positive operators is atmost $O(n^{-2})$. Thus, if we want to have a better order of approximation for smoother functions, we have to slacken the positivity condition. Several investigations indicated that even when a sequence or class of linear positive operators is saturated with a certain order of approximation, some carefully chosen linear combinations of its members give a better order of approximation for smoother functions. The first attempt at some how improving the order of approximation was made by Butzer [24], who showed that by taking a linear combination of the Bernstein polynomials, the order of approximation considerably improves for smoother functions. More general combinations have been studied by Rathore [115] and May [99] for other sequence of linear positive operators. Micchelli [103] offered yet another approach for improving the order of approximation by Bernstein polynomials by considering the iterative combinations $U_{n,k} = \left[I - (I - B_n)^k\right]$. Agrawal and Gupta [5] applied his technique to improve the order of approximation by Phillips operators.

For obvious reasons, summation type operators as such are not L_p-approximation methods. Nevertheless, several linear positive operators of summation type have been appropriately modified to become L_p-approximation method. The underlying idea behind such a modification is to replace, in the expression for the operator, the function value at a nodal point by an average value(in the sense of integration) of the function in an appropriate neighbourhood of the point. The first such modification was made by Kantorovich [86] for the case of Bernstein polynomials. Another modification of Bernstein polynomials was introduced by Durrmeyer [54] and later studied extensively by Derrienic [41], Sahai and Prasad [117]. Mazhar and Totik [100] modified Lupas and Szász operators respectively in an analogous manner to make it possible to approximate functions in $L_p[0, \infty), p \geq 1$.

0.7 Statistical Approximation of Linear Positive Operators

One of the most recently studied subject in Approximation Theory is the approximation of continuous functions by linear positive operators using the statistical convergence or the matrix summability method.

Statistical convergence was introduced in connection with problems of series summation. The main idea of statistical convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ is that the majority, in a certain sense, of its elements converges and we do not care what is going on with other elements. At the time, it is known that sequences that come from real life sources are not convergent in the strictly mathematical sense. This way, the advantage of replacing the uniform convergence by statistical convergence consists in the fact the second convergence models and improves the technique of signal approximation in different function spaces.

The first research which deals with the statistical convergence for sequences of linear positive operators was attempted in the year 2002 by A.D. Gadjiev and C. Orhan. The research field was proved to be extremely fertile, many researchers approaching this subject.

Motivated by this research direction, our interest is to construct different classes of linear positive operators of discrete or integral type and to study their statistical approximation properties. We know that any convergent sequence is statistically convergent but the converse is not true. The aim is to construct such sequences of operators which approximated the function in statistical sense.

0.8 Contents of the Thesis

The present thesis consists of six chapters, whose contents are as described below:

Chapter 1 is a study of some theorems on approximation of the r-th derivative of a given function f by corresponding r-th derivative of the generalized Bernstein operator.

Chapter 2 In this chapter, we give another modification of Baskakov operators and Balázs operators and obtain the approximation properties of these operators. Then we show that our modified operators have a better error estimation. Then we also study rate of convergence as well as their Voronovskaya type results.

Chapter 3 We consider the multidimensional Bernstein operators $G_n(f, x, y)$ and its Durrmeyer variants $Q_n(f, x, y)$ on a simplex. We characterize the rate of approximation by means of $K-$functionals and estimate the order of convergence by means of a semi-norm $\phi(f)$. At the end of the chapter we establish an inverse theorem of approximation.

Chapter 4 In this chapter, we obtain quantitative estimates for generalized double Baskakov operators. We calculate global results for these operators using Lipscitz-type spaces and estimate the error using modulus of continuity.

Chapter 5 This chapter is a study of linear combinations of Beta-Durrmeyer operators $J_n(f, x)$. We consider the direct theorem in terms of higher order modulus of continuity in simultaneous approximation and inverse theorem for these operators in ordinary approximation.

Chapter 6 In this chapter, we consider general Beta operators, which is a general sequence of integral type operators including Beta function and an operator introduced by Jain [83] with the help of Poisson type distribution. We study the King type Beta operators which preserves the third test function x^2 then we obtain some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators.

Chapter 1

Simultaneous Approximation On Generalized Bernstein-Durrmeyer Operators

1.1 Introduction

Denoted by $B[0,1]$, the space of bounded functions on the interval $[0,1]$, with the sup-norm: $\|.\|$ and by $C[0,1]$, the subspace of continuous functions, the Bernstein operators $B_n : B[0,1] \to \mathbb{R}^{[0,1]}, n \in \mathbb{N}$ are given by (0.3.1). Consider the monomial functions $e_j(x) = x^j, t \in [0,1], j = 0,1,2,...$ we have

(i) $B_n(e_0, x) = e_0$,

(ii) $B_n(e_1, x) = e_1$,

(iii) $B_n(e_2, x) = e_2 + \frac{e_1(e_0 - e_1)}{n}$.

In 1967, J.L. Durrmeyer [54] introduced Bernstein-Durrmeyer operators $\bar{B}_n$ (0.3.4) and independently, by Lupaş [[97],pg.68] associate with each function f integrable on $I = [0,1]$. They result from the classical Bernstein operators (0.3.1) in which the discrete values $f\left(\frac{k}{n}\right)$ is replaced by the integral $\int_0^1 b_{n,k}(t)f(t)dt$ in order to approximate L_p functions $(1 \le p < \infty)$.

Later Derriennic [41], Gupta [65], Gupta & Srivastava [73] and Heilmann [76] studied so called Bernstein Durrmeyer operators $\bar{B}_n$ in detail and established many interesting properties of these operators.

Various generalizations of the Bernstein operators defined on $C[0,1]$ by the relation (0.3.1) have been given. Besides the convergence and approximation Bernstein polynomial preserve some properties of original function:

(*i*) if $f(x)$ is non-decreasing, then for all $n \geq 1$, $(B_n(f;x))$ are also non-decreasing,

(*ii*) if $f(x)$ is convex then for all $n \geq 1$ the $(B_n(f;x))$ are convex and

$$B_n(f;x) \geq B_{n+1}(f;x) \geq f(x), x \in [0,1],$$

(*iii*) if $f \in Lip_\mu A$ then for all $n \geq 1, B_n(f;x) \in Lip_A\mu$.

In 1964, E.W. Cheney and A. Sharma [28] generalized the Bernstein polynomials by the relation

$$(A_n f)(x) = (1+nt_n)^{-n} \sum_{k=0}^{n} \binom{n}{k} x(x+kt_n)^{k-1}(1-x+(n-k)t_n)^{n-k} f\left(\frac{k}{n}\right), \quad (1.1.1)$$

where $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of positive real numbers.

The operator A_n (1.1.1) is called n^{th} Bernstein Cheney Sharma operator and constitute a different generalization of the Bernstein operator which can be obtained by $t_n = 0$.

Very recently Deo et al. [39] introduced modified Bernstein operator M_n defined as:

$$(M_n f)(x) = \sum_{k=0}^{n} d_{n,k}(x) f\left(\frac{k}{n}\right), \quad (1.1.2)$$

where

$$d_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k} \text{ and } x \in \left[0, 1-\frac{1}{n+1}\right].$$

In this context Deo [31] has studied direct as well as converse results for the Beta operators and in [32, 33] Deo has given Voronovskaya type results for exponential operators. Some basic results for the operator M_n (1.1.2) are, for $n \geq 1$:

(i) $(M_n 1)(x) = 1,$

(ii) $(M_n t)(x) = \left(1 + \frac{1}{n}\right) x,$

(iii) $(M_n t^2)(x) = \left(1 + \frac{1}{n} - \frac{1}{n^2} - \frac{1}{n^3}\right) x^2 + \left(\frac{1}{n} + \frac{1}{n^2}\right) x.$

We study the following Durrmeyer variant of the operator (1.1.2) as:

$$\left(\bar{M}_n f\right)(x) = n\left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{n} d_{n,k}(x) \int_0^{\frac{n}{n+1}} d_{n,k}(t) f(t) dt, \tag{1.1.3}$$

where $d_{n,k}(x)$ is defined in (1.1.2) above. In the operators (1.1.3), the interval of the definition of integrability of function has been contracted from class $[0,1]$ to $\left[0, 1 - \frac{1}{n+1}\right]$. Very recently, Jung et al. [84] have given some interesting results for modified Bernstein operators. Also see([29], [62], [135], [136]) for simultaneous approximation of various operators.

In this chapter, we prove some theorems on the approximation of r-th derivative of a function f by the corresponding operators $\left(\bar{M}_n^{(r)}\right)$.

1.2 Auxiliary Results

In this section, we shall mention some definitions and certain lemmas to prove our main theorems.

Lemma 1.2.1. *If f is differentiable r times on $\left[0, 1 - \frac{1}{n+1}\right]$, then we get*

$$\left(\bar{M}_n^{(r)} f\right)(x) = \frac{(n+1)^{2(r+1)}}{n^{2r+1}} \frac{(n!)^2}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} d_{n-r,k}(x) \int_0^{\frac{n}{n+1}} d_{n+r,k+r}(t) f^{(r)}(t) dt. \tag{1.2.1}$$

Proof. We have by Leibniz's theorem

$$\left(\bar{M}_n^{(r)} f\right)(x) = \frac{(n+1)^{n+2}}{n^{n+1}} \sum_{i=0}^{r} \sum_{k=0}^{n-r+i} \begin{pmatrix} r \\ i \end{pmatrix} \frac{(-1)^{r-i} n! x^{k-i}}{(k-i)!(n-k-r+i)!}$$

$$\cdot \left(\frac{n}{n+1} - x\right)^{n-k-r+i} \int_0^{\frac{n}{n+1}} d_{n,k}(t) f(t) dt$$

$$= \frac{(n+1)^{n+2}}{n^{n+1}} \sum_{i=0}^{r} \sum_{k=0}^{n-r+i} \begin{pmatrix} r \\ i \end{pmatrix} (-1)^{r-i} \frac{n!}{k!\,(n-k)!} \left(k(k-1)...(k-i+1)\right) x^{k-i}$$

$$\left((n-k)(n-k-1)...(n-k-r+i+1)\right) \left(\frac{n}{n+1} - x\right)^{n-k-r+i}$$

$$\cdot \int_0^{\frac{n}{n+1}} d_{n,k}(t) f(t) dt$$

$$= \frac{(n+1)^{r+2}}{n^{r+1}} \sum_{k=i}^{n-r+i} \sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} \frac{(-1)^{r-i} n!}{(n-r)!} d_{n-r,k-i}(x) \int_0^{\frac{n}{n+1}} d_{n,k}(t) f(t) dt$$

$$= \frac{(n+1)^{n+2}}{n^{n+1}} \sum_{i=0}^{r} \sum_{k=i}^{n-r+i} \begin{pmatrix} r \\ i \end{pmatrix} (-1)^{r-i} \frac{n!}{(k-i)!\,(n-k-r+i)!} x^{k-i}$$

$$\left(\frac{n}{n+1} - x\right)^{n-k-r+i} \int_0^{\frac{n}{n+1}} d_{n,k}(t) f(t) dt$$

$$= \frac{(n+1)^{n+2}}{n^{n+1}} \left(\frac{n}{n+1}\right)^{n-r} \sum_{i=0}^{r} \sum_{k=0}^{n-r+i} \begin{pmatrix} r \\ i \end{pmatrix} (-1)^{r-i} \frac{n!}{(n-r)!} \left(\frac{n+1}{n}\right)^{n-r}$$

$$\begin{pmatrix} n-r \\ k \end{pmatrix} x^k \left(\frac{n}{n+1} - x\right)^{n-k-r} \int_0^{\frac{n}{n+1}} d_{n,k+i}(t) f(t) dt$$

$$\cdot \int_0^{\frac{n}{n+1}} d_{n,k+i}(t) f(t) dt$$

$$= \frac{(n+1)^{r+2}}{n^{r+1}} \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} (-1)^r d_{n-r,k}(x) \int_0^{\frac{n}{n+1}} \sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} (-1)^i d_{n,k+i}(t) f(t) dt.$$

Again using Leibniz's theorem

$$\frac{d^r}{dt^r} d_{n+r,k+r}(t) = \sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} (-1)^i \frac{(n+r)!}{n!} \left(\frac{n}{n+1}\right)^r d_{n,k+i}(t)$$

$$\left(\bar{M}_n^{(r)} f\right)(x) = \frac{(n+1)^{2(r+1)}}{n^{2r+1}} \frac{(n!)^2}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} d_{n-r,k}(x) \int_0^{\frac{n}{n+1}} (-1)^r d_{n+r,k+r}^{(r)}(t) f(t) dt.$$

Further integrating by parts r times, we get the required result. □

Lemma 1.2.2. *Let $r, m \in \mathbb{N}^0$ (the set of non-negative integers), $n \in \mathbb{N}$ and $x \in [0, \infty)$. Let the m-th order moments are defined by if*

$$\mu_{n,m}(x) = (n+r+1)\left(1+\frac{1}{n}\right) \sum_{k=0}^{n-r} d_{n-r,k}(x) \int_0^{\frac{n}{n+1}} d_{n+r,k+r}(t)(t-x)^m dt,$$

then we get

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{(1+r)\{n-2x(n+1)\}}{(n+1)(n+r+2)}, \tag{1.2.2}$$

$$\mu_{n,2}(x) = \frac{(r+1)(r+2)\{n-2x(n+1)\}^2}{(n+1)^2(n+r+2)(n+r+3)} + \frac{2x\{n-x(n+1)\}}{(n+r+2)(n+r+3)} \tag{1.2.3}$$

and

$$(m+n+r+2)\mu_{n,m+1}(x) = (1+m+r)\left(\frac{n}{n+1} - 2x\right)\mu_{n,m}(x) + 2mx\left(\frac{n}{n+1} - x\right)\mu_{n,m-1}(x) + x\left(\frac{n}{n+1} - x\right)\mu'_{n,m}(x). \tag{1.2.4}$$

Consequently,

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree $\leq m$,

(ii) $\mu_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, where $[\alpha]$ denotes the integral part of α.

Proof. The values of $\mu_{n,0}$ and $\mu_{n,1}$ can easily follows from the definition. We prove the recurrence relation as follows:

$$\mu'_{n,m}(x) = (n+r+1)\left(1+\frac{1}{n}\right) \sum_{k=0}^{n-r} d'_{n-r,k}(x) \int_0^{\frac{n}{n+1}} d_{n+r,k+r}(t)(t-x)^m dt - m\mu_{n,m-1}(x)$$

Using the following relation

$$x\left(\frac{n}{n+1}-x\right)d'_{n,k}(x)=n\left(\frac{k}{n+1}-x\right)d_{n,k}(x), \tag{1.2.5}$$

then we get

$$\begin{aligned}
&x\left(\frac{n}{n+1}-x\right)\left\{\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right\}\\
&=(n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r}x\left(\frac{n}{n+1}-x\right)d'_{n-r,k}(x)\int_0^{\frac{n}{n+1}}d_{n+r,k+r}(t)(t-x)^m dt\\
&=(n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r}d_{n-r,k}(x)\int_0^{\frac{n}{n+1}}\left\{\frac{nk}{n+1}-(n-r)x\right\}d_{n+r,k+r}(t)(t-x)^m dt\\
&=(n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r}d_{n-r,k}(x)\int_0^{\frac{n}{n+1}}\left[\left((k+r)\left(\frac{n}{n+1}\right)-t\,(n+r)\right)\right.\\
&\quad\left.-r\left(\frac{n}{n+1}-2x\right)+(n+r)\,(t-x)\,d_{n+r,k+r}(t)(t-x)^m dt\right]\\
&=(n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r}d_{n-r,k}(x)\int_0^{\frac{n}{n+1}}t\left(\frac{n}{n+1}-t\right)d'_{n+r,k+r}(t)(t-x)^m dt\\
&\quad-r\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)+(n+r)\mu_{n,m+1}(x)\\
&=(n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r}d_{n-r,k}(x)\int_0^{\frac{n}{n+1}}\left[-(t-x)^{m+2}+\left(\frac{n}{n+1}-2x\right)(t-x)^{m+1}\right.\\
&\quad\left.+x\left(\frac{n}{n+1}-x\right)(t-x)^m\right]d'_{n+r,k+r}(t)dt-r\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)+(n+r)\mu_{n,m+1}(x)\\
&=(m+2)\mu_{n,m+1}(x)-\left(\frac{n}{n+1}-2x\right)(m+1)\mu_{n,m}(x)-mx\left(\frac{n}{n+1}-x\right)\mu_{n,m-1}(x)\\
&\quad-r\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)+(n+r)\mu_{n,m+1}(x).
\end{aligned}$$

This completes the proof of the recurrence relation. The values of $\mu_{n,1}(x)$ and $\mu_{n,2}(x)$ can be easily obtained from the above recurrence relation. □

1.3 Main Results

In this section we shall prove the following main results.

Theorem 1.3.1. *If $f^{(r)}$ is a bounded and integrable in $\left[0,1-\frac{1}{n+1}\right]$ and admits $(r+2)$-th derivative at a point $x \in \left[0,1-\frac{1}{n+1}\right]$, then*

$$\lim_{n\to\infty} n\left[\frac{n^{2r}(n-r)!(n+r+1)!}{(n+1)^{2r+1}(n!)^2}\left(\bar{M}_n^{(r)}f\right)(x) - f^{(r)}(x)\right] = (1-2x)(1+r)f^{(r+1)}(x) + x(1-x)f^{(r+2)}(x). \quad (1.3.1)$$

Proof. By Taylor's formula, we have

$$f^{(r)}(t) = f^{(r)}(x) + (t-x)f^{(r+1)}(x) + \frac{(t-x)^2}{2}f^{(r+2)}(x) + \frac{(t-x)^2}{2}\zeta(t-x), \quad (1.3.2)$$

where $\zeta(u) \to 0$ as $u \to 0$ and ζ is a bounded and integrable function on $[-x, 1-x]$.

Now, using (1.3.2) and by Lemma 1.2.2, we get

$$\frac{n^{2r}(n-r)!(n+r+1)!}{(n+1)^{2r+1}(n!)^2}\left(\bar{M}_n^{(r)}f\right)(x) - f^{(r)}(x) = \left\{\frac{(1+r)\{n-2x(n+1)\}}{(n+1)(n+r+2)}\right\}f^{(r+1)}(x) + \frac{1}{2}\left\{\frac{(r+1)(r+2)\{n-2x(n+1)\}^2}{(n+1)^2(n+r+2)(n+r+3)} + \frac{2x\{n-x(n+1)\}}{(n+r+2)(n+r+3)}\right\}f^{(r+2)}(x) + R_{n,r}(x),$$

where

$$R_{n,r}(x) = \frac{1}{2}\sum_{k=0}^{n-r} d_{n-r,k}(x)\int_0^{\frac{n}{n+1}} d_{n+r,k+r}(t)(t-x)^2\zeta(t-x)dt.$$

Now we have to show that $nR_{n,r} \to 0$ as $n \to \infty$. Let $K = \sup_{u\in[-x,1-x]}|\zeta(u)|$ and let $\varepsilon > 0$. Choose $\delta > 0$ such that $|\zeta(u)| < \varepsilon$ when $|u| \le \delta$. So for all $t \in \left[0,1-\frac{1}{n+1}\right]$, we have $|\zeta(t-x)| < \varepsilon + K\frac{(t-x)^2}{\delta^2}$.
Clearly

$$|nR_{n,r}(x)| < \frac{n\varepsilon}{2}\bar{M}_n^{(r)}(t-x)^2(x) + \frac{Kn}{2\delta^2}\bar{M}_n^{(r)}(t-x)^4(x) = \frac{n\varepsilon}{2}\left\{\frac{(r+1)(r+2)\{n-2x(n+1)\}^2}{(n+1)^2(n+r+2)(n+r+3)} + \frac{2x\{n-x(n+1)\}}{(n+r+2)(n+r+3)}\right\} + \frac{K}{2\delta^2}O\left(\frac{1}{n}\right),$$

since $\varepsilon > 0$ is arbitrary, this implies that $|nR_{n,r}(x)| \to 0$ as $n \to \infty$. Thus as $n \to \infty$, we get the required result from (1.3.2). This completes the proof of the theorem. $\square$

Theorem 1.3.2. *If* $f^{(r+1)} \in C\left[0, 1 - \frac{1}{n+1}\right]$ *and let* $\omega\left(f^{(r+1)}; .\right)$ *be the moduli of continuity of* $f^{(r+1)}$*. Then for* $n \geq r$*,* $(r = 0, 1, 2, ...)$*, we have*

$$\left\|\bar{M}_n^{(r)} - f^{(r)}\right\| \leq \left\|f^{(r+1)}\right\| + \frac{1}{2\sqrt{n}}\left\{\sqrt{\lambda}r + \frac{\lambda r}{2}\right\}\omega\left(f^{(r+1)}; \frac{1}{\sqrt{n}}\right), \tag{1.3.3}$$

where the norm is sup-norm over $\left[0, 1 - \frac{1}{n+1}\right]$ *and* $\lambda r = 1 + \frac{r}{2}$*.*

Proof. Following [119] and by the Taylor formula

$$f^{(r)}(t) - f^{(r)}(x) = (t - x)f^{(r+1)}(x) + \int_x^t \left\{(f^{(r+1)}(y) - f^{(r+1)}(x)\right\}.$$

Now, applying (1.2.1)to the above and using the inequality

$$\left|f^{(r+1)}(y) - f^{(r+1)}(x)\right| \leq \left\{1 + \frac{|y - x|}{\delta}\right\}\omega\left(f^{(r+1)}; \delta\right)$$

and the results (1.2.2) and (1.2.3), we have

$$\begin{aligned}
&\left|\left(\bar{M}_n^{(r)} f\right)(x) - f^{(r)}(x)\right| \\
&\leq \left|f^{(r+1)}(x)\right|\left|\bar{M}_n^{(r)}(t - x)(x)\right| + \omega\left(f^{(r+1)}; \delta\right)\bar{M}_n^{(r)}\left[\left|\int_x^t 1 + \frac{|y - x|}{\delta} dy\right|\right](x), \\
&\leq \left|f^{(r+1)}(x)\right|\left|\bar{M}_n^{(r)}(t - x)(x)\right| + \omega\left(f^{(r+1)}; \delta\right)\left\{\sqrt{\bar{M}_n^{(r)}(t - x)^2(x)} + \frac{1}{2\delta}\bar{M}_n^{(r)}(t - x)^2(x)\right\}.
\end{aligned}$$

Choosing $\delta = \frac{1}{\sqrt{n}}$ and using the result (1.2.3), we get the required result (1.3.3). This completes the proof. □

Chapter 2

Better Approximation for Positive Linear Operators

In 2003, King [91] introduced an exotic sequence of positive linear operators H_n : $C([0,1]) \to C([0,1])$, which modifies the Bernstein operators:

$$(H_n f)(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (c_n(x))^k (1 - c_n(x))^{n-k} f\left(\frac{k}{n}\right), f \in C([0,1]), x \in [0,1],$$

where $c_n(x) : [0,1] \to [0,1]$ are continuous function,

$$c_n(x) = \begin{cases} x^2, & \text{n=1,} \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & \text{n=2,3,...} \end{cases} \tag{2.0.1}$$

This sequence preserves two test functions e_0, e_2 and $(H_n e_1)(x) = c_n(x)$. He also proved that the operators H_n have a better rate of convergence than the classical Bernstein polynomials whenever $0 \leq x \leq 1/3$. After this several researchers have studied that many approximating operators, L, possess these properties, i.e., $L(e_i, x) = e_i(x)$ where $e_i(x) = x^i (i = 0, 1)$ or $(i = 0, 2)$. For example similar problems were accompolished for Szász-Mirakyan operators [51], Szász-Mirakyan-Beta operators by Duman, Özarslan and Aktuglu [52], Meyer-König and Zeller operators by Özarslan and Duman [107], Bernstein-Chlodovsky operators [8], q-Bernstein operators [98], q-analogue of complex summation-integral type operators [6] and some other kinds of summation-type positive linear operators [9]. Local approximation properties of modified Szász-Mirakyan operators have been investigated in [108]. Certain results on modified Szász-Mirakyan operators have been calculated by Finta, Govil and Gupta in [56].

Recently, Rempulska and Tomczak [116] have investigated the King type operators on an appropriate weighted space and they have given some significant applications, such as modifications of Baskakov operators, Post-Widder and Stancu operators.

Very recently Deo and Singh [40] have given another modification of Baskakov operators and studied Voronovskaya type results. Deo [31, 33], Pop [114, 113] and Srivastava and Gupta [120] have studied Voronovskaya formula for other positive linear operators.

2.1 Some Approximation Results for Durrmeyer Operators

Now we consider Heilmann's operator which is defined as:

Definition 2.1.1. [32, 76] The n-th operator D_n of Baskakov-Durrmeyer operator, $n \in \mathbb{N},\ c \in \mathbb{N}^0,\ n > c$, is defined by

$$\left(D_n f\right)(x) = (n-c)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t) f(t)dt \tag{2.1.1}$$

with $x \in [0,\infty)\,,\ p_{n,k}(x) = (-1)^k \frac{x^k}{k!}\phi_n^{(k)}(x)$, where

$$\phi_n(x) = \begin{cases} e^{-nx} & \text{for the interval } [0,\infty) \text{ with } c = 0, \\ (1+cx)^{-n/c}, & \text{for the interval } [0,\infty) \text{ with } c > 0, \end{cases}$$

and f is a function for which the right side of (2.1.1) makes sense. It is easy to see that D_n are Szász-Durrmeyer operators [89, 100], Lupaş-Durrmeyer [117] and Baskakov-Durrmeyer operators [76] for $c = 0, c = 1$ and $c > 0$, respectively.

Lemma 2.1.1. *[32, 76] Let $e_i(x) = x^i,\ i = 0, 1, 2,$ then for $x \in [0, \infty)$, $n \in \mathbb{N}$ and $n > 3c$, we have*

(i) $\left(D_n e_0\right)(x) = 1,$

(ii) $\left(D_n e_1\right)(x) = \frac{nx+1}{n-2c},$

(iii) $\left(D_n e_2\right)(x) = \frac{n(n+c)x^2+4nx+2}{(n-2c)(n-3c)}.$

Lemma 2.1.2. *For $x \in [0, \infty),\ n \in \mathbb{N},\ n > 3c$ and $\varphi_x(t) = t - x$, we have*

(i) $\left(D_n \varphi_x\right)(x) = -\frac{1+2cx}{n-2c},$

(ii) $\left(D_n \varphi_x^2\right)(x) = \frac{2\{(n+3c)x(1+cx)+1\}}{(n-2c)(n-3c)}.$

The linear functions, i.e., for $h(t) = ct + d$, where c, d any real constants, we get $\left(D_n h\right)(x) = h(x)$.

2.1.1 Construction of the Operators and Basic Results

Let $\{q_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0, \infty)$ with $0 \leq q_n(x) < \infty$, for $x \in [0, \infty),\ n \in \mathbb{N}$ then we have

$$\left(\hat{D}_n f\right)(x) = (n-c)\sum_{k=0}^{\infty} p_{n,k}(q_n(x)) \int_0^{\infty} p_{n,k}(t) f(t)\, dt \tag{2.1.2}$$

with $x \in [0, \infty),\ p_{n,k}(q_n(x)) = (-1)^k \frac{(q_n(x))^k}{k!} \phi_n^{(k)}(q_n(x))$, where

$$\phi_n\left(q_n(x)\right) = \begin{cases} e^{-nq_n(x)}, & \text{for the interval } [0, \infty) \text{ with } c = 0, \\ \left(1 + cq_n(x)\right)^{-n/c}, & \text{for the interval } [0, \infty) \text{ with } c > 0, \end{cases}$$

and

$$q_n(x) = \frac{(n-2c)x - 1}{n}.$$

We obtain the following results at once.

Lemma 2.1.3. *Let $e_i(x) = x^i,\ i = 0, 1, 2$ then for each $x \geq 0$ and $n > 3c$, we have*

(i) $\left(\hat{D}_n e_0\right)(x) = 1,$

(ii) $\left(\hat{D}_n e_1\right)(x) = x,$

(iii) $\left(\hat{D}_n e_2\right)(x) = \frac{(n+c)(n-2c)^2x^2+2(n-c)(n-2c)x-(n-c)}{n(n-2c)(n-3c)}.$

Lemma 2.1.4. *For $x \in [0,\infty)$, $n \in \mathbb{N}$, $n > 3c$ and $\varphi_x(t) = t - x$, we have*

(i) $\left(\hat{D}_n \varphi_x\right)(x) = 0,$

(ii) $\left(\hat{D}_n \varphi_x^2\right)(x) = \frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)},$

(iii) $\left(\hat{D}_n \varphi_x^m\right)(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right).$

2.1.2 Voronovskaya Type Results & Better Error Estimation

In this section we compute the rates of convergence and Voronovskaya type results of these operators $\hat{D}_n$ given by (2.1.2).

Theorem 2.1.5. *Let $f \in C_B[0,\infty)$, then for every $x \in [0,\infty)$ and for $C > 0$, $n > 0$ and $n > 3c$, we have*

$$\left|\left(\hat{D}_n f\right)(x) - f(x)\right| \le C\omega_2\left(f, \sqrt{\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}}\right). \tag{2.1.3}$$

Proof. Let $g \in W_\infty^2$. Using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

From Lemma 2.1.4, we have

$$\left(\hat{D}_n g\right)(x) - g(x) = \left(\hat{D}_n \int_x^t (t-u)g''(u)du\right)(x).$$

We know that

$$\left|\int_x^t (t-u)g''(u)du\right| \le (t-u)^2 \left\|g''\right\|.$$

Therefore

$$\left|\left(\hat{D}_n g\right)(x) - g(x)\right| \leq \left(\hat{D}_n (t-u)^2\right)(x) \left\|g''\right\| = \frac{(n-c)\left\{2x\left(1+cx\right)\left(n-2c\right)-1\right\}}{n(n-2c)(n-3c)} \left\|g''\right\|.$$

By Lemma 2.1.3, we have

$$\left|\left(\hat{D}_n f\right)(x)\right| \leq (n-c)\sum_{k=0}^{\infty} p_{n,k}\left(q_n(x)\right)\int_0^{\infty} p_{n,k}(t)|f(t)|dt \leq \|f\|.$$

Hence

$$\begin{aligned}\left|\left(\hat{D}_n f\right)(x) - f(x)\right| &\leq \left|\left(\hat{D}_n (f-g)\right)(x) - (f-g)(x)\right| + \left|\left(\hat{D}_n g\right)(x) - g(x)\right| \\ &\leq 2\left\|f-g\right\| + \frac{(n-c)\left\{2x\left(1+cx\right)\left(n-2c\right)-1\right\}}{n(n-2c)(n-3c)}\left\|g''\right\|.\end{aligned}$$

Taking the infimum on the right side over all $g \in W_\infty^2$ and using Peetre's K_2 functional (0.2.2), we get the required result. □

Theorem 2.1.6. *If a function f is such that its first and second derivative are bounded in $[0,\infty)$, then we get*

$$\lim_{n\to\infty} n\left\{\left(\hat{D}_n f\right)(x) - f(x)\right\} = x\left(1+cx\right)f''(x). \tag{2.1.4}$$

Proof. Using Taylor's theorem we write that

$$f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \frac{(t-x)^2}{2!}\xi(t,x), \tag{2.1.5}$$

where $\xi(t,x)$ is a bounded function $\forall\, t,x$ and $\lim_{t\to x}\xi(t,x) = 0$.

Now applying (2.1.2) and (2.1.5), we get

$$\left(\hat{D}_n f\right)(x) - f(x) = f'(x)\hat{D}_n\left(\varphi_x, x\right) + \frac{f''(x)}{2}\hat{D}_n\left(\varphi_x^2, x\right) + I_1,$$

where

$$I_1 = \frac{1}{2}\hat{D}_n\left(\varphi_x^2, x\right)\xi(t,x).$$

Using Lemma 2.1.4, we get

$$n\left\{\left(\hat{D}_n f\right)(x) - f(x)\right\} = \frac{f''(x)}{2}\left\{\frac{(n-c)\left\{2x\left(1+cx\right)(n-2c)-1\right\}}{(n-2c)(n-3c)}\right\} + nI_1.$$

Now, we have to show that as $n \to \infty$, the value of $nI_1 \to 0$. Let $\varepsilon > 0$ be given since $\xi(t,x) \to 0$ as $t \to 0$, then there exists $\delta > 0$ such that when $|t-x| < \delta$ we have $|\xi(t,x)| < \varepsilon$ and when $|t-x| \geq \delta$, we write

$$|\xi(t,x)| \leq C < C\frac{(t-x)^2}{\delta^2}.$$

Thus, for all $t, x \in [0,\infty)$

$$|\xi(t,x)| \leq \varepsilon + C\frac{(t-x)^2}{\delta^2}$$

and

$$\begin{aligned} nI_1 &\leq n\left(\hat{D}_n\varphi_x^2\left(\varepsilon + \frac{C\varphi_x^2}{\delta^2}\right)\right)(x) \\ &\leq \varepsilon n\left(\hat{D}_n\varphi_x^2\right)(x) + \frac{C}{\delta^2}n\left(\hat{D}_n\varphi_x^4\right)(x). \end{aligned}$$

Using Lemma 2.1.4, we get that,

$$nI_1 \to 0 \;\text{ as }\; n \to \infty.$$

This leads to (2.1.4). □

Remark 2.1.7. *We may note here that under the conditions of Theorem 2.1.6, we have*

$$\lim_{n\to\infty} n\left\{\left(D_n f\right)(x) - f(x)\right\} = -(1+2cx)f'(x) + x\left(1+cx\right)f''(x). \tag{2.1.6}$$

Theorem 2.1.8. *If $g \in C_B^2[0,\infty)$ then we have*

$$\left|\left(\hat{D}_n g\right)(x) - g(x)\right| \leq \sigma_n(x)\left\|g\right\|_{C_B^2}, \tag{2.1.7}$$

where

$$\sigma_n(x) = \frac{(n-c)\left\{2x\left(1+cx\right)\left(n-2c\right)-1\right\}}{2n(n-2c)(n-3c)},\ n > 3c.$$

Proof. We write that

$$g(t) - g(x) = (t-x)g'(x) + \frac{1}{2}(t-x)^2 g''(\zeta), \tag{2.1.8}$$

where $t \leq \zeta \leq x$. From Lemma 2.1.4 and (2.1.8), we have

$$\begin{aligned}\left|\left(\hat{D}_n g\right)(x) - g(x)\right| &\leq \|g'\| \left|\left(\hat{D}_n \varphi_x\right)(x)\right| + \frac{1}{2}\|g''\| \left|\left(\hat{D}_n \varphi_x^2\right)(x)\right| \\ &\leq \frac{(n-c)\left\{2x\left(1+cx\right)\left(n-2c\right)-1\right\}}{2n(n-2c)(n-3c)} \|g''\| \\ &= \sigma_n(x) \|g\|_{C_B^2}.\end{aligned}$$

This proves the theorem completely. □

Remark 2.1.9. *Under the same conditions of Theorem 2.1.8, we obtain*

$$\left|\left(D_n g\right)(x) - g(x)\right| \leq \sigma_n^*(x) \|g\|_{C_B^2}, \tag{2.1.9}$$

where

$$\sigma_n^*(x) = \frac{(n+3c)x(1+cx)+1}{(n-2c)(n-3c)}.$$

Theorem 2.1.10. *For $f \in C_B[0,\infty)$, we obtain*

$$\left|\left(\hat{D}_n f\right)(x) - f(x)\right| \leq A\left\{\omega_2\left(f, \frac{\sqrt{\sigma_n(x)}}{2}\right) + \min\left(1, \frac{\sigma_n(x)}{2}\right)\|f\|_{C_B}\right\}, \tag{2.1.10}$$

where $n > 3c$ and constant A depends on f & $\sigma_n(x)$.

Proof. For $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$ we write

$$\left(\hat{D}_n f\right)(x) - f(x) = \left(\hat{D}_n f\right)(x) - \left(\hat{D}_n g\right)(x) + \left(\hat{D}_n g\right)(x) - g(x) + g(x) - f(x).$$

By using (2.1.7) and Peetre's K_2-functional, we get

$$\begin{aligned}
\left|\left(\hat{D}_n f\right)(x) - f(x)\right| &= \left|\left(\hat{D}_n f\right)(x) - \left(\hat{D}_n g\right)(x)\right| + \left|\left(\hat{D}_n g\right)(x) - g(x)\right| + |g(x) - f(x)| \\
&\leq \left\|\hat{D}_n f\right\| \|f - g\| + \sigma_n(x) \|g\|_{C_B^2} + \|f - g\| \\
&\leq 2\|f - g\| + \sigma_n(x) \|g\|_{C_B^2} \\
&\leq 2\left\{\|f - g\| + \frac{1}{2}\sigma_n(x) \|g\|_{C_B^2}\right\} \leq 2K_2\left\{f, \frac{1}{2}\sigma_n(x)\right\} \\
&\leq 2A\left\{\omega_2\left(f, \frac{1}{2}\sqrt{\sigma_n(x)}\right) + \min\left(1, \frac{1}{2}\sigma_n(x)\right)\|f\|_{C_B}\right\}.
\end{aligned}$$

This completes the proof of the theorem. □

Remark 2.1.11. *Under the same conditions of Theorem 2.1.10, we get*

$$|(D_n f)(x) - f(x)| \leq 2A\left\{\omega_2\left(f, \sqrt{\sigma_n^*(x)}\right) + \min\left(1, \sigma_n^*(x)\right)\|f\|_{C_B}\right\}. \quad (2.1.11)$$

Theorem 2.1.12. *For every $f \in C[0, \infty)$, $x \in [0, \infty)$, we obtain*

$$\left|\left(\hat{D}_n f\right)(x) - f(x)\right| \leq 2\omega(f, \delta_x), \quad (2.1.12)$$

where

$$\delta_x = \sqrt{\frac{(n-c)\{2x(1+cx)(n-2c)-1\}}{n(n-2c)(n-3c)}}, \ n > 3c$$

and $\omega(f, \delta_x)$ is the modulus of continuity of f.

Proof. Using linearity and monotonicity of $\hat{D}_n$, we easily obtain, for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$\left|\left(\hat{D}_n f\right)(x) - f(x)\right| \leq \omega(f, \delta)\left\{1 + \frac{1}{\delta}\sqrt{\hat{D}_n(\varphi_x^2, x)}\right\}.$$

By using Lemma 2.1.4 and choosing $\delta = \delta_x$ the proof is completed. □

Remark 2.1.13. *For the original operator D_n defined in, we may write that, for every $f \in C[0, \infty)$*

$$|(D_n f)(x) - f(x)| \leq 2\omega(f, \nu_x), \quad (2.1.13)$$

where

$$\nu_x = \sqrt{\frac{2\left\{(n+3c)x(1+cx)+1\right\}}{(n-2c)(n-3c)}}$$

and $\omega\left(f, \nu_x\right)$ *is the modulus of continuity of* f. *The error estimate in Theorem 2.1.12 is better than that of (2.1.13) for* $f \in C\left[0, \infty\right)$ *and* $x \in \left[0, \infty\right)$, *we get* $\delta_x \leq \nu_x$.

Now we compute rate of convergence of the operators of $\hat{D}_n$ by means of the Lipschitz class $Lip_M(\gamma)$, $(0 < \gamma \leq 1)$. As usual, we say that $f \in C_B\left[0, \infty\right)$ belongs to $Lip_M(\gamma)$ if the inequality (0.2.4) holds.

Theorem 2.1.14. *If* $f \in Lip_M(\gamma)$, $x \in \left[0, \infty\right)$ *and* $n > 3c$, *we have*

$$\left|\left(\hat{D}_n f\right)(x) - f(x)\right| \leq M\left[\frac{(n-c)\left\{2x\left(1+cx\right)\left(n-2c\right)-1\right\}}{n(n-2c)(n-3c)}\right]^{\gamma/2}.$$

Proof. Since $f \in Lip_M(\gamma)$ and $x \geq 0$, from inequality (0.2.4) and applying the Holder inequality with $p = \frac{2}{\gamma}, q = \frac{2}{2-\gamma}$, we have

$$\begin{aligned}
\left|\left(\hat{D}_n f\right)(x) - f(x)\right| &\leq \left(\hat{D}_n\left|f(t) - f(x)\right|\right)(x) \\
&\leq M\left(\hat{D}_n|t-x|^{\gamma}\right)(x) \\
&\leq M\left\{\left(\hat{D}_n \varphi_x^2\right)(x)\right\}^{\gamma/2} \\
&\leq M\left[\frac{(n-c)\left\{2x\left(1+cx\right)\left(n-2c\right)-1\right\}}{n(n-2c)(n-3c)}\right]^{\gamma/2}.
\end{aligned}$$

This completes the proof. □

Remark 2.1.15. *If using Lemma 2.1.2, for the original operator* D_n, *then we get the following result*

$$\left|\left(D_n f\right)(x) - f(x)\right| \leq M\left\{\frac{2\left\{(n+3c)x(1+cx)+1\right\}}{(n-2c)(n-3c)}\right\}^{\gamma/2}$$

for every $f \in Lip_M(\gamma)$, $x \geq 0$ *and* $n \geq 1$.

2.2 A Better Error Estimation On Balázs Operators

Let f be a real-valued continuous function on the closed unit interval $[0,1]$ of the real line and let $n \in \mathbb{N} = \{1,2,3,...\}$. Then $n-th$ Bernstein polynomial of f is defined as (0.3.1). It is well known that the sequence $\{B_n(f)\}_{n\in\mathbb{N}}$ converges uniformly to f on $[0,1]$ and the Bernstein polynomials and their generalization as well as modification have an important role in approximation theory (see, for instance, [13], [14], [39], [42], [54], [121], [131]).

In view of these concernments, Katalin Balázs [17] introduced and studied several approximation properties of the Bernstein type rational functions defined as follows:

Let f be a real single valued function defined on $[0,\infty)$ and $n \in \mathbb{N}$, and define

$$R_n(f;x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (a_n x)^k f\left(\frac{k}{b_n}\right), \quad x \in [0,\infty), \tag{2.2.1}$$

where $a = \{a_n\}_{n\in\mathbb{N}}$ and $b = \{b_n\}_{n\in\mathbb{N}}$ are suitable chosen sequence of positive real numbers, independent of x.

In this section we study the approximation properties of modified Balázs [17] operators and obtain better error estimation, rate of convergence and Voronovskaya result.

Lemma 2.2.1. *[17, 48] Let $e_i(t) = t^i,\ i = 0,1,2,3,4$ then for $x \geq 0$ and $n \in \mathbb{N}$, we have*

(i) $R_n(e_0;x) = 1,$

(ii) $R_n(e_1;x) = \frac{n}{b_n}\left(\frac{a_n x}{1+a_n x}\right),$

(iii) $R_n(e_2;x) = \frac{n(n-1)}{b_n^2}\left(\frac{a_n x}{1+a_n x}\right)^2 + \frac{n}{b_n^2}\left(\frac{a_n x}{1+a_n x}\right),$

(iv) $R_n(e_3;x) = \frac{n(n-2)}{b_n^3}\frac{(a_n x)^2\{1+na_n x\}}{(1+a_n x)^3} + \frac{n}{b_n^3}\frac{a_n x\{1+2na_n x\}}{(1+a_n x)^2},$

(iv) $R_n(e_4;x) = \frac{n(n-3)}{b_n^4}\frac{(a_n x)^2\left\{1+(3n-1)a_n x+(na_n x)^2\right\}}{(1+a_n x)^4} + \frac{n}{b_n^4}\frac{a_n x\left\{1+2(3n-1)a_n x+3(na_n x)^2\right\}}{(1+a_n x)^3}.$

Thus

$$R_n(\varphi_x;x) = -\frac{a_n x^2}{1+a_n x} \quad \textit{and} \quad R_n(\varphi_x^2;x) = \frac{x\left\{1+n(a_n x)^3\right\}}{b_n(1+a_n x)^2},$$

where $\varphi_x(e_1) = e_1 - e_0 x$ *and* $a_n = b_n/n,\ b_n > 0$ *is an arbitrary real number.*

Agratini [7], İspir & Atakut [78] and Gupta [67] have studied and given some interesting results on the variant of Balázs operators. Many researchers have given King type modification for different operators (see, for instance, [36], [49], [52], [51], [70], [71], [91], [111], [116]), now we consider Balázs [17] operators for this type of modification.

2.2.1 Construction of the Operators

We assume that $\{u_n(x)\}$ is a sequence of real-valued continuous functions defined on $[0,\infty)$ with $0 \leq u_n(x) < \infty$, for $x \in \left[1, \frac{n}{b_n}\right)$ $\left(\frac{n}{b_n} \to \infty \text{ as } n \to \infty\right)$, $n \in \mathbb{N}$ and $n \geq 2$ then we have

$$\hat{R}_n(f;x) = \frac{1}{(1+a_n u_n(x))^n} \sum_{k=0}^{n} \binom{n}{k} (a_n u_n(x))^k f\left(\frac{k}{b_n}\right), \tag{2.2.2}$$

where

$$u_n(x) = \frac{b_n}{a_n} \frac{x}{(n-b_n x)}, x \neq \frac{n}{b_n}.$$

We obtain the following results at once.

Lemma 2.2.2. *Let* $e_i(x) = x^i,\ i = 0,1,2,3,4$ *then for* $x \in \left[1, \frac{n}{b_n}\right)$ *and* $n \geq 2$ *where* $n \in \mathbb{N}$*, we have*

(*i*) $\hat{R}_n(e_0;x) = 1,$

(*ii*) $\hat{R}_n(e_1;x) = x,$

(*iii*) $\hat{R}_n(e_2;x) = \left(\frac{n-1}{n}\right)x^2 + \frac{1}{b_n}x,$

(*iv*) $\hat{R}_n(e_3;x) = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{nb_n}x^2 + \frac{x}{b_n^2},$

(*v*) $\hat{R}_n(e_4;x) = \frac{(n-3)(n-2)(n-1)}{n^3}x^4 + \frac{6(n-1)(n-2)}{n^2 b_n}x^3 + \frac{7(n-1)}{nb_n^2}x^2 + \frac{x}{b_n^3}.$

Lemma 2.2.3. *For* $x \in \left[1, \frac{n}{b_n}\right)$, $n \geq 2$ *where* $n \in \mathbb{N}$ *and* $\varphi_x(t) = e_1 - e_0 x$*, we have*

(*i*) $\hat{R}_n(\varphi_x;x) = 0,$

(ii) $\hat{R}_n(\varphi_x^2;x) = \frac{x(1-a_nx)}{na_n}$,

(iii) $\hat{R}_n(\varphi_x^3;x) = \frac{2x^3}{n^2} + \frac{3(2n-1)}{nb_n}x^2 + \frac{x}{b_n^2}$,

(iv) $\hat{R}_n(\varphi_x^4;x) = \frac{3(n-2)}{n^3}x^4 - \frac{6(n-2)}{n^2b_n}x^3 + \frac{(3n-7)}{nb_n^2}x^2 + \frac{x}{b_n^3}$.

The operators $\hat{R}_n$ preserve the linear functions, i.e., for $h(t) = ct + d$, where c, d any real constants, we obtain $\hat{R}_n(h;x) = h(x)$.

Throughout, in this section we have taken $\lambda(x) = \frac{x(1-a_nx)}{na_n}$ ($a_n = n^{-1/3}$ & $b_n = n^{2/3}$) and $n \geq 2,$ *where* $n \in \mathbb{N}$.

2.2.2 Voronovskaya Type Results

In this section first we establish a direct local approximation theorem for the modified operators $\hat{R}_n$ in ordinary approximation then we compute the rates of convergence and Voronovskaya type result of these operators (2.2.2).

Theorem 2.2.4. *Let* $f \in C_B[0,\infty)$*, then for every* $x \in \left[1, \frac{n}{b_n}\right)$ *and for* $C > 0,\ n \geq 2$ *where* $n \in \mathbb{N}$*, we have*

$$\left|\left(\hat{R}_nf\right)(x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\lambda(x)}\right). \tag{2.2.3}$$

Proof. Let $g \in W_\infty^2$. Using Taylor's expansion

$$g(y) = g(x) + g'(x)(y-x) + \int_x^y (y-u)g''(u)du.$$

From Lemma 2.2.3, we have

$$\left(\hat{R}_ng\right)(x) - g(x) = \left(\hat{R}_n\int_x^y (y-u)g''(u)du\right)(x).$$

We know that

$$\left|\int_x^y (y-u)g''(u)du\right| \leq (y-u)^2\left\|g''\right\|.$$

Therefore

$$\left|\left(\hat{R}_ng\right)(x) - g(x)\right| \leq \left(\hat{R}_n(y-u)^2\right)(x)\left\|g''\right\| = \lambda(x)\left\|g''\right\|.$$

By Lemma 2.2.2, we have

$$\left|\left(\hat{R}_n f\right)(x)\right| \leq \frac{1}{(1+a_n r_n(x))^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_n r_n(x))^k \leq \|f\|.$$

Hence

$$\begin{aligned}\left|\left(\hat{R}_n f\right)(x)-f(x)\right| &\leq \left|\left(\hat{R}_n(f-g)\right)(x)-(f-g)(x)\right|+\left|\left(\hat{R}_n g\right)(x)-g(x)\right| \\ &\leq 2\|f-g\|+\lambda(x)\|g''\|.\end{aligned}$$

Taking the infimum on the right side over all $g \in W_\infty^2$ and using (0.2.2), we get the required result. □

Remark 2.2.5. *Under the same conditions of Theorem 2.2.4, we obtain*

$$|(R_n f)(x)-f(x)| \leq C\omega_2\left(f, \sqrt{\frac{x\left\{1+n(a_n x)^3\right\}}{b_n(1+a_n x)^2}}\right). \tag{2.2.4}$$

Theorem 2.2.6. *If a function f is such that its first and second derivative are bounded in $[0,\infty)$ then for $C>0$ and $n \geq 2$, where $n \in \mathbb{N}$, we get*

$$na_n\left[\left(\hat{R}_n f\right)(x)-f(x)\right]=\frac{x(1-a_n x)}{2}f''(x). \tag{2.2.5}$$

Proof. Applying Taylor's theorem we write that

$$f(t)-f(x)=(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+\frac{(t-x)^2}{2!}\xi(t,x), \tag{2.2.6}$$

where $\xi(t,x)$ is a bounded function $\forall\, t,x$ and $\lim_{t\to x}\xi(t,x)=0$.

Using (2.2.2) and (2.2.6), we obtain

$$\left(\hat{R}_n f\right)(x)-f(x)=f'(x)\hat{R}_n(\varphi_x,x)+\frac{f''(x)}{2}\hat{R}_n\left(\varphi_x^2,x\right)+\frac{1}{2}\hat{R}_n\left(\varphi_x^2,x\right)\xi(t,x).$$

From Lemma 2.2.3, we get

$$na_n\left[\left(\hat{R}_n f\right)(x)-f(x)\right]=\frac{f''(x)}{2}\left(x\left(1-a_nx\right)\right)+\frac{na_n}{2}\hat{R}_n\left(\varphi_x^2,x\right)\xi(t,x).$$

Now, we have to show that as $n\to\infty$, the value of $I=\frac{na_n}{2}\hat{R}_n\left(\varphi_x^2,x\right)\xi(t,x)\to 0$. Let $\varepsilon\geq 0$ be given since $\xi(t,x)\to 0$ as $t\to x$, then there exists $\delta>0$ such that when $|t-x|<\delta$ we have $|\xi(t,x)|<\varepsilon$ and when $|t-x|\geq\delta$, we write

$$|\xi(t,x)|\leq C<C\frac{(t-x)^2}{\delta^2}.$$

Thus, for all $t,x\in[0,\infty)$

$$|\xi(t,x)|\leq\varepsilon+C\frac{(t-x)^2}{\delta^2}$$

and

$$\begin{aligned}I&\leq na_n\left(\hat{R}_n\varphi_x^2\left(\varepsilon+\frac{C\varphi_x^2}{\delta^2}\right)\right)(x)\\&\leq na_n\varepsilon\left(\hat{R}_n\varphi_x^2\right)(x)+\frac{na_nC}{\delta^2}\left(\hat{R}_n\varphi_x^4\right)(x)\\&=E_1+E_2,\end{aligned}$$

by Lemma 2.2.3 and choosing ε arbitrarily as 0, we obtain E_1 and E_2 tends to 0 as $n\to\infty$.

Hence

$$I\to 0\ \text{ as }\ n\to\infty.$$

This leads to (2.2.5). □

Theorem 2.2.7. *If $g\in C_B^2[0,\infty)$ then we have*

$$\left|\left(\hat{R}_n g\right)(x)-g(x)\right|\leq\frac{\lambda(x)}{2}\|g\|_{C_B^2}. \tag{2.2.7}$$

Proof. We have

$$g(t)-g(x)=(t-x)g'(x)+\frac{1}{2}(t-x)^2g''(\zeta) \tag{2.2.8}$$

where $t \leq \zeta \leq x$. From Lemma 2.2.3 and (2.2.8), we get

$$\begin{aligned}\left|\left(\hat{R}_n g\right)(x) - g(x)\right| &\leq \|g'\| \left|\left(\hat{R}_n \varphi_x\right)(x)\right| + \frac{1}{2}\|g''\| \left|\left(\hat{R}_n \varphi_x^2\right)(x)\right| \\ &\leq \frac{\lambda(x)}{2}\|g''\| = \frac{\lambda(x)}{2}\|g\|_{C_B^2}.\end{aligned}$$

This proves the theorem. □

Remark 2.2.8. *Under the same conditions of Theorem 2.2.7, we obtain*

$$|(R_n g)(x) - g(x)| \leq \frac{x\left\{1 + n(a_n x)^3\right\}}{2b_n(1 + a_n x)^2}\|g\|_{C_B^2}. \tag{2.2.9}$$

Theorem 2.2.9. *For $f \in C_B[0,\infty)$, we obtain*

$$\left|\left(\hat{R}_n f\right)(x) - f(x)\right| \leq A\left\{\omega_2\left(f, \frac{1}{2}\sqrt{\lambda(x)}\right) + \min\left(1, \frac{\lambda(x)}{4}\right)\|f\|_{C_B}\right\}, \tag{2.2.10}$$

where constant A depends on f & $\left\{\frac{\lambda(x)}{2}\right\}$.

Proof. For $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$ we write

$$\left(\hat{R}_n f\right)(x) - f(x) = \left(\hat{R}_n f\right)(x) - \left(\hat{R}_n g\right)(x) + \left(\hat{R}_n g\right)(x) - g(x) + g(x) - f(x).$$

From (2.2.7) and Peetre's K_2-functional(0.2.2), we get

$$\begin{aligned}\left|\left(\hat{R}_n f\right)(x) - f(x)\right| &= \left|\left(\hat{R}_n f\right)(x) - \left(\hat{R}_n g\right)(x)\right| + \left|\left(\hat{R}_n g\right)(x) - g(x)\right| \\ &\quad + |g(x) - f(x)| \\ &\leq \left\|\hat{R}_n f\right\|\|f - g\| + \frac{\lambda(x)}{2}\|g\|_{C_B^2} + \|f - g\| \\ &\leq 2\|f - g\| + \frac{\lambda(x)}{2}\|g\|_{C_B^2} \\ &= 2\left\{\|f - g\| + \frac{\lambda(x)}{4}\|g\|_{C_B^2}\right\} \leq 2K_2\left\{f, \frac{\lambda(x)}{4}\right\} \\ &\leq 2A\left\{\omega_2\left(f, \frac{1}{2}\sqrt{\lambda(x)}\right) + \min\left(1, \frac{\lambda(x)}{4}\right)\|f\|_{C_B}\right\}.\end{aligned}$$

This completes the proof. □

Remark 2.2.10. *By the same conditions of Theorem 2.2.9, we get*

$$\begin{aligned} &\left|\left(R_n f\right)(x)-f(x)\right| \qquad (2.2.11)\\ &\leq 2A\left\{\omega_2\left(f,\frac{1}{2}\sqrt{\frac{x\left\{1+n(a_nx)^3\right\}}{b_n(1+a_nx)^2}}\right)+\min\left(1,\frac{x\left\{1+n(a_nx)^3\right\}}{4b_n(1+a_nx)^2}\right)\|f\|_{C_B}\right\}. \end{aligned}$$

Theorem 2.2.11. *For every $f\in C\left[0,\infty\right),\ x\in\left[1,\frac{n}{b_n}\right)$ and $n\geq 2$ where $n\in\mathbb{N}$, we obtain*

$$\left|\left(\hat{R}_n f\right)(x)-f(x)\right|\leq 2\omega\left(f,\delta_x\right), \qquad (2.2.12)$$

where $\omega\left(f,\delta_x\right)$ is the modulus of continuity of f.

Proof. Let $f\in C\left[0,\infty\right),\quad x\in\left[1,\frac{n}{b_n}\right)$ and $n\geq 2$, where $n\in\mathbb{N}$. Using linearity and monotonicity of $\hat{R}_n$, we obtain, for every $\delta_x>0$, that

$$\left|\left(\hat{R}_n f\right)(x)-f(x)\right|\leq\omega(f,\delta_x)\left\{1+\frac{1}{\delta_x}\sqrt{\hat{R}_n\left(\varphi_x^2,x\right)}\right\}.$$

By using Lemma 2.2.3 and choosing $\delta_x=\sqrt{\lambda\left(x\right)}$ this completes the proof. □

Remark 2.2.12. *For the original operator R_n defined in, we may write that, for every $f\in C\left[0,\infty\right)$*

$$\left|\left(R_n f\right)(x)-f(x)\right|\leq 2\omega\left(f,\nu_x\right), \qquad (2.2.13)$$

where

$$\nu_x=\sqrt{\frac{x\left\{1+n(a_nx)^3\right\}}{b_n(1+a_nx)^2}}$$

and $\omega\left(f,\nu_x\right)$ is the modulus of continuity of f. The error estimate in Theorem 2.2.11 is better than that of (2.2.13) for $f\in C\left[0,\infty\right)$ and $x\in\left[1,\frac{n}{b_n}\right)$, we get $\delta_x\leq\nu_x$.

For rate of convergence of these operators by means of the Lipschitz class $Lip_M(\gamma)$, $(0<\gamma\leq 1)$ we have, $f\in C_B\left[0,\infty\right)$ belongs to $Lip_M(\gamma)$ if the inequality (0.2.4) holds.

Theorem 2.2.13. *If $f\in Lip_M(\gamma)$ and $x\in\left[1,\frac{n}{b_n}\right)$, we have*

$$\left|\left(\hat{R}_n f\right)(x)-f(x)\right|\leq M[\lambda\left(x\right)]^{\gamma/2}.$$

Proof. For $f \in Lip_M(\gamma)$, $x \in \left[1, \frac{n}{b_n}\right)$ and $n \geq 2$,where $n \in \mathbb{N}$, from inequality (0.2.4) and using the Hölder inequality with $p = \frac{2}{\gamma}, q = \frac{2}{2-\gamma}$, we get

$$\begin{aligned}\left|\left(\hat{R}_n f\right)(x) - f(x)\right| &\leq \left(\hat{R}_n \left|f(t) - f(x)\right|\right)(x) \\ &\leq M\left(\hat{R}_n |t-x|^{\gamma}\right)(x) \\ &\leq M\left\{\left(\hat{R}_n \varphi_x^2\right)(x)\right\}^{\gamma/2} \\ &\leq M[\lambda(x)]^{\gamma/2}.\end{aligned}$$

This leads to the result. □

Remark 2.2.14. *From Lemma 2.2.1, under the same conditions of Theorem 2.2.13 for the original operator R_n, then we have the following result*

$$\left|\left(R_n f\right)(x) - f(x)\right| \leq M\left\{\frac{x\left\{1 + n(a_n x)^3\right\}}{2b_n(1 + a_n x)^2}\right\}^{\gamma/2}$$

for every $f \in Lip_M(\gamma)$, $x \in \left[1, \frac{n}{b_n}\right)$.

2.3 Better Error Estimation of Modified Baskakov Operators

For $f \in C[0, \infty)$, the Baskakov operator was introduced by V.A. Baskakov [19] as (0.3.3).

Several modifications of Baskakov operators have been studied by many mathematicians (see [34], [66], [76], [117]). Now we have given another modification of Baskakov operators as:

$$P_n(f, x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n+1}\right), \tag{2.3.1}$$

where

$$p_{n,k}(x) = \left(1 - \frac{1}{n+1}\right)^n \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{x^k}{\left(1 - \frac{1}{n+1} + x\right)^{n+k}}.$$

It is a generalized form of Baskakov operators, i.e., if n is sufficient large then our operators convert in the classical Baskakov operators (0.3.3).

Durrmeyer variants of these operators (2.3.1) are:

$$L_n\left(f, x\right) = \frac{(n^2-1)}{n} \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) dt. \tag{2.3.2}$$

This modification is based on recent modification of Bernstein operator, which is given by Deo et al. [39] and they established a Voronovskaya type asymptotic formula and obtain an estimate of error in terms of modulus of continuity in simultaneous approximation by the linear combinations of these operators. In [37], Deo and Singh have given some theorems on the approximation of the r-th derivative of a function f by same operators.

2.3.1 Construction of the Operators

Now we consider King type modification of these operators (2.3.2).

$$\hat{L}_n\left(f, (r_n(x))\right) = \frac{(n^2-1)}{n} \sum_{k=0}^{\infty} p_{n,k}\left(r_n(x)\right) \int_0^{\infty} p_{n,k}(t) f(t) dt, \tag{2.3.3}$$

where

$$p_{n,k}\left(r_n(x)\right) = \left(1 - \frac{1}{n+1}\right)^n \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{\left(r_n(x)\right)^k}{\left(1 - \frac{1}{n+1} + r_n(x)\right)^{n+k}}$$

and

$$r_n(x) = \frac{(n+1)(n-2)x - n}{n(n+1)}.$$

In this section, we give some ordinary as well as simultaneous approximation properties and we also study some ordinary approximation results including Voronovskaya type results and better error estimates for these operators (2.3.2).

2.3.2 Properties and Basic Results

In this section we write some basic results to prove our theorem.

Lemma 2.3.1. *For $n \geq 1$ one obtains,*

$$P_n(1, x) = 1,$$

$$P_n(t, x) = x,$$

$$P_n\left(t^2, x\right) = \left(1 + \frac{1}{n}\right) x^2 + \frac{x}{n+1}.$$

Lemma 2.3.2. *For $m \in \mathbb{N}^0$ (the set of non-negative integers), the m-th order moment of the operator is defined as*

$$T_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m.$$

Consequently, $T_{n,0}(x) = 1$ and $T_{n,1}(x) = x/n$. There holds the recurrence relation

$$nT_{n,m+1}(x) = x\left(1 - \frac{1}{n+1} + x\right)\left[T'_{n,m}(x) + mT_{n,m-1}(x)\right].$$

Proof. It is easily observed that

$$x\left(1 - \frac{1}{n+1} + x\right) p'_{n,k}(x) = n\left(\frac{k}{n+1} - x\right) p_{n,k}(x). \tag{2.3.4}$$

We have

$$T_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m$$

$$T'_{n,m}(x) = \sum_{k=0}^{\infty} p'_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m - mT_{n,m-1}(x)$$

$$x\left(1 - \frac{1}{n+1} + x\right)T'_{n,m}(x) = \sum_{k=0}^{\infty} x\left(1 - \frac{1}{n+1} + x\right)p'_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m - mx\left(1 - \frac{1}{n+1} + x\right)T_{n,m-1}(x)$$

Using (2.3.4),

$$\begin{aligned} x\left(1 - \frac{1}{n+1} + x\right)T'_{n,m}(x) &= n\sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m \\ &\quad - mx\left(1 - \frac{1}{n+1} + x\right)T_{n,m-1}(x) \\ &= nT_{n,m+1}(x) - mx\left(1 - \frac{1}{n+1} + x\right)T_{n,m-1}(x). \end{aligned}$$

Hence the result. Thus

(i) $T_{n,m}(x)$ is a polynomial in x of degree $\leq m$,

(ii) For every $x \in [0, \infty)$, $T_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, where $[\beta]$ denotes the integral part of β.

□

Lemma 2.3.3. *Let the $m-th$ order moment be defined by*

$$U_{r,n,m} = (n-r-1)\left(1 + \frac{1}{n}\right)\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^{\infty} p_{n-r,k+r}(t)(t-x)^m dt,$$

then

$$U_{r,n,0}(x) = 1, \quad U_{r,n,1}(x) = \frac{\{n(1+2x)+2x\}(1+r)}{(n+1)(n-r-2)}, \quad n-r > 2 \tag{2.3.5}$$

$$U_{r,n,2}(x) = \frac{2\left[(1+r)^2\{n(1+2x)+2x\}^2 + x\{n(1+x)+x\}(n^2-1)\right]}{(n+1)^2(n-r-2)(n-r-3)} \tag{2.3.6}$$

and

$$(n-r-m-2)U_{r,n,m+1}(x) = (m+r+1)\left(1-\frac{1}{n+1}+2x\right)U_{r,n,m}(x) \qquad (2.3.7)$$
$$+x\left(1-\frac{1}{n+1}+x\right)\left[U'_{r,n,m}(x)+2mU_{r,n,m-1}(x)\right].$$

Further, for all $x \in [0,\infty)$

$$U_{r,n,m}(x) = O\left(n^{-\left[\frac{(m+1)}{2}\right]}\right). \qquad (2.3.8)$$

Proof. We can easily obtain (2.3.5) and (2.3.6) by using the definition of $T_{n,m}(x)$. For the proof of (2.3.7), we proceed as follows. First

$$x\left(1-\frac{1}{n+1}+x\right)U'_{r,n,m}(x) = (n-r-1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{\infty}x\left(1-\frac{1}{n+1}+x\right)p'_{n+r,k}(x)$$
$$\cdot\int_0^{\infty}p_{n-r,k+r}(t)(t-x)^m dt - mx\left(1-\frac{1}{n+1}+x\right)U_{r,n,m-1}(x).$$

Now, using inequality (2.3.4) two times, then we get

$$x\left(1-\frac{1}{n+1}+x\right)\left[U'_{r,n,m}(x)+mU_{r,n,m-1}(x)\right]$$
$$=\left(1+\frac{1}{n}\right)(n-r-1)\sum_{k=0}^{\infty}p_{n+r,k}(x)\left[\frac{nk}{n+1}-(n+r)x\right]\int_0^{\infty}p_{n-r,k+r}(t)(t-x)^m dt$$
$$=\left(1+\frac{1}{n}\right)(n-r-1)\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^{\infty}\left[\frac{n(k+r)}{n+1}-(n-r)t\right]p_{n-r,k+r}(t)(t-x)^m dt$$
$$-r\left(1-\frac{1}{n+1}+2x\right)U_{r,n,m}(x)+(n-r)U_{r,n,m+1}(x)$$
$$=-r\left(1-\frac{1}{n+1}+2x\right)U_{r,n,m}(x)+(n-r)U_{r,n,m+1}(x)+\left(1+\frac{1}{n}\right)(n-r-1)$$
$$\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^{\infty}\left[\left(1-\frac{1}{n+1}+2x\right)(t-x)+(t-x)^2+x\left(1-\frac{1}{n+1}+x\right)\right]$$
$$p'_{n-r,k+r}(t)(t-x)^m dt$$

$$= -r\left(1-\frac{1}{n+1}+2x\right)U_{r,n,m}(x)+(n-r)U_{r,n,m+1}(x)-\left(1-\frac{1}{n+1}+2x\right)$$
$$(m+1)U_{r,n,m}(x)-(m+2)U_{r,n,m+1}(x)-x\left(1-\frac{1}{n+1}+x\right)mU_{r,n,m-1}(x).$$

This leads to (2.3.7). The proof of (2.3.8) easily follows from (2.3.5) and (2.3.7). □

Corollary 2.3.4. *When $r = 0$, we conclude from Lemma 2.3.3*

$$U_{n,0}(x) = 1, \quad U_{n,1}(x) = \frac{n(1+2x)+2x}{(n+1)(n-2)}, \quad n > 2 \tag{2.3.9}$$

$$U_{n,2}(x) = \frac{2\left[\{n(1+2x)+2x\}^2 + x\{n(1+x)+x\}(n^2-1)\right]}{(n+1)^2(n-2)(n-3)}. \tag{2.3.10}$$

Further, for all $x \in [0,\infty)$

$$U_{n,m}(x) = O\left(n^{-\left[\frac{(m+1)}{2}\right]}\right). \tag{2.3.11}$$

Corollary 2.3.5. *Let $e_i(x) = x^i,\ i = 0,1,2,$ then for $x \in [0,\infty)$ and $n \in \mathbb{N}$, we have*

(*i*) $(L_n e_0)(x) = 1,$

(*ii*) $(L_n e_1)(x) = \frac{n\{1+(1+n)x\}}{(n+1)(n-2)},$

(*iii*) $(L_n e_2)(x) = \frac{n(n+1)^3x^2+4n^2(1+n)x+2n^2}{(n+1)^2(n-2)(n-3)}.$

Lemma 2.3.6. *Let f be r times differentiable on $[0,\infty)$ such that $f^{(r-1)} = O(t^\alpha)$, for some $\alpha > 0$ as $t \to \infty$ then for $r = 1,2,3,...$ and $n > \alpha + r$, we have*

$$\left(L_n^{(r)} f\right)(x) = \frac{(n+r-1)!(n-r-1)!(n+1)}{n!(n-2)!}\sum_{k=0}^{\infty} p_{n+r,k}(x)\int_0^\infty p_{n-r,k+r}(t)f^{(r)}(t)dt. \tag{2.3.12}$$

Proof. We have by Leibniz theorem

$$\left(L_n^{(r)} f\right)(x) = \left(\frac{n}{n+1}\right)^n\left(\frac{n^2-1}{n}\right)\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(-1)^{r-i}(n+k+r-i-1)}{(n-1)!(k-i)!}$$

$$\frac{x^{k-i}}{\left(1-\frac{1}{n+1}+x\right)^{n+k+r-i}}\cdot\int_0^\infty p_{n,k}(t)f(t)dt$$

$$=\frac{(n+r-1)!}{(n-2)!}\left(\frac{n+1}{n}\right)^{r+1}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^\infty\sum_{i=0}^{r}(-1)^{r-i}\begin{pmatrix} r \\ i \end{pmatrix}p_{n,k+i}(t)f(t)dt.$$

Again applying Leibniz theorem

$$p^{(r)}_{n-r,k+r}(t)=\sum_{i=0}^{r}\left(\frac{n+1}{n}\right)^{r}\frac{(n-1)!}{(n-r-1)!}(-1)^{i}\begin{pmatrix} r \\ i \end{pmatrix}p_{n,k+i}(t)$$

$$\left(L_n^{(r)}f\right)(x)=\frac{(n+r-1)!(n-r-1)!(n+1)}{n!(n-2)!}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^\infty(-1)^{r}p^{(r)}_{n-r,k+r}(t)f(t)dt.$$

Further integrating by parts r times, we get the required result. □

Lemma 2.3.7. *Let* $e_i(x)=x^i,\ i=0,1,2$ *then for each* $x\geq 0$*, we have*

(i) $\left(\hat{L}_n e_0\right)(x)=1,$

(ii) $\left(\hat{L}_n e_1\right)(x)=x,$

(iii) $\left(\hat{L}_n e_2\right)(x)=\frac{(n+1)(n-2)\left[(n+1)^2(n-2)x^2+2nx(n-1)\right]-n^2(n-1)}{n(n+1)^2(n-2)(n-3)}.$

Lemma 2.3.8. *For* $x\in[0,\infty),\ n\in N$ *and* $\varphi_x(t)=t-x,$ *we have*

(i) $\left(\hat{L}_n\varphi_x\right)(x)=0,$

(ii) $\left(\hat{L}_n\varphi_x^2\right)(x)=\frac{(n-1)\left[2(n+1)(n-2)x\{(n+1)x+n\}-n^2\right]}{n(n+1)^2(n-2)(n-3)},$

(iii) $\left(\hat{L}_n\varphi_x^m\right)(x)=O\left(n^{-\left[\frac{m+1}{2}\right]}\right).$

Operator preserves the linear functions, i. e., for $h(t)=ct+d$, where c,d any real constants, we get $\left(\hat{L}_n h\right)(x)=h(x)$.

2.3.3 Main Results

Let $f\in C_B[0,\infty)$ be the space of all real valued continuous bounded functions on $[0,\infty)$, equipped with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(t)|$. The Peetre's $K-$functional is defined by (0.2.2) where $W_\infty=\{g\in C_B[0,\infty)\}$. From [42], there exists a positive constant C such that

$$K(f,\delta) \leq C\omega\left(f,\sqrt{\delta}\right). \tag{2.3.13}$$

Theorem 2.3.9. *If a function f is such that its first and second order derivatives are bounded in $[0,\infty)$, then*

$$\lim_{n\to\infty}(n+1)\left\{\left(L_n f\right)(x) - f(x)\right\} = f'(x)(1+2x) + x(1+x)f''(x). \tag{2.3.14}$$

Proof. Using Taylor's theorem we write that

$$f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \frac{(t-x)^2}{2!}\eta(t,x), \tag{2.3.15}$$

where $\eta(t,x)$ is a bounded function $\forall t,x$ and $\lim_{t\to x}\eta(t,x) = 0$.

Now applying (2.3.2) and (2.3.15), we get

$$\left(L_n f\right)(x) - f(x) = f'(x)\left(L_n(t-x)\right)(x) + \frac{f''(x)}{2}\left(L_n(t-x)^2\right)(x) + I_1,$$

where

$$I_1 = \frac{1}{2}\left(L_n(t-x)^2\eta(t,x)\right)(x).$$

Using (2.3.9) and (2.3.10), we get

$$\begin{aligned}
\left(L_n f\right)(x) - f(x) &= f'(x)U_{n,1}(x) + \frac{f''(x)}{2}U_{n,2}(x) + I_1\\
&= f'(x)\left\{\frac{n(1+2x)+2x}{(n+1)(n-2)}\right\}\\
&\quad + f''(x)\frac{\left[\left\{n(1+2x)+2x\right\}^2 + x\left\{n(1+x)+x\right\}(n^2-1)\right]}{(n+1)^2(n-2)(n-3)} + I_1\\
\Rightarrow (n+1)\left\{\left(L_n f\right)(x) - f(x)\right\} &= f'(x)\left\{\frac{n(1+2x)+2x}{n-2}\right\}\\
&\quad + f''(x)\frac{\left[\left\{n(1+2x)+2x\right\}^2 + x\left\{n(1+x)+x\right\}(n^2-1)\right]}{(n+1)(n-2)(n-3)}\\
&\quad + (n+1)I_1.
\end{aligned}$$

Now, we have to show that as $n \to \infty$, the value of $(n+1)I_1 \to 0$. Let $\varepsilon > 0$ be given since $\eta(t,x) \to \infty$ as $t \to 0$, then there exists $\delta > 0$ such that when $|t-x| < \delta$ we have $|\eta(t,x)| < \varepsilon$ and when $|t-x| \geq \delta$, we write

$$|\eta(t,x)| \leq M < M\frac{(t-x)^2}{\delta^2}.$$

Thus, for all $t, x \in [0,\infty)$

$$\begin{aligned}|\eta(t,x)| &\leq \varepsilon + M\frac{(t-x)^2}{\delta^2}\\ (n+1)I_1 &\leq (n+1)\left\{L_n(t-x)^2\left(\varepsilon + \frac{M(t-x)^2}{\delta^2}\right)\right\}(x)\\ &\leq \varepsilon(n+1)\left\{L_n(t-x)^2\right\}(x) + \frac{M}{\delta^2}(n+1)\left\{L_n(t-x)^4\right\}(x).\end{aligned}$$

Using (2.3.9) and (2.3.11), we see that,

$$(n+1)I_1 \to 0 \ \text{ as } \ n \to \infty.$$

This leads to (2.3.14). □

Remark 2.3.10. *Under the same conditions of Theorem 2.3.9, we have*

$$\lim_{n\to\infty}(n+1)\left\{\left(\hat{L}_n f\right)(x) - f(x)\right\} = x(1+x)f''(x). \tag{2.3.16}$$

Theorem 2.3.11. *If $g \in C_B^2[0,\infty)$ then we have*

$$|(L_n g)(x) - g(x)| \leq \lambda_n(x)\|g\|_{C_B^2}, \tag{2.3.17}$$

where

$$\lambda_n(x) = \frac{n(1+2x)+2x}{(n+1)(n-2)}.$$

Proof. We write that

$$g(t) - g(x) = (t-x)g'(x) + \frac{1}{2}(t-x)^2 g''(\xi) \tag{2.3.18}$$

where $t \leq \xi \leq x$. Now applying (2.3.2) on (2.3.18)

$$\begin{aligned}
|(L_n g)(x) - g(x)| &\leq \|g'\| \, |L_n(t-x)(x)| + \frac{1}{2} |g''| \left| L_n(t-x)^2(x) \right| \\
&\leq \frac{n(1+2x)+2x}{(n+1)(n-2)} \|g'\| \\
&\quad + \frac{\left[\{n(1+2x)+2x\}^2 + x \{n(1+x)+x\} (n^2-1) \right]}{(n+1)^2(n-2)(n-3)} \|g''\| \\
&\leq \lambda_n(x) \{ \|g'\| + \|g''\| \} \leq \lambda_n(x) \|g\|_{C_B^2} .
\end{aligned}$$

□

Remark 2.3.12. *We may note here that under the conditions of Theorem 2.3.11, we obtain*

$$\left| \left(\hat{L}_n g \right)(x) - g(x) \right| \leq \lambda_n^*(x) \|g\|_{C_B^2} , \tag{2.3.19}$$

where

$$\lambda_n^*(x) = \frac{(n-1)\left[2(n+1)(n-2)x \{(n+1)x+n\} - n^2 \right]}{n(n+1)^2(n-2)(n-3)}.$$

Theorem 2.3.13. *For $f \in C_B[0,\infty)$, we obtain*

$$|(L_n f)(x) - f(x)| \leq A \left\{ \omega_2 \left(f, \frac{\sqrt{\lambda_n(x)}}{2} \right) + \min \left(1, \frac{\lambda_n(x)}{2} \right) \|f\|_{C_B} \right\}, \tag{2.3.20}$$

where constant A depends on f and $\lambda_n(x)$.

Proof. For $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$ we write

$$(L_n f)(x) - f(x) = (L_n f)(x) - (L_n g)(x) + (L_n g)(x) - g(x) + g(x) - f(x).$$

By using (2.3.17) and Peetre's K_2-functional (0.2.2), we get

$$\begin{aligned}
|(L_n f)(x) - f(x)| &= |(L_n f)(x) - (L_n g)(x)| + |(L_n g)(x) - g(x)| + |g(x) - f(x)| \\
&\leq \|L_n f\| \, \|f-g\| + \lambda_n(x) \|g\|_{C_B^2} + \|f-g\|
\end{aligned}$$

$$\leq 2\|f-g\|+\lambda_n(x)\|g\|_{C_B^2}$$
$$\leq 2\left\{\|f-g\|+\frac{1}{2}\lambda_n(x)\|g\|_{C_B^2}\right\}$$
$$\leq 2K\left\{f;\frac{1}{2}\lambda_n(x)\right\}$$
$$\leq 2A\left\{\omega_2\left(f,\frac{1}{2}\sqrt{\lambda_n(x)}\right)+\min\left(1,\frac{1}{2}\lambda_n(x)\right)\|f\|_{C_B}\right\}.$$

This completes the proof of the theorem. □

Remark 2.3.14. *In the proof of Theorem 2.3.13, if we use Lemma 2.3.8, then we get following result for the operators*

$$\left|\left(\hat{L}_n f\right)(x)-f(x)\right|\leq A_1\left\{\omega_2\left(f,\frac{\sqrt{\lambda_n^*(x)}}{2}\right)+\min\left(1,\frac{\lambda_n^*(x)}{2}\right)\|f\|_{C_B}\right\},\quad (2.3.21)$$

where constant A_1 depends on f and $\lambda_n^*(x)$.

Chapter 3

Some Approximation Theorems For Multivariate Bernstein Operators

3.1 Introduction

The Bernstein polynomials are given by (0.3.1). Some approximation properties of multivariate Bernstein operators were studied by several researchers see ([27], [26], [44], [46], [122] and [139]). Guo [64] and Li [95] studied two-dimensional Baskakov operators and Szász–Durrmeyer operators respectively and gave some interesting results. It may be observed that bivariate Bernstein operators play an important role in theory of approximation.

In [24], P.L. Butzer also introduced two dimensional Bernstein polynomials $L_{n,m}(f, x, y)$ on the square $\square := \{(x, y) : 0 \le x, y \le 1\}$ and defined for $f : [0, 1] \times [0, 1] \to \mathbb{R}$, as follows:

$$L_{n,m}(f, x, y) = \sum_{k=0}^{n} \sum_{l=0}^{m} b_{n,k}(x) b_{m,l}(y) f\left(\frac{k}{n}, \frac{l}{m}\right), \quad (x, y) \in [0, 1] \times [0, 1].$$

D.D. Stancu [122] defined another bivariate Bernstein operators on the triangle

$\triangle := S = \{(x,y) : x+y \leq 1, 0 \leq x,y \leq 1\}$ for the functions $f : S \to \mathbb{R}$. More precisely, in [122] there is considered $G_n(f,x,y)$ with

$$\begin{aligned} G_n(f,x,y) &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}x^k y^l(1-x-y)^{n-k-l} f\left(\frac{k}{n},\frac{l}{n}\right) \\ &\equiv \sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y) f\left(\frac{k}{n},\frac{l}{n}\right), \quad (x,y) \in S. \end{aligned} \tag{3.1.1}$$

If Π_n denotes the linear space of all real polynomials of degree $\leq n$, the one-dimensional Bernstein-Durrmeyer operators $\bar{B}_n : C[0,1] \to \Pi_n$ are given by

$$\bar{B}_n(f,x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 b_{n,k}(t)f(t)dt, \quad x \in [0,1] \ and \ f \in C[0,1].$$

In 1992, Zhou [138] defined the two-dimensional Bernstein-Durrmeyer operators $Q_n : f \to Q_n(f,\cdot,\cdot),\ f \in C(S)$, as

$$Q_n(f,x,y) = (n+1)(n+2)\sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y)\int_0^1\int_0^{1-t} b_{n,k,l}(s,t)f(s,t)ds\, dt, \tag{3.1.2}$$

and studied its rate of approximation by means of K-functionals and smoothness of functions.

Very recently, Deo et al. [39] has given another modification of Bernstein operators B_n and studied Voronovskaya type asymptotic formula as well as error estimate in terms of modulus of continuity in simultaneous approximation by the linear combinations of these operators. In [33, 32], author has also studied direct result and Voronovskaya type asymptotic formula for the exponential-type operators in simultaneous approximation. Now in this chapter, we study some direct results for two dimensional Bernstein operators $G_n(f,x,y)$ by using the multivariate decomposition skills and in the second part we give equivalent theorems for two dimensional Bernstein-Durrmeyer operators in $C(S)$.

3.2 Auxiliary Results

First we need some results for one-dimensional Bernstein operators, definitions and some notational convention, which are necessary to prove the main result.

$$C_{a,b,c,d}(\mathbb{R}_+^2) = \{f : f \in C(\mathbb{R}_+^2),\ wf \in L_\infty(\mathbb{R}_+^2)\},$$

$$C^0_{a,b,c,d}(\mathbb{R}_+^2) = \{f : f \in C_{a,b,c,d}(\mathbb{R}_+^2), f(x,0) = f(0,y) = 0\}$$

and
$$w(x,y) = x^a(1-x)^b \left(\frac{y}{1-x}\right)^c \left(1 - \frac{y}{1-x}\right)^d,$$

where $0 < a,\ c < 1;\ \ b, d < 0;\ \ w(x) = x^a(1-x)^b$ and the norm $\|.\|_\infty$ is defined as

$$\|f\|_\infty = \sup_{(x,y)\in\mathbb{R}_+^2} |f(x,y)|\,.$$

Also the weighted norm is given by

$$\|f\|_w = \sup_{(x,y)\in\mathbb{R}_+^2} \{|w(x,y)f(x,y)| + |f(x,0)| + |f(0,y)|\}\,.$$

For $0 < \lambda < 1$, we define the Peetre's $K-$functional as

$$K_{t,\phi^\lambda}(f,t) = \inf_{g\in D}\{\|f-g\|_w + t\Phi(g)\},$$

where

$$D = \left\{g : g \in C(\mathbb{R}_+^2),\ \Phi(g) < \infty,\ \ g_x, g_y \in A.C_{loc}\right\},$$

and with $\phi^2(x) = x(1-x)$,

$$\Phi(g) = max\left\{\left\|\phi^{2\lambda} g_{xx}\right\|_w,\ \left\|\phi^{2\lambda} g_{yy}\right\|_w,\ \ \left\|\phi^{2\lambda} g_{xy}\right\|_w\right\}.$$

Throughout this chapter we consider C as a positive constant, but it is not necessarily the same in each occurrence.

Lemma 3.2.1. *For the bivariate operators $G_n(f,x,y)$, we have*

$$G_n(f,x,y)=\sum_{k=0}^{n} b_{n,k}(x)\sum_{l=0}^{n-k} b_{n-k,l}\left(\frac{y}{1-x}\right) f\left(\frac{k}{n},\frac{l}{n}\right), \tag{3.2.2a}$$

$$G_n(f,x,y)=\sum_{l=0}^{n} b_{n,m}(y)\sum_{k=0}^{n-l} b_{n-l,k}\left(\frac{x}{1-y}\right) f\left(\frac{k}{n},\frac{l}{n}\right). \tag{3.2.2b}$$

Proof. We have

$$\begin{aligned}
G_n(f,x,y) &= \sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y) f\left(\frac{k}{n},\frac{l}{n}\right)\\
&= \sum_{k=0}^{n} b_{n,k}(x)\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{y^l}{(1-x)^{n-k}}(1-x-y)^{n-k-l} f\left(\frac{k}{n},\frac{l}{n}\right)\\
&= \sum_{k=0}^{n} b_{n,k}(x)\sum_{l=0}^{n-k} b_{n-k,l}\left(\frac{y}{1-x}\right) f\left(\frac{k}{n},\frac{l}{n}\right).
\end{aligned}$$

Similarly, we can prove (3.2.2b). □

Remark 3.2.2. *By properties of one dimensional Bernstein operators, we have*

$$\sum_{l=0}^{n-k} b_{n-k,l}\left(\frac{y}{1-x}\right)=1 \quad \textit{and} \quad \sum_{k=0}^{n-l} b_{n-l,k}\left(\frac{x}{1-y}\right)=1, \quad \textit{where } x,y\in[0,1).$$

Lemma 3.2.3. *[138] Suppose $n\in\mathbb{N}$ and $(x,y)\in\mathbb{R}^2_+$. Then it is easily verified from previous lemma that*

(i) $G_n(1,x,y)=1$,

(ii) $G_n(s,x,y)=x$, *for* $f(s,t)=s$,

(iii) $G_n(t,x,y)=y$, *for* $f(s,t)=t$,

(iv) $G_n((s-x)^2,x,y)=\frac{x(1-x)}{n}$,

(v) $G_n((t-y)^2,x,y)=\frac{y(1-y)}{n}$,

(vi) $G_n\left(s^2,x,y\right)=x^2+\frac{x(1-x)}{n}$,

(vii) $G_n\left(t^2,x,y\right)=y^2+\frac{y(1-y)}{n}$,

(viii) $G_n\left(st,x,y\right)=\left(1-\frac{1}{n}\right)xy$, *(ix)* $G_n((s-x)(t-y),x,y)=\frac{-xy}{n}$.

Remark 3.2.4. *By Lemma 3.2.3 and using Hölder's inequality, we have*

$$G_n(|s-x|\,,x,y)=O(\phi(x)n^{-1/2})$$

and

$$G_n(|t-y|\,,x,y)=O(\phi(y)n^{-1/2}).$$

Lemma 3.2.5. *(i) If* $(x,y)\in\left[\frac{1}{n},1-\frac{1}{n}\right]\times[0,1]$, *then*

$$G_n((s-x)^{2l},x,y)\leq Cn^l(\phi(x))^{2l}.$$

(ii) If $(x,y)\in[0,1]\times\left[\frac{1}{n},1-\frac{1}{n}\right]$, *then*

$$G_n((t-y)^{2l},x,y)\leq Cn^l(\phi(y))^{2l}.$$

Proof. By using the properties of one dimensional Bernstein operators and (3.2.1), we have

$$\begin{aligned}G_n((s-x)^{2l},x,y)&=\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^{2l}b_{n,k}(x)\sum_{l=0}^{n-k}b_{n-k,l}\left(\frac{y}{1-x}\right)\\&=\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^{2l}b_{n,k}(x)\leq Cn^l(\phi(x))^{2l}.\end{aligned}$$

The proof of (ii) is similar, we omit the details. □

Lemma 3.2.6. *For* $0<\lambda<1,\ \phi^2(u)=u(1-u),\ u\in[0,1],\ t\in[0,1]$, *we get*

$$\left|\int_x^t|t-z|\,\phi^{-2\lambda}(z)dz\right|\leq C(t-u)^2\left(\phi^{-2\lambda}(u)+u^{-\lambda}(1-u)^{-\lambda}\right). \tag{3.2.5}$$

Proof. Suppose $z=t+\mu(u-t),\ 0\leq\mu\leq1$, then

$$\left|\int_x^t|t-z|\,\phi^{-2\lambda}(z)dz\right|\leq\int_x^t\frac{|t-z|}{z^\lambda}dz\left\{\frac{1}{(1-u)^\lambda}+\frac{1}{(1-t)^\lambda}\right\}$$

$$
\begin{aligned}
&\leq \int_x^t \frac{\mu(t-u)^2}{(\mu u+(1-\mu)t)^\lambda} d\mu \left\{ \frac{1}{(1-u)^\lambda} + \frac{1}{(1-t)^\lambda} \right\} \\
&\leq (t-u)^2 \int_0^1 \frac{\mu^{1-\lambda}}{u^\lambda} d\mu \left\{ \frac{1}{(1-u)^\lambda} + \frac{1}{(1-t)^\lambda} \right\} \\
&\leq \frac{1}{2-\lambda} (t-u)^2 \big(\phi^{-2\lambda}(u) + u^{-\lambda}(1-u)^{-\lambda} \big).
\end{aligned}
$$

This completes the proof of the theorem. □

Lemma 3.2.7.

$$
\sum_{k=1}^{n} \sum_{l=1}^{n-k} b_{n,k,l}(x,y) \frac{w(x,y)}{w\left(\frac{k}{n}, \frac{l}{n}\right)} \leq C. \tag{3.2.6}
$$

Proof. By [137], we have

$$
\sum_{k=1}^{n} b_{n,k}(x) \frac{w(x)}{w\left(\frac{k}{n}\right)} \leq C \ \textit{and} \ \sum_{l=1}^{n-k} b_{n,k}(x,y) \left(\frac{w(x)}{w\left(\frac{k}{n}\right)} \right)^2 \leq C.
$$

Using multivariate decomposition skills, we obtain

$$
\sum_{k=1}^{n} \sum_{l=1}^{n-k} b_{n,k,l}(x,y) \frac{w(x,y)}{w\left(\frac{k}{n}, \frac{l}{n}\right)} = \sum_{k=1}^{n} b_{n,k}(x) \frac{w(x)}{w(\frac{k}{n})} \sum_{l=1}^{n-k} b_{n-k,l} \left(\frac{y}{1-x} \right) \frac{w\left(\frac{y}{1-x}\right)}{w(\frac{l}{n-k})} \leq C.
$$

□

Lemma 3.2.8. *If* $f \in C^0_{a,b,c,d}(\mathbb{R}^2_+)$, *then*

$$
\big\| G_n(f) \big\|_\infty \leq \big\| f \big\|_\infty. \tag{3.2.7}
$$

Proof. By (3.2.1) and (3.2.7), we get

$$
\begin{aligned}
&\left| w(x,y) \sum_{k=1}^{n} \sum_{l=1}^{n-k} b_{n,k,l}(x,y) f\left(\frac{k}{n}, \frac{l}{n} \right) \right| \\
&\qquad \leq \big\| wf \big\|_\infty \sum_{k=1}^{n} b_{n,k}(x) \frac{w(x)}{w\left(\frac{k}{n}\right)} \sum_{l=1}^{n-k} b_{n-k,l} \left(\frac{y}{1-x} \right) \frac{w\left(\frac{y}{1-x}\right)}{w\left(\frac{l}{n-k}\right)} \leq C \big\| wf \big\|_\infty.
\end{aligned}
$$

This completes the proof. □

Lemma 3.2.9. *For $Q_n(f,x,y)$ given by (3.1.2) and for $f(s,t)=s,t,1-s-t$, we have*

(i) $Q_n(1,x,y)=1$,

(ii) $Q_n(s,x,y)=\frac{1+nx}{n+3}$,

(iii) $Q_n(t,x,y)=\frac{1+ny}{n+3}$,

(iv) $Q_n\left(s^2,x,y\right)=\frac{n(n-1)x^2+4nx+2}{(n+3)(n+4)}$,

(v) $Q_n\left(t^2,x,y\right)=\frac{n(n-1)y^2+4ny+2}{(n+3)(n+4)}$,

(vi) $Q_n\left(st,x,y\right)=\frac{n(n-1)xy+n(x+y)+1}{(n+3)(n+4)}$,

(vii) $Q_n\left((s-x)^2,x,y\right)=\frac{2\left[(6-n)x^2+(n-4)x+1\right]}{(n+3)(n+4)}$,

(viii) $Q_n\left((t-y)^2,x,y\right)=\frac{2\left[(6-n)y^2+(n-4)y+1\right]}{(n+3)(n+4)}$,

(ix) $Q_n(1-s-t,x,y)=\frac{n(1-x-y)+1}{n+3}$,

(x) $Q_n\left((1-s-t)^2,x,y\right)=\frac{n(n-1)(x+y)^2-2n(n+1)(x+y)+(n^2+3n+2)}{(n+3)(n+4)}$.

Lemma 3.2.10. *If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exists, then we have*

(i) $\frac{\partial}{\partial x}Q_n(f,x,y)=\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{n-k}\left(\frac{k}{x}-\frac{n-k-l}{1-x-y}\right)F(x,y)$,

(ii) $\frac{\partial}{\partial y}Q_n(f,x,y)=\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{n-k}\left(\frac{l}{y}-\frac{n-k-l}{1-x-y}\right)F(x,y)$,

(iii) $\frac{\partial^2}{\partial x^2}Q_n(f,x,y)=\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{n-k}\left[\frac{k(k-1)}{x^2}-2\frac{k(n-k-l)}{x(1-x-y)}+\frac{(n-k-l)(n-k-l-1)}{(1-x-y)^2}\right]F(x,y)$,

(iv) $\frac{\partial^2}{\partial y^2}Q_n(f,x,y)=\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{n-k}\left[\frac{l(l-1)}{y^2}-2\frac{l(n-k-l)}{y(1-x-y)}+\frac{(n-k-l)(n-k-l-1)}{(1-x-y)^2}\right]F(x,y)$,

(v) $\frac{\partial^2}{\partial x\partial y}Q_n(f,x,y)=\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{n-k}\left[\frac{lk}{xy}-\left(\frac{k}{x}+\frac{l}{y}\right)\frac{(n-k-l)}{(1-x-y)}+\frac{(n-k-l)(n-k-l-1)}{(1-x-y)^2}\right]F(x,y)$,

where

$$F(x,y)=(n+1)(n+2)b_{n,k,l}(x,y)\int_0^1\int_0^{1-t}b_{n,k,l}(s,t)f(s,t)ds\,dt.$$

Result of the Lemma is obvious, we omit the details.

Definition 3.2.1. The subspace G of $C(S)$ is the collection of $f\in C(S)$ for which the seminorm

$$\phi(f)=\max\left(\left|x\frac{\partial^2}{\partial x^2}f(x,y)\right|,\left|y\frac{\partial^2}{\partial y^2}f(x,y)\right|,\left|xy\frac{\partial^2}{\partial x\partial y}f(x,y)\right|\right)$$

is bounded, where f is locally twice differentiable in the interior of S and that f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are locally absolutely continuous for both the variables.

Definition 3.2.2. The interpolation space $(C,G)_\beta$ is the collection of all $f\in C(S)$ for which $H(f,t)\leq L(f)t^\beta$ for all $t\leq t_0$, where $H(f,t)=\inf_{g\in G}\left(\|f-g\|+t\phi(g)\right)$.

Lemma 3.2.11. *Let $f(x,y)\in C^2(S)$ such that*

$$\left\|\frac{\partial f}{\partial x}\right\|\leq C,\left\|\frac{\partial f}{\partial y}\right\|\leq C,\left\|\frac{\partial^2 f}{\partial x^2}\right\|\leq C,\left\|\frac{\partial^2 f}{\partial y^2}\right\|\leq C\ \text{ and}\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|\leq C$$

then we get $|Q_n(f,x,y)-f(x,y)|\leq Cn^{-1}[x(1-x)+y(1-y)]$, where $\|.\|$ means $\|.\|_{C(S)}$.

Lemma 3.2.12. *If $f\in C(S)$, then $\phi\left(Q_n\left(f\right)\right)\leq L_n\left\|f\right\|$.*

Lemma 3.2.13. *For $f\in C(S), f\in C^2$ locally in interior of S and $f\in G$, we have $\phi\left(Q_n\left(f\right)\right)\leq L\left\|\phi(f)\right\|$, where L is a constant independent of f and n.*

3.3 Main Theorem

Theorem 3.3.1. *If $f\in C^0_{a,b,c,d}(\mathbb{R}^2_+)$, $0<\lambda<1$ then*

$$w(x,y)\big|G_n(f,x,y)-f(x,y)\big|\leq C.K_{2,\phi^\lambda}(f,n^{-1}(\phi^{2(1-\lambda)}(x)+\phi^{2(1-\lambda)}(y))),\qquad(3.3.1)$$

where C is positive constant independent from n,x,y.

Proof. Initially

$$G_n\left((s-x)^2(1-s)^{-\lambda};x,y\right)\leq Cn^{-1}\phi^2(x)(1-x)^{-\lambda},\qquad(3.3.2)$$

$$G_n\left((t-x)^2(1-t)^{-\lambda};x,y\right)\leq Cn^{-1}\phi^2(y)(1-y)^{-\lambda},\qquad(3.3.3)$$

$$G_n\left((s-x)(1-s)^{-\frac{\lambda}{2}}(t-y)(1-t)^{-\frac{\lambda}{2}},x,y\right)\le C.n^{-1}\phi(x)\phi(y)(1-x)^{-\frac{\lambda}{2}}(1-y)^{-\frac{\lambda}{2}}. \tag{3.3.4}$$

Using Hölder inequality, Schwartz's inequality, (3.2.1) and (3.2.5), for $(x,y)\in I_n=\left[\frac{1}{n},1-\frac{1}{n}\right)\times\left[\frac{1}{n},1-\frac{1}{n}\right)$, we get

$$\begin{aligned}G_n\left((s-x)^2(1-s)^{-\lambda};x,y\right)&\le\left(G_n(s-x)^4,x,y\right)^{\frac{1}{2}}\left(G_n(1-s)^{-2\lambda},x,y\right)^{\frac{1}{2}}\\&\le\left(G_n(s-x)^4,x,y\right)^{\frac{1}{2}}\left(G_n(1-s)^{-2},x,y\right)^{\frac{\lambda}{2}}\\&\le C.n^{-1}\phi^2(x)(1-x)^{-\lambda}.\end{aligned}$$

Now for $(x,y)\in I_n^c$, we get

$$\begin{aligned}G_n\left((s-x)^2(1-s)^{-\lambda};x,y\right)&\le\left(G_n(s-x)^2,x,y\right)=n^{-1}\phi^2(x)=n^{-1}\phi^2(x)\frac{(1-x)^\lambda}{(1-x)^\lambda}\\&\le Cn^{-1}\phi^2(x)(1-x)^{-\lambda}.\end{aligned}$$

Similarly we can prove (3.3.3) and (3.3.4).

Now again using Lemma 3.2.1, Lemma 3.2.5 and (3.2.5), (3.3.1), (3.3.2), (3.3.3) and applying the Taylor's formula as well as Hardy-Littlewood majorant, for $g\in D$, we obtain

$$\begin{aligned}&w(x,y)\big|G_n(g,x,y)-g(x,y)\big|\\&\le w(x,y)\left|G_n\left(\int_x^s(s-\eta)\frac{\partial^2 g(\eta,y)}{\partial^2 x}d\eta,x,y\right)\right|+w(x,y)\left|G_n\left(\int_y^t(t-\xi)\frac{\partial^2 g(x,\xi)}{\partial^2 x}d\xi,x,y\right)\right|\\&\quad+w(x,y)\left|G_n\left(\int_x^s(s-\eta)\int_y^t(t-\xi)\frac{\partial^2 g(\eta,\xi)}{\partial^2 x}d\eta d\xi,x,y\right)\right|\\&\le C\Big[\big\|w\phi^{2\lambda}g_{xx}\big\|_\infty\phi^{-2\lambda}(x)G_n\big((s-x)^2,x,y\big)+x^{-\lambda}G_n\big((s-x)^2(1-s)^{-\lambda},x,y\big)\\&\quad+\big\|w\phi^{2\lambda}g_{yy}\big\|_\infty\phi^{-2\lambda}(y)G_n\big((t-y)^2,x,y\big)+x^{-\lambda}G_n\big((t-y)^2(1-s)^{-\lambda},x,y\big)\\&\quad+\big\|w\phi^{2\lambda}g_{xy}\big\|_\infty\phi^{-\lambda}(x)\phi^{-\lambda}(y)G_n\big((s-x)(t-y),x,y\big)\\&\quad+x^{-\lambda}G_n\big((t-x)^2(1-t)^{-\lambda},x,y\big)\Big]\end{aligned}$$

$$\leq C.n^{-1}\big(\phi^{2(1-\lambda)}(x)+\phi^{2(1-\lambda)}(y)+\phi^{1-\lambda}(x)\phi^{1-\lambda}(y)\big)\Phi(g)$$
$$\leq C.n^{-1}\big(\phi^{2(1-\lambda)}(x)+\phi^{2(1-\lambda)}(y)\big)\Phi(g). \tag{3.3.5}$$

Thus, from (3.2.7) and (3.3.5), for $g \in D$ *and* $f \in C^0_{a,b,c,d}(\mathbb{R}^2_+)$ we have

$$\begin{aligned} w(x,y)\big|G_n(f,x,y)-f(x,y)\big| &\leq \big|w(x,y)G_n\big((f-g),x,y\big)\big| + w(x,y)\big|f(x,y)-g(x,y)\big| \\ &\quad + w(x,y)\big|G_n(g,x,y)-g(x,y)\big| \\ &\leq C.K_{2,\phi^\lambda}\big(f, n^{-1}(\phi^{2(1-\lambda)}(x)+\phi^{2(1-\lambda)}(y))\big). \end{aligned}$$

This completes the proof. □

Theorem 3.3.2. *For $f \in C(S)$, which is twice continuously differentiable in the interior of S, then we have $|Q_n(f,x,y)-f(x,y)| \leq Cn^{-n}(1+\phi(f))$, where C is independent of n.*

Proof. Using Taylor's formula

$$\begin{aligned} f(s,t) = f(x,y) &+ (s-x)\frac{\partial}{\partial x}f(x,y) + (t-y)\frac{\partial}{\partial y}f(x,y) \\ &+ (s-x)^2\int_0^1 u\frac{\partial^2}{\partial x^2}f(s+u(x-s),t+u(y-t))du \\ &+ (t-y)^2\int_0^1 u\frac{\partial^2}{\partial y^2}f(s+u(x-s),t+u(y-t))du \\ &+ (s-x)(t-y)\int_0^1 u\frac{\partial^2}{\partial x\partial y}f(s+u(x-s),t+u(y-t))du, \end{aligned}$$

since

$$\begin{aligned} (s-x)^2\int_0^1 u\frac{\partial^2}{\partial x^2}f(s+u(x-s),t+u(y-t))du &\leq (s-x)^2\int_0^1 \frac{u\phi(f)}{|s+u(x-s)|}du \\ &= \phi(f)\int_s^x \frac{\xi-s}{\xi}d\xi \leq \phi(f)\frac{(x-s)^2}{x}, \end{aligned}$$

as for ξ between x and s, $\left|\frac{\xi-s}{\xi}\right| \leq \left|\frac{x-s}{x}\right|$. Similarly,

$$(t-y)^2\int_0^1 u\frac{\partial^2}{\partial y^2}f(s+u(x-s),t+u(y-t))du \leq \phi(f)\frac{(y-t)^2}{y},$$

and

$$\begin{aligned}
&\left|(x-s)(y-t)\int_0^1 u\frac{\partial^2}{\partial x\partial y}f(s+u(x-s),t+u(y-t))du\right| \\
&\leq \left|\phi(f)\int_0^1 \frac{u(x-s)(y-t)du}{|s+u(x-s)|^{1/2}\,|t+u(y-t)|^{1/2}}\right| \\
&\leq \phi(f)\left[\int_s^x \frac{(\xi-s)^2}{\xi}d\xi\right]^{1/2}\left[\int_s^x \frac{(\eta-t)^2}{\eta}d\eta\right]^{1/2} \\
&\leq \frac{\phi(f)\,|s-x|\,|y-t|}{\sqrt{xy}}.
\end{aligned}$$

We have

$$\left|Q_n(s-t,x,y)\frac{\partial}{\partial x}f(x,y)\right| \leq C_1,\quad \left|Q_n(t-y,x,y)\frac{\partial}{\partial y}f(x,y)\right| \leq C_2\frac{1}{n},$$

$$\begin{aligned}
Q_n\left((x-s)^2,x,y\right)\frac{\phi(f)}{x(1-x)} &\leq \frac{\phi(f)}{x(1-x)}\frac{2\left[x(1-x)(n-6)+(1+2x)\right]}{(n+3)(n+4)} \\
&\leq \frac{\phi(f)}{x(1-x)}\left[\frac{2x(1-x)(n-6)}{(n+3)(n+4)}\right] \leq C_3\frac{1}{n}\phi(f).
\end{aligned}$$

Similarly

$$\begin{aligned}
Q_n\left((y-t)^2,x,y\right)\frac{\phi(f)}{y(1-y)} &\leq C_4\frac{1}{n}\phi(f), \\
Q_n\left(\frac{|x-s|\,|y-t|}{\sqrt{xy}};x,y\right)\phi(f) &\leq \frac{\phi(f)}{2}\left\{\frac{Q_n\left((x-s)^2;x,y\right)}{x}+\frac{Q_n\left((y-t)^2;x,y\right)}{y}\right\} \\
&\leq C_5\frac{1}{n}\phi(f),
\end{aligned}$$

then, we get

$$|Q_n\left(f,x,y\right)-f(x,y)| \leq Cn^{-n}\left(1+\phi(f)\right),$$

where

$$C_1=\max\left(\left|\frac{\partial f}{\partial x}\right|,\left|\frac{\partial f}{\partial y}\right|\right), C=\max\left(C_2,C_3\right).$$

□

Theorem 3.3.3. *If $f\in(C(S),G)_\beta\,, 0<\beta<1$, then we have*

$$\|Q_n(f,x,y)-f(x,y)\| \leq C_1 n^{-\beta}. \tag{3.3.6}$$

Proof. For $t = \frac{1}{n}$ and $H\left(f, \frac{1}{n}\right) \leq L\left(f\right) t^{\beta}$, we have $g \in G$ such that

$$\|f-g\| + n^{-1}\phi(g) \leq 2L(f)n^{-\beta}$$

$$\text{or} \quad \|f-g\| \leq 2L(f)n^{-\beta} \text{ and } \phi(g) \leq 2L(f)n^{1-\beta},$$

then we have

$$\begin{aligned}|Q_n\left(f,x,y\right) - f(x,y)| &\leq |Q_n\left(f-g,x,y\right) - f(x,y) + g(x,y)| + |Q_n\left(g,x,y\right) - g(x,y)| \\ &\leq |Q_n\left(f-g,x,y\right)| + \|f-g\| + n^{-n}C\left(1+\phi(f)\right) \\ &\leq 2\left\|f-g\right\| + n^{-n}C\left(1+\phi(f)\right) \leq 2L(f)\left(1+C\right)n^{-\beta} + Cn^{-1}.\end{aligned}$$

This leads to (3.3.6). □

Now we will prove the inverse theorem.

Theorem 3.3.4. *For $f \in C(S), 0 < \beta < 2$ and $\|Q_n(f,x,y) - f(x,y)\| \leq C_1 n^{-\beta}$, then we have $f \in \left(C(S), G\right)_{\beta}$.*

Proof. Since $Q_n(f,x,y)$ belongs to C^2 locally in the interior of S, therefore

$$H(f,t) \leq \|f(x,y) - Q_n(f,x,y)\| + t\phi\left(Q_n(f)\right).$$

By Lemma 3.2.12 and Lemma 3.2.13, we get

$$\begin{aligned}t\phi(Q_n,f) &= t\phi\left(Q_n(f-g) + Q_n(g)\right) \\ &\leq t\left[\phi\left(Q_n(f-g) + \phi\left(Q_n(g)\right)\right)\right] \\ &\leq t\left[L_n\left\|f-g\right\| + L\phi(g)\right] \leq tnL\left[\|f-g\| + n^{-1}\phi(g)\right], \quad g \in G,\end{aligned}$$

then we have

$$t\phi(Q_n f) \leq tnLH\left(f,n^{-1}\right), \quad H(f,t) \leq Cn^{-\beta} + tnLH\left(f,n^{-1}\right),$$

by Lorentz-Hermann Lemma, for $t \in (0,1)$, we have

$$H(f,t) = O\left(n^{-\beta}\right).$$

This completes the proof of the theorem. □

Chapter 4

Quantitative Global Estimates for Generalized Two Dimensional Operators

4.1 Introduction

In [109], Özarslan and Duman have introduced a different approach in order to get a faster approximation without preserving the test functions for positive linear operators. In [105], Özarslan and Aktuğlu have calculated quantitative global estimates for double Szász-Mirakyan operators. Obtaining better error estimations in approximation to a function by a sequence of positive linear operators is an important problem in the approximation theory. So far, some relating results have been presented for Bernstein polynomials [91], Szász-Mirakyan operators [51], Szász-Mirakyan-Stancu Durrmeyer operators [61], Meyer-König and Zeller operators [107] and Bernstein-Chlodovsky operators [8] by preserving some test functions in the approximation. Recently, Agratini [9] has applied similar idea to more general summation-type positive linear operators. In [109], without preserving the test functions, a different approach in order to get a faster approximation have been introduced.

Let $\mathbb{R}^+ := [0, \infty)$ and $\mathbb{R}_b^+ := [0, b]$ with $b > 0$. Consider the function space$E(\mathbb{R}^+)$

defined by

$$E\left(\mathbb{R}^{+}\right) := \left\{ f \in C\left(\mathbb{R}^{+}\right) : \lim_{x\to\infty} \frac{f(x)}{1+x^{2}} \;\; is \;\; finite \right\}$$

endowed with the norm

$$\|f\|_{+} = \sup_{x\in\mathbb{R}^{+}} \frac{f(x)}{1+x^{2}}.$$

However, for the bounded interval $\mathbb{R}_b^{+}$, consider the function space $C\left(\mathbb{R}_b^{+}\right)$ and the usual maximum norm $\|.\|$ on $\mathbb{R}_b^{+}$.

Assume that a sequence $\{L_n\}$ of positive linear operators defined on $E\left(\mathbb{R}^{+}\right)$ or $C\left(\mathbb{R}_b^{+}\right)$ satisfies the following conditions:

$$L_n\left(f_0;x\right) = 1, L_n\left(f_1;x\right) = a_n x + b_n, L_n\left(f_2;x\right) = c_n x^2 + d_n x + e_n \tag{4.1.1}$$

where $(a_n), (b_n), (c_n), (d_n)$ and (e_n) are sequences of non-negative real numbers satisfying the following conditions:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 1\left(c_n \neq 0\right), \lim_{n\to\infty} b_n = \lim_{n\to\infty} d_n = \lim_{n\to\infty} e_n = 0. \tag{4.1.2}$$

Many well-known approximation operators, such as Bernstein polynomials, Szász-Mirakyan operators, Bernstein-Kantorovich operators etc.,satisfy the conditions (4.1.1) and (4.1.2). In this chapter, we construct positive linear operators $\{L_n\}$ as given above.

Now consider the lattice homomorphism $T_b : C\left(\mathbb{R}^{+}\right) \to C\left(\mathbb{R}_b^{+}\right)$ defined by $T_b\left(f\right) := f|_{\mathbb{R}_b^{+}}$ for every $f \in C\left(\mathbb{R}^{+}\right)$. In this case, from the classical Korovkin theorem (see [94], p.14) that

$$\lim_{n\to\infty} T_b\left(L_n\left(f\right)\right) = T_b\left(f\right) \;\; uniformly \;\; on\; \mathbb{R}_b^{+}. \tag{4.1.3}$$

On the other hand, with the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4(vi) of [13], p.199) we have the following:

"Let X be a compact set and H be cofinal subspace of $C(X)$. If E is a Banach lattice, $S : C(X) \to E$ is a lattice homomorphism and if $\{L_n\}$ is a sequence of positive linear operators from $C(X)$ into E such that $\lim\limits_{n\to\infty} L_n(h) = S(h)$ for all $h \in H$, then $\lim\limits_{n\to\infty} L_n(f) = f$ provided that f belongs to the Korovkin closure of H."

Hence, by using (4.1.3) and the above property, there is a result.

Theorem 4.1.1. *[109] Let $\{L_n\}$ be a sequence of positive linear operators defined on $E(\mathbb{R}^+)$ (resp.$C(\mathbb{R}_b^+)$) satisfying the conditions in (4.1.1) and (4.1.2). Then, for all $f \in E(\mathbb{R}^+)$ (resp. for all $f \in C(\mathbb{R}_b^+)$) , we have $\lim\limits_{n\to\infty} L_n(f) = f$ uniformly on the interval $\mathbb{R}_b^+$ with $b > 0$.*

We have used following definitions in this chapter for global results of the multi-dimensional operators.

Ötto Szász [124] earlier considered this space of bivariate extension of Lipschitz-type space, given as:

$$Lip_M^*(\alpha)$$
$$:= \left\{ f \in C([0,\infty)\times[0,\infty)) : |f(t) - f(x)| \le M \frac{\|t-x\|^\alpha}{(\|t\| + x + y)^{\alpha/2}}; t, s; x, y \in (0,\infty) \right\},$$

where $\mathbf{t} = (t,s), \mathbf{x} = (x,y)$ and M is any positive constant and $0 < \alpha \le 1$.
For all $f \in C([0,\infty)\times[0,\infty))$, the modulus of f denoted by $\omega(f;\delta)$ is defined as

$$\omega(f;\delta)$$
$$:= \sup\left\{ |f(t,s) - f(x,y)| : \sqrt{(t-x)^2 + (s-y)^2} < \delta, (t,s), (x,y) \in [0,\infty)\times[0,\infty) \right\}.$$

4.2 Estimation for Generalized Double Baskakov Operators

The Baskakov operator $V_n(f;x)$ was introduced by V.A. Baskakov [18] given by (0.3.3) , $f \in C_B[0,\infty), C_B[0,\infty)$ is the set of function which is bounded.

By now, a number of results about the operator have been obtained ([4], [20], [31],

[36], [45], [64]). In this section, we address the investigation for the multivariate Baskakov operator defined as follows.
Let $V_n^*(f;x,y)$ are the following two variate Baskakov operators:

$$V_n^*(f;x,y) = \sum_{k,l=0}^{\infty} v_{n,k}(x)\, v_{n,l}(y) f\left(\frac{k}{n},\frac{l}{n}\right), \tag{4.2.1}$$

where $0 \leq k \leq n, 0 \leq l \leq n, f(x,y) \in C_B[0,\infty;0,\infty)$.

Consider the classical Baskakov operators defined by (0.3.3). Since

$$V_n(f_0;x) = 1, V_n(f_1;x) = x, V_n(f_2;x) = \left(1+\frac{1}{n}\right)x^2 + \frac{x}{n}.$$

Following the similar arguments as used in [109], the best error estimation among all the general double Baskakov operators can be obtained from the case by taking

$$a_n = 1, b_n = e_n = 0, c_n = 1+\frac{1}{n}, d_n = \frac{1}{n}$$

for all $n \in \mathbb{N}$ where $(a_n),(b_n),(c_n),(d_n)$ and (e_n) are sequences of non-negative real numbers as defined above.
Now observe that

$$u_n(x) = \frac{2a_n x - d_n}{2c_n} = \frac{2nx-1}{2(n+1)} \in [0,\infty),$$

where u_n is a functional sequence, $u_n : I \to A$, where A denotes $\mathbb{R}^+$ and assume that I be subinterval of A.
So, $u_n(x) \in \mathbb{R}^+$ if and only if $x \geq \frac{1}{2n}$ and $n \geq 1$. Hence, choosing

$$I = \left[\frac{1}{2},\infty\right) \subset \mathbb{R}^+.$$

The best error estimation among all the general double Baskakov operators can be obtained from the case

$$u_n(x) = \frac{2nx-1}{2(n+1)}, v_n(y) = \frac{2ny-1}{2(n+1)}; n \in \mathbb{N}$$

for all $f \in C_B([0,\infty) \times C_B[0,\infty))$ and $x, y \in \left[\frac{1}{2}, \infty\right)$. Hence, (4.2.1) becomes

$$V_n^{**}(f;x,y) : V_n^*(f;u_n(x), v_n(y)) \tag{4.2.2}$$
$$= \sum_{k,l=0}^{\infty} \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \begin{pmatrix} n+l-1 \\ l \end{pmatrix} f\left(\frac{k}{n}, \frac{l}{n}\right)$$
$$(u_n(x))^k (1+u_n(x))^{-(n+k)} (v_n(y))^l (1+v_n(y))^{-(n+l)},$$
$$f \in C_B([0,\infty) \times C_B[0,\infty)).$$

For the operators $V_n^{**}(f;x,y)$, we have following Lemma:

Lemma 4.2.1. *Let* $\boldsymbol{x} = (x,y), \boldsymbol{t} = (t,s); e_{i,j}(x) = x^i y^j, i,j = 0,1,2$ *and* $\psi_x^2(t) = \|t-x\|^2$. *Then, for each* $x, y \geq 0$ *and* $n > 1$, *we have*

(i) $V_n^{**}(e_{0,0};x,y) = 1$,

(ii) $V_n^{**}(e_{1,0};x,y) = u_n(x)$,

(iii) $V_n^{**}(e_{0,1};x,y) = v_n(y)$,

(iv) $V_n^{**}(e_{2,0}+e_{0,2};x,y) = \left(1+\frac{1}{n}\right)\left(u_n^2(x) + v_n^2(y)\right) + \frac{u_n(x)+v_n(y)}{n}$,

(v) $V_n^{**}(\psi_x^2(t);x,y) = (u_n(x)-x)^2 + (v_n(y)-y)^2 + \frac{1}{n}\left(u_n^2(x) + v_n^2(y) + u_n(x) + v_n(y)\right)$.

4.3 Global Results

Now, for the space $Lip_M^*(\alpha)$ with $0 < \alpha \leq 1$, we have the following approximation result.

Theorem 4.3.1. *For any* $f \in Lip_M^*(\alpha), \alpha \in (0,1]$, *and for each* $x, y \in (0,\infty), n \in \mathbb{N}$, *we have*

$$|V_n^{**}(f;x,y)-f(x,y)| \tag{4.3.1}$$
$$\leq \frac{M}{(x+y)^{\alpha/2}}\left[(u_n(x)-x)^2+(v_n(y)-y)^2+\frac{1}{n}\left(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y)\right)\right]^{\alpha/2}.$$

Proof. Let $\alpha=1$. For each $x,y\in(0,\infty)$ and for $f\in Lip_M^*(1)$, we have

$$\begin{aligned}|V_n^{**}(f;x,y)-f(x,y)| &\leq V_n^{**}(|f(t,s)-f(x,y)|;x,y)\\ &\leq MV_n^{**}\left(\frac{\|t-x\|}{(\|t\|+x+y)^{1/2}};x,y\right)\\ &\leq \frac{M}{(x+y)^{1/2}}V_n^{**}(\|t-x\|;x,y).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}|V_n^{**}(f;x,y)-f(x,y)| &\leq \frac{M}{(x+y)^{1/2}}\sqrt{V_n^{**}(\psi_x^2(t);x,y)}\\ &= \frac{M}{(x+y)^{1/2}}\sqrt{(u_n(x)-x)^2+(v_n(y)-y)^2+\frac{1}{n}(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y))}.\end{aligned}$$

Now, let $0<\alpha<1$. Then for each $x,y\in(0,\infty)$ and for $f\in Lip_M^*(\ alpha)$, we obtain

$$\begin{aligned}|V_n^{**}(f;x,y)-f(x,y)| &\leq V_n^{**}(|f(t,s)-f(x,y)|;x,y)\\ &\leq MV_n^{**}\left(\frac{\|t-x\|^\alpha}{(\|t\|+x+y)^{\alpha/2}};x,y\right)\\ &\leq \frac{M}{(x+y)^{\alpha/2}}V_n^{**}(\|t-x\|^\alpha;x,y).\end{aligned}$$

For Holder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, for any $f\in Lip_M^*(\alpha)$, we have

$$\begin{aligned}|V_n^{**}(f;x,y)-f(x,y)| &\leq \frac{M}{(x+y)^{\alpha/2}}\left[V_n^{**}\left(\psi_x^2(t);x,y\right)\right]^{\alpha/2}\\ &= \frac{M}{(x+y)^{\alpha/2}}\left[(u_n(x)-x)^2+(v_n(y)-y)^2+\frac{1}{n}\left(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y)\right)\right]^{\alpha/2}\end{aligned}$$

which is the required result. $\square$

Lemma 4.3.2. *For each $x, y > 0$,*

$$V_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right) \tag{4.3.2}$$
$$\leq \frac{1}{\sqrt{x}}\sqrt{\left(u_n\left(x\right)-x\right)^2+\frac{u_n^2\left(x\right)+u_n\left(x\right)}{n}}+\frac{1}{\sqrt{y}}\sqrt{\left(v_n\left(y\right)-y\right)^2+\frac{v_n^2\left(y\right)+v_n\left(y\right)}{n}}.$$

Proof. We have $\sqrt{c+d} \leq \sqrt{c}+\sqrt{d}\,(c, d \geq 0)$, therefore

$$V_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$
$$=\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l}\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}$$
$$\left(u_n\left(x\right)\right)^k\left(1+u_n\left(x\right)\right)^{-(n+k)}\left(v_n\left(y\right)\right)^l\left(1+v_n\left(y\right)\right)^{-(n+l)}$$
$$\leq \sum_{k=0}^{\infty}\binom{n+k-1}{k}\left|\sqrt{\frac{k}{n}}-\sqrt{x}\right|\left(u_n\left(x\right)\right)^k\left(1+u_n\left(x\right)\right)^{-(n+k)}$$
$$+\sum_{l=0}^{\infty}\binom{n+l-1}{l}\left|\sqrt{\frac{l}{n}}-\sqrt{y}\right|\left(v_n\left(y\right)\right)^l\left(1+v_n\left(y\right)\right)^{-(n+l)}$$
$$=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\left|\frac{k}{n}-x\right|}{\sqrt{\frac{k}{n}}+\sqrt{x}}\left(u_n\left(x\right)\right)^k\left(1+u_n\left(x\right)\right)^{-(n+k)}$$
$$+\sum_{l=0}^{\infty}\binom{n+l-1}{l}\frac{\left|\frac{l}{n}-y\right|}{\sqrt{\frac{l}{n}}+\sqrt{y}}\left(v_n\left(y\right)\right)^l\left(1+v_n\left(y\right)\right)^{-(n+l)}$$
$$=\frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left|\frac{k}{n}-x\right|\left(u_n\left(x\right)\right)^k\left(1+u_n\left(x\right)\right)^{-(n+k)}$$
$$+\frac{1}{\sqrt{y}}\sum_{l=0}^{\infty}\binom{n+l-1}{l}\left|\frac{l}{n}-y\right|\left(v_n\left(y\right)\right)^l\left(1+v_n\left(y\right)\right)^{-(n+l)}.$$

Using the Cauchy-Schwarz inequality,

$$V_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$
$$\leq \frac{1}{\sqrt{x}}\sqrt{\sum_{k=0}^{\infty}\left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\left(\frac{k}{n}-x\right)^2(u_n(x))^k(1+u_n(x))^{-(n+k)}}$$
$$+\frac{1}{\sqrt{y}}\sqrt{\sum_{l=0}^{\infty}\left(\begin{array}{c} n+l-1 \\ l \end{array}\right)\left(\frac{l}{n}-y\right)^2(v_n(y))^l(1+v_n(y))^{-(n+l)}}.$$

Using Lemma(4.2.1),

$$V_n^{**}\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$
$$\leq \frac{1}{\sqrt{x}}\sqrt{(u_n(x)-x)^2+\frac{u_n^2(x)+u_n(x)}{n}}+\frac{1}{\sqrt{y}}\sqrt{(v_n(y)-y)^2+\frac{v_n^2(y)+v_n(y)}{n}}$$

which is the desired result. □

Theorem 4.3.3. *Let $g(x,y)=f(x^2,y^2)$. Then we have for each $x,y>0$,*

$$|V_n^{**}(f;x,y)-f(x,y)|\leq 2\omega(g;\delta_n(x,y))$$

where $\delta_n(x,y)=\frac{1}{\sqrt{x}}\sqrt{(u_n(x)-x)^2+\frac{u_n^2(x)+u_n(x)}{n}}+\frac{1}{\sqrt{y}}\sqrt{(v_n(y)-y)^2+\frac{v_n^2(y)+v_n(y)}{n}}$.

Proof. We have

$$|V_n^{**}(f;x,y)-f(x,y)|\leq V_n^{**}(|f(t,s)-f(x,y)|;x,y)$$
$$=V_n^{**}\left(\left|g\left(\sqrt{t},\sqrt{s}\right)-g\left(\sqrt{x},\sqrt{y}\right)\right|;x,y\right)$$
$$\leq V_n^{**}\left(\omega\left(g;\sqrt{(t-x)^2+(s-y)^2}\right);x,y\right)$$

$$= \sum_{k,l=0}^{\infty} \binom{n+k-1}{k} \binom{n+l-1}{l}$$

$$\omega \left(g; \sqrt{\left(\sqrt{\frac{k}{n}} - \sqrt{x} \right)^2 + \left(\sqrt{\frac{l}{n}} - \sqrt{y} \right)^2}; x, y \right)$$

$$(u_n(x))^k (1+u_n(x))^{-(n+k)} (v_n(y))^l (1+v_n(y))^{-(n+l)}$$

$$= \sum_{k,l=0}^{\infty} \binom{n+k-1}{k} \binom{n+l-1}{l}$$

$$\omega \left(g; \frac{\sqrt{\left(\sqrt{\frac{k}{n}} - \sqrt{x} \right)^2 + \left(\sqrt{\frac{l}{n}} - \sqrt{y} \right)^2}}{V_n^{**}\left(\sqrt{\left(\sqrt{t} - \sqrt{x} \right)^2 + \left(\sqrt{s} - \sqrt{y} \right)^2}; x, y \right)} \right.$$

$$\left. V_n^{**}\left(\sqrt{\left(\sqrt{t} - \sqrt{x} \right)^2 + \left(\sqrt{s} - \sqrt{y} \right)^2}; x, y \right); x, y \right).$$

Now, we have

$$\omega(f; \lambda\delta) \leq (1+\lambda)\, \omega(f; \delta).$$

Therefore,

$$|V_n^{**}(f; x, y) - f(x, y)| \leq \omega \left(g; V_n^{**}\left(\sqrt{\left(\sqrt{t} - \sqrt{x} \right)^2 + \left(\sqrt{s} - \sqrt{y} \right)^2}; x, y \right) \right)$$

$$\times \sum_{k,l=0}^{\infty} \binom{n+k-1}{k} \binom{n+l-1}{l} \left[1 + \frac{\sqrt{\left(\sqrt{\frac{k}{n}} - \sqrt{x} \right)^2 + \left(\sqrt{\frac{l}{n}} - \sqrt{y} \right)^2}}{V_n^{**}\left(\sqrt{\left(\sqrt{t} - \sqrt{x} \right)^2 + \left(\sqrt{s} - \sqrt{y} \right)^2}; x, y \right)} \right]$$

$$(u_n(x))^k (1+u_n(x))^{-(n+k)} (v_n(y))^l (1+v_n(y))^{-(n+l)}$$

$$\leq 2\omega \left(g; V_n^{**}\left(\sqrt{\left(\sqrt{t} - \sqrt{x} \right)^2 + \left(\sqrt{s} - \sqrt{y} \right)^2}; x, y \right) \right).$$

Now, using Lemma(4.3.2), completes the proof. □

Theorem 4.3.4. *Let* $g(x, y) = f(x^2, y^2)$. *Let*

$$g \in Lip_M(\alpha) := \left\{ g \in C_{\boldsymbol{B}}([0,\infty) \times [0,\infty)) : |g(t) - g(x)| \leq M \|t - x\|^{\alpha}; t, s; x, y \in (0,\infty) \right\},$$

where $\boldsymbol{t} = (t, s)$, $\boldsymbol{x} = (x, y)$ *and* M *is any positive constant and* $0 < \alpha \leq 1$.
Then,

$$\left|V_n^{**}(f; x, y) - f(x, y)\right| \leq M\delta_n^{\alpha}(x, y), \tag{4.3.3}$$

where $\delta_n(x, y)$ *is the same as in Theorem (4.3.3).*

Proof. We have

$$\begin{aligned}
\left|V_n^{**}(f; x, y) - f(x, y)\right| &\leq V_n^{**}\left(\left|f(t, s) - f(x, y)\right|; x, y\right) \\
&= V_n^{**}\left(\left|g\left(\sqrt{t}, \sqrt{s}\right) - g\left(\sqrt{x}, \sqrt{y}\right)\right|; x, y\right) \\
&\leq MV_n^{**}\left(\left(\left(\sqrt{t} - \sqrt{x}\right)^2 + \left(\sqrt{s} - \sqrt{y}\right)^2\right)^{\alpha/2}; x, y\right) \\
= M\sum_{k,l=0}^{\infty} \binom{n+k-1}{k}\binom{n+l-1}{l} &\left(\left(\sqrt{\frac{k}{n}} - \sqrt{x}\right)^2 + \left(\sqrt{\frac{l}{n}} - \sqrt{y}\right)^2\right)^{\alpha/2} \\
&(u_n(x))^k(1 + u_n(x))^{-(n+k)}(v_n(y))^l(1 + v_n(y))^{-(n+l)}.
\end{aligned}$$

For Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\left|V_n^{**}(f; x, y) - f(x, y)\right| \leq M\left[V_n^{**}\left(\sqrt{\left(\sqrt{t} - \sqrt{x}\right)^2 + \left(\sqrt{s} - \sqrt{y}\right)^2}; x, y\right)\right]^{\alpha}.$$

By using Lemma(4.3.2), completes the proof. □

Chapter 5

Direct and Inverse Theorems for Beta Durrmeyer Operators

5.1 Introduction and Definitions

Gupta and Ahmad [69] defined the Beta operators as:

$$F_n(f,x) = \frac{1}{n}\sum_{k=0}^{\infty} l_{n,k}(x) f\left(\frac{k}{n+1}\right), \qquad x \in [0,\infty) \tag{5.1.1}$$

where

$$l_{n,k}(x) = \frac{(n+k)!}{k!(n-1)!}\frac{x^k}{(1+x)^{n+k+1}},$$

and Durrmeyer variant

$$J_n(f(t);x) = \frac{1}{n}\sum_{k=0}^{\infty} l_{n,k}(x)\int_0^{\infty} l_{n,k}(t)f(t)dt = \int_0^{\infty} W_n(t,x)f(t)dt, \tag{5.1.2}$$

of these operators is studied by Deo [32].

The norm $\|\bullet\|_\alpha$ on the space $C_\alpha[0,\infty)$ denotes the class of continuous functions on $[0,\infty)$ satisfying growth condition $|f(t)| \le Mt^\alpha,\ M>0,\ \alpha>0$ with the norm

$$\|f\|_\alpha = \sup_{0\le t<\infty} |f(t)| t^{-\alpha}.$$

To improve the saturation order $O(n^{-1})$ for the operator (5.1.2), we use the technique of linear combination as described by May [99] for a sequence of positive linear operators. We consider the linear combination of the operators (5.1.2) as described below:

The linear combination $J_n(f,(d_0,d_1,d_2,...,d_k),x)$ of $J_{d_jn}(f,x),\ j=0,1,2,...,k$ are defined by

$$J_n(f,(d_0,d_1,d_2,...,d_k),x)=\sum_{j=0}^{k}C(j,k)J_{d_jn}(f,x),$$

where $d_0,d_1,d_2,...,d_k$ are arbitrary but fixed distinct positive integers and

$$C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i}\ for\ k\neq 0\ \&\ C(0,0)=1.$$

In this chapter we obtain direct theorem in terms of higher order modulus of continuity in simultaneous approximation with the help of properties of Steklov means and in the last section of this chapter we give inverse theorem for these linear combination of the operators J_n in ordinary approximation.

5.2 Preliminary Results

In order to prove the Theorem, we shall require the following results:

Lemma 5.2.1. *[31] Let $m\in\mathbb{N}^0$ (the set of nonnegative integers) and the $m-th$ moment for the operators (5.1.1) be defined by:*

$$U_{n,m}(x)=\sum_{k=0}^{\infty}\left(\frac{k}{n+1}-x\right)^m l_{n,k}(x),$$

then

$$(n+1)U_{n,m+1}(x)=x(1+x)\left[U'_{n,m}(x)+mU_{n,m-1}(x)\right],\ \ (x\geq 0).$$

Consequently

(i) *$U_{n,m}(x)$ is a polynomial in x of degree $\leq m$,*

(ii) *$U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$ where $[\beta]$ denotes the integer part of β.*

Lemma 5.2.2. *[32] Let $m \in \mathbb{N}^0$, we define the function $T_{n,m}(x)$ as:*

$$T_{n,m}(x) = \frac{1}{(n+r)} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} p_{n-r,k+r}(t)(t-x)^m dt$$

then $T_{n,0}(x) = 1,\ \ T_{n,1}(x) = \frac{(1+2x)(1+r)}{n-r-1}$ and

$$(n-m-r-1)T_{n,m+1}(x) = x(1+x)\big[T'_{n,m}(x) + 2mT_{n,m-1}(x) + (1+2x)(r+m+1)T_{n,m}(x)\big].$$

Further, for all $x \in [0,\infty)$

$$T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Lemma 5.2.3. *[32] If f is r times $(r = 1,2,3,...)$ differentiable on $[0,\infty)$ such that $f^{(r-1)}$ is absolutely continuous with $f^{(r-1)}(t) = O(t^{\alpha})$ for some $\alpha > 0$ as $t \to \infty$ and $n > \alpha + r$, then we have*

$$J_n^{(r)}(f,x) = \frac{(n-r-1)!(n+r-1)!}{n!(n-1)!} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int_0^{\infty} p_{n-r,k+r}(t) f^{(r)}(t)dt. \quad (5.2.1)$$

Lemma 5.2.4. *[96] There exist polynomials $q_{i,j,r}(x)$ independent of n and k such that*

$$\{x(1+x)\}^r \frac{d^r}{dx^r}[l_{n,k}(x)] = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+1)^i \left|k-(n+1)x\right|^j q_{i,j,r}(x) l_{n,k}(x).$$

Lemma 5.2.5. *Let $f \in C_{\alpha}[0,\infty)$, if $f^{(2k+r+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} n^{k+1}\left\{J_n^{(r)}(f,(d_0,d_1,d_2,...,d_k),x) - f^{(r)}(x)\right\} = \sum_{i=r}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x),$$

where $Q(i,k,r,x)$ are certain polynomial in x of degree i.

The proof of Lemma 5.2.5 follows along the lines of [72].

Lemma 5.2.6. *Let δ and γ be any two positive numbers and $[a,b] \subset [0,\infty)$. Then, for any $m > 0$ there exists a constant M_m such that*

$$\left\| \int_{\left|t-x\right| \geq \delta} W_n(t,x) t^{\gamma} dt \right\|_{C[a,b]} \leq M_m n^{-m}.$$

The proof of this result follows easily by using Schwarz inequality and Lemma 2.7 from [11].

5.3 Direct Theorem

In this section we study direct result in terms of higher order modulus of continuity in simultaneous approximation for the operators (5.1.2).

Theorem 5.3.1. *Let $f^{(r)} \in C_\alpha[0,\infty)$ and $0 < a < a_1 < b_1 < b < \infty$. Then for all n sufficiently large, we have*

$$\left\| J_n^{(r)}\left(f, (d_0, d_1, d_2, ..., d_k), \bullet\right) - f^{(r)} \right\|_{C[a_1,b_1]} \leq Max\left\{ C_1 \omega_{2k+2}(f^{(r)}; n^{-1/2}, a, b) + C_2 n^{-(k+1)} \left\| f \right\|_{\alpha} \right\},$$

where $C_1 = C_1(k,r)$ and $C_2 = C_2(k,r,f)$.

Proof. Using linearity property

$$\begin{aligned} \left\| J_n^{(r)}\left(f, (d_0, d_1, d_2, ..., d_k), \bullet\right) - f^{(r)} \right\|_{C[a_1,b_1]} & \\ \leq \left\| J_n^{(r)}\left((f - f_{2k+2,\eta}), (d_0, d_1, d_2, ..., d_k), \bullet\right) \right\|_{C[a_1,b_1]} & \\ + \left\| J_n^{(r)}\left(f_{2k+2,\eta}, (d_0, d_1, d_2, ..., d_k), \bullet\right) - f^{(r)}_{2k+2,\eta} \right\|_{C[a_1,b_1]} & \\ + \left\| f^{(r)} - f^{(r)}_{2k+2,\eta} \right\|_{C[a_1,b_1]} & \\ := E_1 + E_2 + E_3. & \end{aligned}$$

Since, $f^{(r)}_{2k+2,\eta}(t) = \left(f^{(r)}\right)_{2k+2,\eta}(t)$, by property (iii) of Steklov mean, we obtain

$$E_3 \leq C_1\omega_{2k+2}(f^{(r)}, \eta, a, b).$$

By Lemma 5.2.5, we get

$$E_2 \leq C_2 n^{-(k+1)} \sum_{j=r}^{2k+r+2} \left\| f^{(j)}_{2k+2,\eta} \right\|_{C[a,b]}.$$

Using the interpolation property due to Goldberg and Meir [60] for each $j = r, r + 1, ..., 2k + r + 2$, we get

$$\left\| f^{(r)}_{2k+2,\eta} \right\|_{C[a,b]} \leq C_3 \left\{ \| f_{2k+2,\eta} \|_{C[a,b]} + \left\| f^{(2k+r+2)}_{2k+2,\eta} \right\|_{C[a,b]} \right\}.$$

Now using properties (ii) and (iv) of Steklov mean, we obtain

$$E_2 \leq C_4 n^{-(k+1)} \Big\{ \big\| f \big\|_{\alpha} + \eta^{-(2k+2)} \omega_{2k+2}(f^{(r)}, \eta) \Big\}.$$

To estimate E_1, choosing $a^{'}, b^{'}$ such that

$$0 < a < a' < a_1 < b_1 < b' < b < \infty.$$

Also let $\psi(t)$ be the characteristic function of the interval $[a^{'}, b^{'}]$, then

$$\begin{aligned} E_1 \leq & \left\| J^{(r)}_n \left(\psi(t) \left(f(t) - f_{2k+2,\eta}(t)\right) (d_0, d_1, d_2, ..., d_k), \bullet\right) \right\|_{C[a_1,b_1]} \\ & + \left\| J^{(r)}_n \left((1 - \psi(t)) \left(f(t) - f_{2k+2,\eta}(t)\right) (d_0, d_1, d_2, ..., d_k), \bullet\right) \right\|_{C[a_1,b_1]} \\ & := E_4 + E_5. \end{aligned}$$

We note that in order to estimate E_4 and E_5, it is sufficient to consider their expressions without the linear combination. It is clear that by Lemma 5.2.3, we obtain

$$\begin{aligned}
&J_n^{(r)}\left(\psi(t)\left(f(t)-f_{2k+2,\eta}(t)\right),x\right)\\
&=\frac{(n-r-1)!(n+r-1)!}{n!(n-1)!}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^{\infty}p_{n-r,k+r}(t)\psi(t)\left(f^{(r)}(t)-f_{2k+2,\eta}^{(r)}(t)\right)dt.
\end{aligned}$$

Hence

$$\left\|J_n^{(r)}\left(\psi(t)\left(f(t)-f_{2k+2,\eta}(t)\right),\bullet\right)\right\|_{C[a,b]}\leq C_5\left\|f^{(r)}-f_{2k+2,\eta}^{(r)}\right\|_{C[a',b']}.$$

Now for $x\in[a_1,b_1]$ and $t\in[0,\infty)\,/\,\left[a',b'\right]$ we can choose an η_1 satisfying $\left|t-x\right|\geq\eta_1$. Therefore by Lemma 5.2.4 and Schwarz inequality, we obtain

$$\begin{aligned}
I&\equiv\left|J_n^{(r)}\left((1-\psi(t))\left(f(t)-f_{2k+2,\eta}(t)\right),x\right)\right|\\
&\leq\frac{1}{n}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\frac{\left|q_{i,j,r}(x)\right|}{x^r}\sum_{k=0}^{\infty}l_{n,k}(x)\left|k-(n+1)x\right|^j\int_0^{\infty}l_{n,k}(t)\left(1-\psi(t)\right)\left|f(t)-f_{2k+2,\eta}(t)\right|dt\\
&\leq C_6\left\|f\right\|_{\alpha}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^{i-1}\sum_{k=0}^{\infty}l_{n,k}(x)\left|k-(n+1)x\right|^j\int_{\left|t-x\right|\geq\eta_1}l_{n,k}(t)dt\\
&\leq C_6\left\|f\right\|_{\alpha}\eta_1^{-2s}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^{i-1}\sum_{k=0}^{\infty}l_{n,k}(x)\left|k-(n+1)x\right|^j\left(\int_0^{\infty}l_{n,k}(t)dt\right)^{1/2}\\
&\qquad\qquad\left(\int_0^{\infty}l_{n,k}(t)(t-x)^{4s}dt\right)^{1/2}\\
&\leq C_6\left\|f\right\|_{\alpha}\eta_1^{-2s}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\left(\frac{1}{n}\sum_{k=0}^{\infty}l_{n,k}(x)(k-(n+1)x)^{2j}\right)^{1/2}\\
&\qquad\qquad\left(\frac{1}{n}\sum_{k=0}^{\infty}l_{n,k}(x)\int_0^{\infty}l_{n,k}(t)(t-x)^{4s}dt\right)^{1/2}.
\end{aligned}$$

Hence by Lemma 5.2.1 and Lemma 5.2.2, we have

$$I\leq C_7\left\|f\right\|_{\alpha}\sum n^{(i+\frac{j}{2}-s)}\leq C_7n^{-q}\left\|f\right\|_{\alpha},$$

where $q=(s-r/2)$. Now choose $s>0$ such that $q\geq k+1$, then $I\leq C_7n^{-(k+1)}\left\|f\right\|_{\alpha}$.

So by property (iii) of Steklov mean, we have

$$
\begin{aligned}
E_1 &\leq C_8 \left\| f^{(r)} - f^{(r)}_{2k+2,\eta} \right\|_{C[a',b']} + C_7 n^{-(k+1)} \left\| f \right\|_{\alpha} \\
&\leq C_9 \omega_{2k+2}(f^{(r)}, \eta, a, b) + C_7 n^{-(k+1)} \left\| f \right\|_{\alpha}.
\end{aligned}
$$

Hence with $\eta = n^{-1/2}$, the theorem follows. □

5.4 Inverse Theorem

In this section we shall prove the following inverse result.

Theorem 5.4.1. *If* $0 < \alpha < 2,\ 0 < a_1 < a_2 < b_2 < b_1 < \infty$ *and suppose* $f \in C_\alpha[0,\infty)$, *then in the following statements* $(i) \Rightarrow (ii)$

(*i*) $\left\| J_n(f,(d_0,d_1,d_2,...,d_k),\bullet) - f \right\|_{C[a_1,b_1]} = O\left(n^{-\alpha(k+1)/2}\right)$, *where* $f \in C_\alpha[a,b]$,

(*ii*) $f \in Lip(\alpha, k+1, a_2, b_2)$,

where $Lip^*(\alpha, a_2, b_2)$ *denotes the Zygmund class satisfying* $\omega_2(f,\eta,a_2,b_2) \leq M\eta^\alpha$.

Proof. Let us choose points a', a'', b', b'' in such a way that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. Also suppose $g \in C_0^\infty$ with supp $g \subset (a'', b'')$ and $g(x) = 1$ on the interval $x \in [a_2, b_2]$. To prove the assertion, it is sufficient to show that

$$
\left\| J_n(fg,(d_0,d_1,d_2,...,d_k),\bullet) - (fg) \right\|_{C[a',b']} = O\left(n^{-\alpha(k+1)/2}\right) \Rightarrow (ii). \qquad (5.4.1)
$$

Using F in place of fg for all the values of $h > 0$, we get

$$
\begin{aligned}
\left\| \triangle_h^{2k+2} F \right\|_{C[a'',b'']} &\leq \left\| \triangle_h^{2k+2}(F - J_n(F,(d_0,d_1,d_2,...,d_k),\bullet)) \right\|_{C[a'',b'']} \\
&\quad + \left\| \triangle_h^{2k+2} J_n(F,(d_0,d_1,d_2,...,d_k),\bullet) \right\|_{C[a'',b'']}. \qquad (5.4.2)
\end{aligned}
$$

Therefore, by definition of $\triangle_h^{2k+2}$,

$$
\left\| \triangle_h^{2k+2} J_n\left(F,(d_0,d_1,d_2,...,d_k),\bullet\right) \right\|_{C[a'',b'']}
$$

$$= \left\| \int_0^h \dots \int_0^h J_n\Big(F, (d_0, d_1, d_2, ..., d_k), \bullet + \sum_{i=1}^{2k+2} x_i\Big) dx_1 ... dx_{2k+2} \right\|_{C[a'',b'']}$$
$$\le h^{2k+2} \left\| J_n^{(2k+2)} (F, (d_0, d_1, d_2, ..., d_k), \bullet) \right\|_{C[a'',b''+(2k+2)h]}$$
$$\le h^{2k+2} \Big\{ \left\| J_n^{(2k+2)} (F - F_{\eta,2k+2}, (d_0, d_1, d_2, ..., d_k), \bullet) \right\|_{C[a'',b''+(2k+2)h]}$$
$$+ \left\| J_n^{(2k+2)} (F_{\eta,2k+2}, (d_0, d_1, d_2, ..., d_k), \bullet) \right\|_{C[a'',b''+(2k+2)h]} \Big\}, \qquad (5.4.3)$$

where $F_{\eta,2k+2}$ is the Steklov mean of $(2k+2)-th$ order corresponding to F. By Lemma 3 from [11], we get

$$\int_0^\infty \left| \frac{\partial^{2k+2}}{\partial x^{2k+2}} W_n(t,x) dt \right| \le \sum_{\substack{2i+j \le 2k+2 \\ i,j \ge 0}} \frac{1}{n} \sum_{k=0}^{\infty} (n+1)^i \left| k - (n+1)x \right|^j$$
$$\frac{|q_{i,j,2k+2}(x)|}{\{x(1+x)\}^{2k+2}} l_{n,k}(x) \int_0^\infty l_{n,k}(t) dt.$$

Since $\int_0^\infty l_{n,k}(t)dt = 1$. By Lemma 5.2.1, we have

$$\sum_{k=0}^{\infty} l_{n,k}(x) \big(k - (n+1)x\big)^{2j} = (n+1)^{2j} \sum_{k=0}^{\infty} l_{n,k}(x) \left(\frac{k}{n+1} - x \right)^{2j} = O(n^j). \qquad (5.4.4)$$

Using Schwarz inequality, we obtain

$$\left\| J_n^{(2k+2)} (F - F_{\eta,2k+2}, (d_0, d_1, d_2, ..., d_k), \bullet) \right\|_{C[a'',b''+(2k+2)h]}$$
$$\le K_1 n^{k+1} \left\| F - F_{\eta,2k+2} \right\|_{C[a'',b'']}. \qquad (5.4.5)$$

By Lemma 2 from [11], we get

$$\int_0^\infty \left[\frac{\partial^k}{\partial x^k} W_n(t,x) \right] (t-x)^i dt = 0, \ \ for \ k > i. \qquad (5.4.6)$$

By Taylor's expansion, we obtain

$$F_{\eta,2k+2}(t) = \sum_{i=0}^{2k+1} \frac{F_{\eta,2k+2}^{(i)}(x)}{i!} (t-x)^i + F_{\eta,2k+2}^{(2k+2)}(\xi) \frac{(t-x)^{2k+2}}{(2k+2)!}, \qquad (5.4.7)$$

where $t < \xi < x$. By (5.4.6) and (5.4.7), we get

$$\left\| \frac{\partial^{2k+2}}{\partial x^{2k+2}} J_n \left(F_{\eta,2k+2}, (d_0, d_1, d_2, ..., d_k), \bullet \right) \right\|_{C[a'',b''+(2k+2)h]}$$
$$\leq \sum_{j=0}^{k} \frac{|C(j,k)|}{(2k+2)!} \left\| F^{(2k+2)}_{\eta,2k+2} \right\|_{C[a'',b'']} \left\| \int_0^\infty \left[\frac{\partial^{2k+2}}{\partial x^{2k+2}} W_{d_j n}(t,x) \right] (t-x)^{2k+2} dt \right\|_{C[a'',b'']}.$$

Again applying Schwarz inequality for integration and summation and Lemma 3 from [11], we obtain

$$\begin{aligned}
I \equiv & \int_0^\infty \left| \frac{\partial^{2k+2}}{\partial x^{2k+2}} W_n(t,x) \right| (t,x)^{2k+2} dt \\
\leq & \frac{1}{n} \sum_{\substack{2i+j \leq 2k+2 \\ i,j \geq 0}} \sum_{k=0}^{\infty} (n+1)^i l_{n,k}(x) \left| k-(n+1)x \right|^j \frac{|q_{i,j,2k+2}(x)|}{\{x(1+x)\}^{2k+2}} \int_0^\infty l_{n,k}(t)(t-x)^{2k+2} dt \\
\leq & \sum_{\substack{2i+j \leq 2k+2 \\ i,j \geq 0}} (n+1)^i \frac{|q_{i,j,2k+2}(x)|}{\{x(1+x)\}^{2k+2}} \left\{ \sum_{k=0}^{\infty} l_{n,k}(x)(k-(n+1)x)^{2j} \right\}^{1/2} \\
& \times \left\{ \frac{1}{n} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^\infty l_{n,k}(t)(t-x)^{4k+4} dt \right\}^{1/2}. \qquad (5.4.8)
\end{aligned}$$

Using Lemma 2 from [11]

$$\frac{1}{n} \sum_{k=0}^{\infty} l_{n,k}(x) \int_0^\infty l_{n,k}(t)(t-x)^{4k+4} dt = T_{n,4k+4}(x) = O\left(n^{-(2k+2)}\right). \qquad (5.4.9)$$

Using (5.4.4) and (5.4.9) in (5.4.8), we obtain

$$I \leq \sum_{\substack{2i+j \leq 2k+2 \\ i,j \geq 0}} (n+1)^i \frac{|q_{i,j,2k+2}(x)|}{\{x(1+x)\}^{k+1}} O(n^{j/2}) O\left(n^{-(k+1)}\right) = O(1).$$

Hence

$$\left\| W_n^{(2k+2)} \left(F_{\eta,2k+2}, (d_0, d_1, d_2, ..., d_k), \bullet \right) \right\|_{C[a'',b''+(2k+2)h]}$$
$$\leq K_2 \left\| F^{(2k+2)}_{\eta,2k+2} \right\|_{C[a'',b'']}. \qquad (5.4.10)$$

On combining (5.4.2), (5.4.3), (5.4.5) and (5.4.10) it follows

$$\left\|\triangle_h^{2k+2}F\right\|_{C[a'',b'']} \leq \left\|\triangle_h^{2k+2}\left(F - J_n(F,(d_0,d_1,d_2,...,d_k),\bullet)\right)\right\|_{C[a'',b'']} + K_3h^{2k+2}\Big(n^{k+1}\left\|F - F_{\eta,2k+2}\right\|_{C[a'',b'']} + \left\|F^{(2k+2)}_{\eta,(2k+2)}\right\|_{C[a'',b'']}\Big).$$

For small value of h, the above relation holds, it follows from the properties of $F_{\eta,2k+2}$ and (5.4.1) that

$$\omega_{2k+2}(F,l,[a'',b'']) \leq K_4\left\{n^{-\alpha(k+1)/2} + l^{2k+2}\left(n^{k+1} + \eta^{-2k+2}\right)_{\omega_{2k+2}}(F,\eta,[a'',b''])\right\}.$$

Choosing η is such a way that $n < \eta^{-2} < 2h$ and following Berens and Lorentz [21], we obtain

$$w_{2k+2}(F,l,[a'',b'']) = O(l^{\alpha(k+1)}). \tag{5.4.11}$$

Since $F(x) = f(x)$ in $[a_2,b_2]$, from (5.4.11) we have

$$w_{2k+2}(f,l,[a_2,b_2]) = O(l^{\alpha(k+1)}), \quad i.e., \; f \in Liz(\alpha,k+1,a_2,b_2).$$

Let us assume (i). Putting $\tau = \alpha(k+1)$, we first consider the case $0 < \tau \leq 1$. For $x \in [a',b']$, we get

$$\begin{aligned} J_n(fg,(d_0,d_1,d_2,...,d_k),x) - f(x)g(x) &= g(x)J_n\left((f(t)-f(x)),(d_0,d_1,d_2,...,d_k),x\right) \\ &+ \sum_{j=0}^{k} C(j,k)\int_{a_1}^{b_1} W_{d_j,n}(t,x)f(x)\left(g(t)-g(x)\right)dt + O\left(n^{-k+1)}\right) \\ &= I_1 + I_2 + O\left(n^{-(k+1)}\right), \end{aligned} \tag{5.4.12}$$

where the $O - term$ holds uniformly for $x \in [a',b']$. Now by assumption

$$\left\|J_n(f,(d_0,d_1,d_2,...,d_k),\bullet) - f\right\|_{C[a_1,b_1]} = O\left(n^{-\tau/2}\right),$$

we have

$$\|I_1\|_{C[a',b']} \leq \|g\|_{C[a',b']}\|J_n(f,(d_0,d_1,d_2,...,d_k),\bullet)-f\|_{C[a',b']} \leq K_5 n^{-\tau/2}. \quad (5.4.13)$$

By mean value theorem, we get

$$I_2 = \sum_{j=0}^{k} C(j,k) \int_{a_1}^{b_1} W_{d_j,n}(t,x) f(t) \{g'(\xi)(t-x)\} \, dt.$$

Again applying Cauchy-Schwarz inequality and Lemma 2 from [11], we get

$$\|I_2\|_{C[a',b']} \ \|f\|_{C[a_1,b_1]}\|g'\|_{C[a',b']} \left(\sum_{j=0}^{k} |C(j,k)|\right) \max_{0\leq j\leq k} \left\| \int_0^\infty W_{d_j,n}(t,x)(t-x)^2 dt \right\|_{C[a',b']}^{1/2}$$
$$= O\left(n^{-\tau/2}\right). \quad (5.4.14)$$

Combining (5.4.12)-(5.4.14), we obtain

$$\|J_n(fg,(d_0,d_1,d_2,...,d_k),\bullet)-fg\|_{C[a',b']} = O\left(n^{-\tau/2}\right), \ for\ 0<\tau\leq 1.$$

Now to prove the implication for $0 < \tau < 2k+2$, it is sufficient to assume it for $\tau \in (m-1,m)$ and prove it for $\tau \in (m,m+1)$, $(m = 1,2,3,...,2k+1)$. Since the result holds for $\tau \in (m-1,m)$, we choose two points x_1, y_1 in such a way that $a_1 < x_1 < a' < b' < y_1 < b_1$. Then in view of assumption $(i) \Rightarrow (ii)$ for the interval $(m-1,m)$ and equivalence of (ii) it follows that $f^{(m-1)}$ exists and belongs to the class $Lip(1-\delta, x_1, y_1)$ for any $\delta > 0$. Let $g \in C_0^\infty$ be such that $g(x) = 1$ on $[a'', b'']$ and $supp\ g \subset [a'', b'']$. Then with $\chi(t)$ denoting the characteristic function of the interval $[x_1, y_1]$, we have

$$\|J_n(f,g,(d_0,d_1,d_2,...,d_k),\bullet)-fg\|_{C[a',b']}$$
$$\leq \|J_n(g(x)f(t)-f(x)),(d_0,d_1,d_2,...,d_k),\bullet)\|_{C[a',b']}$$
$$+\|J_n(f(t)(g(t)-g(x))\chi(t),(d_0,d_1,d_2,...,d_k),\bullet)\|_{C[a',b']} + O\left(n^{-(k+1)}\right). \quad (5.4.15)$$

Now

$$\begin{aligned}&\left\|J_n(g(x)(f(t)-f(x)),(d_0,d_1,d_2,...,d_k),\bullet)\right\|_{C[a',b']}\\&\leq \|g\|_{C[a'',b'']}\left\|J_n(f,(d_0,d_1,d_2,...,d_k),\bullet)-f\right\|_{C[a_1,b_1]}=O\left(n^{-\tau/2}\right).\end{aligned} \tag{5.4.16}$$

Applying Taylor's expansion of f, we have

$$\begin{aligned}I_3&\equiv\left\|J_n(f(t)g(t)-g(x))\chi(t),(d_0,d_1,d_2,...,d_k),\bullet)\right\|_{C[a',b']}\\&=\left\|J_n\left(\left[\sum_{i=0}^{m-1}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{\left\{f^{(m-1)}(\xi)-f^{(m-1)}(x)\right\}}{(m-1)!}\right]\right.\right.\\&\qquad\left.\left.\times\,(g(t)-g(x))\chi(t),(d_0,d_1,d_2,...,d_k),\bullet\right)\right\|_{C[a',b']},\end{aligned}$$

where $t<\xi<x$. Since $f^{(m-1)}\in Lip(1-\delta,x_1,y_1)$,

$$\left|f^{(m-1)}(\xi)-f^{(m-1)}(x)\right|\leq K_6\left|\xi-x\right|^{1-\delta}\leq K_6\left|t-x\right|^{1-\delta},$$

where K_6 is the $Lip(1-\delta,x_1,y_1)$ constant for $f^{(m-1)}$, we have

$$\begin{aligned}I_3&\leq\left\|J_n\left(\sum_{i=0}^{m-1}\frac{f^{(i)}(x)}{i!}(t-x)^i(g(t)-g(x))\chi(t),(d_0,d_1,d_2,...,d_k),\bullet\right)\right\|_{C[a',b']}\\&+\frac{K_6}{(m-1)!}\|g'\|_{C[a'',b'']}\left(\sum_{j=0}^{k}\left|C(j,k)\right|\right)\left\|L_{d_j,n}(\left|t-x\right|^{m+1-\delta}\chi(t),\bullet)\right\|_{C[a',b']}\\&=I_4+I_5\text{ (say).}\end{aligned} \tag{5.4.17}$$

By Taylor's expansion of g and Lemma 5.2.5, we have

$$I_4=O\left(n^{-(k+1)}\right). \tag{5.4.18}$$

Also, by Hölder's expansion of g and Lemma 2 from [11], we have

$$\begin{aligned} I_5 &\leq \frac{K_6}{(m-1)!} \left\| g' \right\|_{C[a'',b'']} \left(\sum_{j=0}^{k} \left| C(j,k) \right| \right) \\ &\quad \max_{0 \leq j \leq k} \left\| \int_{x_1}^{y_1} W_{d_j,n}(t-x) \left| t-x \right|^{m+1-\delta} dt \right\|_{C[a',b']} \\ &\leq K_7 \max_{0 \leq j \leq k} \left\| \int_{x_1}^{y_1} W_{d_j,n}(t-x)(t-x)^{2(m+1)} dt \right\|_{C[a',b']}^{\frac{(m+1-\delta)}{2(m+1)}} \\ &= O\left(n^{-(m+1-\delta)/2}\right) = O\left(n^{\tau/2}\right), \end{aligned} \tag{5.4.19}$$

by choosing such that $0 < \delta < m+1-\delta$. Combining the estimate (5.4.15-5.4.19), we get

$$\left\| J_n(fg, (d_0, d_1, d_2, ..., d_k), \bullet) - fg \right\|_{C[a',b']} = O\left(n^{\tau/2}\right).$$

This completes the proof of the Theorem 5.4.1. □

Chapter 6

Approximation by Statistical Convergence

6.1 Statistical Convergence

In this chapter, we use concept of statistical convergence and study the Korovkin type approximation theorem for the first kind Beta operators $\hat{\beta}_n$ and Jain operators $P_n^{[\beta]}$. Before we present the main results, we shall recall some notations and properties on the statistical and A-statistical convergence.

Let (x_n) be a sequence of numbers. Then (x_n) is called statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_j \frac{\Theta \{n \leq j : |x_n - L| \geq \varepsilon\}}{j} = 0,$$

where ΘD denotes the cardinality of the subset D (see [55], also [59]). We denote this statistical limit by $st - lim_n x_n = L$. Now, let $A = (a_{jn})$ be an infinite summability matrix. Then, the A-transform of x, denoted by $Ax = ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$, provided the series converges for each j. We say that A is regular if $lim_j(Ax)_j = L$ whenever $lim_j x_j = L$ [75]. Assume now that A is a nonnegative regular summability matrix. Then, a sequence (x_n) is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0 \tag{6.1.1}$$

holds (see [57]). It is denoted by $st_A - lim_n x_n = L$. Now we recall some basic properties of A-statistical convergence as follows:

- A-statistical convergence method is mainly based on the concept of A-density. Recall that the $A - density$ of a subset $K \subset \mathbb{N}$, denoted by $\delta_A(K)$, is given by

  $$\delta_A(K) = \lim_j \sum_{n=1}^{\infty} a_{jn} \chi_K(n),$$

  provided that the limit exists, where χ_K is the characteristic function of K; or equivalently,

  $$\delta_A(K) = \lim_j \sum_{n \in K} a_{jn}.$$

  So, by (6.1.1), we easily see that $st_A - limx = L$ if and only if

  $$\delta_A(\{n : |x_n - L| \geq \varepsilon\}) = 0$$

  for every $\varepsilon > 0$.

- If we take $A = C_1 = [c_{jn}]$, where the *Cesáro matrix* is given by

  $$c_{jn} = \begin{cases} \frac{1}{j}, & if \ \ 1 \leq n \leq j \\ 0, & otherwise, \end{cases}$$

  then A-statistical convergence reduces to *statistical convergence*, i.e., $st_{c_1} - lim_n \, x_n = st - lim_n \, x_n = L$.

- Taking $A = I$, the *identity matrix*, A-statistical convergence coincides with the ordinary convergence, i.e.,$st_I - lim \ x = lim \ x = L$.

- Observe that every convergent sequence(in the usual sense) is A-statistically convergent to the same value for any non-negative regular matrix A, but its converse is not always true. Actually, in [92], Kölk proved that A-statistical convergence is stronger than convergence when $A = [a_{jn}]$ is a non-negative regular summability matrix such that $lim_j \ max_n a_{jn} = 0$. So, one can construct a sequence that is A-statistically convergent but non-convergent.

- Not all the properties of convergent sequences are true for A-statistical convergence (or statistical convergence). For instance, although it is well known that a subsequence of a convergent sequence is convergent, that is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however an A-statistical convergent sequence does not need to be bounded.

- A characterization for statistical convergence, i.e., the case of $A = C_1$, was proved by Connor [30]: $st - lim \ x = L$ if and only if there exists a subsequence x_{nk} of x such that $\delta(n_1, n_2, ...) = 1$ and $lim_k \ x_{nk} = L$. It is easy to check that a similar characterization is also valid for A-statistical convergence when A is any non-negative regular summability matrix.

6.2 Statistical Convergence for General Beta Operators

Beta operators were introduced by Lupaş [97] and further modified and studied by Khan [90], Upreti [129], Divis [47] and others.

The Beta approximation $\beta_n(f)$ to a function $f : [0,1] \to \mathbb{R}$ is the operator:

$$\beta_n\left(f;x\right) = \frac{1}{B\left(nx, n(1-x)\right)} \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)dt, \qquad (6.2.1)$$

where $B(u,v)$ is the well-known beta probability density function

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt;\ u,v > 0,$$

with the support $(0,1)$ such that t denotes a value of the random variable T, where $n \in \mathbb{N}$, $x \in (0,1)$ and f is any real measurable, Lebesgue integrable function defined on $[0,1]$. When $x = 0$ or $x = 1$, then $\beta_n(f,x) = f(x)$ for all n.

Now the following Lemmas follow from [127], for the operators β_n mentioned by (6.2.1).

Lemma 6.2.1 ([127]). *Let* $e_i(x) = x^i$, $i = 0,1,2$. *Then, for each* $0 < x < 1$ *and* $n \in \mathbb{N}$, *we have*

(i) $\beta_n(e_0; x) = 1$,

(ii) $\beta_n(e_1; x) = x$,

(iii) $\beta_n(e_2; x) = \frac{x(1+nx)}{n+1}$.

Lemma 6.2.2 ([47]). *For each each* $0 < x < 1$ *and* $n \in \mathbb{N}$ *and* $\varphi_x(t) = t - x$, *we have* $\beta_n(\varphi_x^2; x) = \frac{x(1-x)}{n+1}$.

The aim of this section is to construct a general Beta type operators including the King type Beta operators which preserves the third test function x^2. We study some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators than the original Beta operators $\beta_n(f,x)$. Note that rate of convergence and statistical convergence of many other approximation operators are available in literatures(See [35], [36], [43], [50], [70], [79], [106], [110], [111]).

6.2.1 Modified First Kind Beta Operators

Let $\{\alpha_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0,1]$ with $0 < \alpha_n(x) < 1$. Now consider a sequence of positive linear operators:

$$\hat{\beta}_n(f,x) = \frac{1}{B(n\alpha_n(x), n(1-\alpha_n(x)))} \int_0^1 t^{n\alpha_n(x)-1}(1-t)^{n(1-\alpha_n(x))-1} f(t)dt, \quad (6.2.2)$$

where $x \in [0,1]$, $f \in [0,1]$ and $n \in \mathbb{N}$(set of natural numbers). If $\alpha_n(x)$ replaced by e_1, then we obtain original beta operators (6.2.1). Note that

Lemma 6.2.3. *For each* $0 \leq x \leq 1$ *and* $n \in \mathbb{N}$ *and* $\varphi_x(t) = t - x$*, we have*

(i) $\hat{\beta}_n(e_0; x) = 1,$

(ii) $\hat{\beta}_n(e_1; x) = \alpha_n(x),$

(iii) $\hat{\beta}_n(e_2; x) = \dfrac{\alpha_n(x)(1 + n\alpha_n(x))}{n+1},$

(iv) $\hat{\beta}_n(\varphi_x^2; x) = (\alpha_n(x) - x)^2 + \dfrac{\alpha_n(x)(1 - \alpha_n(x))}{n+1}.$

Now, if we replace $\alpha_n(x)$ by

$$\alpha_n^*(x) = \frac{-1 + \sqrt{1 + 4n(n+1)x^2}}{2n}, \quad x \in [0,1] \text{ and } n \in \mathbb{N},$$

then the operators $\hat{\beta}_n$ defined in (6.2.2) reduce to the operators

$$\beta_n^*(f; x) = \frac{1}{B(n\alpha_n^*(x), n(1 - \alpha_n^*(x)))} \int_0^1 t^{n\alpha_n^*(x)-1}(1-t)^{n(1-\alpha_n^*(x))-1} f(t)dt. \quad (6.2.3)$$

These operators are the King type Beta operators. Furthermore, the following Lemma hold:

Lemma 6.2.4. *The operators defined by (6.2.3) verify the following identities*

(i) $\beta_n^*(e_0; x) = 1,$

(ii) $\beta_n^*(e_1; x) = \frac{-1+\sqrt{1+4n(n+1)x^2}}{2n},$

(iii) $\beta_n^*(e_2; x) = x^2.$

Lemma 6.2.5. *For each* $0 \leq x \leq 1$ *and* $n \in \mathbb{N}$ *and* $\varphi_x(t) = t - x$*, we have*

(i) $\beta_n^*(\varphi_x; x) = \frac{\sqrt{1+4n(n+1)x^2} - (1+2nx)}{2n},$

(ii) $\beta_n^*(\varphi_x^2; x) = \frac{(1+2nx)x - x\sqrt{1+4n(n+1)x^2}}{n}.$

6.2.2 Rate of Convergence

In this section we study the rate of convergence of the operators $\hat{\beta}_n(f; x)$ to $f(x)$ by means of the modulus of continuity (0.2.1) and Peetre's K-functional (0.2.2). It is known that for any $\delta > 0$ and $x, y \in [a, b]$, we have

$$|f(y) - f(x)| \leq \omega(f; \delta) \left(\frac{|y - x|}{\delta} + 1 \right).$$

Theorem 6.2.6. *For every $f \in C[0,1]$ and $0 \leq x \leq 1$, we have*

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq 2\omega(f, \delta_{n,x}),$$

where $\delta_{n,x} = \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n+1}}$ *and $\omega(f, \delta_{n,x})$ is the modulus of continuity of f.*

Proof. Let $f \in C[0,1]$ and $x \in [0,1]$. Since $\hat{\beta}_n(e_0, x) = e_0(x)$, from Cauchy-Schwarz inequality for linear positive operators, we obtain for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq \left[\hat{\beta}_n(e_0; x) + \frac{1}{\delta_{n,x}} \left(\hat{\beta}_n\left((e_1 - x)^2; x \right) \right)^{1/2} \right] \omega(f, \delta_{n,x}).$$

Choosing $\delta_{n,x} = \sqrt{\hat{\beta}_n\left((e_1 - x)^2; x \right)} = \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n+1}}$, we obtain

$$\left| \hat{\beta}_n(f; x) - f(x) \right| \leq 2\omega(f, \delta_{n,x}).$$

□

For the King type Beta operators we have the following Corollary at once:

Corollary 6.2.7. *For every $f \in C[0,1]$ and $0 \leq x \leq 1$, we have*

$$|\beta_n^*(f; x) - f(x)| \leq 2\omega(f, \delta_{n,x}),$$

where $\delta_{n,x} = \sqrt{\frac{(1+2nx)x - x\sqrt{1+4n(n+1)x^2}}{n}}$.

Now we give the rate of convergence for the operators $\hat{\beta}_n(f; x)$ by using the Peetre's K-functional in the space $C^2[0,1]$. The classical Peetre's $K-$functional of a function $f \in C[0,1]$ is defined by (0.2.2) and the norm

$$\|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}.$$

Theorem 6.2.8. *For each* $f \in C[0,1]$

$$\left|\hat{\beta}_n(f;x) - f(x)\right| \leq K\left(f; \left(|\alpha_n(x) - x| + \left|(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\right)\right).$$

Proof. Applying Taylor expansion to the function $g \in C^2[0,1]$, we get

$$\hat{\beta}_n(g,x) - g(x) = g'(x)\hat{\beta}_n((e_1 - x), x) + \frac{1}{2}\hat{\beta}_n\left(g''(\xi)(e_1 - x)^2, x\right); \xi \in (t,x).$$

Hence

$$\begin{aligned}
&\left|\hat{\beta}_n(g;x) - g(x)\right| \\
&\leq \|g'\|_{C[0,1]}\left|\hat{\beta}_n((e_1 - x), x)\right| + \|g''\|_{C[0,1]}\left|\hat{\beta}_n((e_1 - x)^2, x)\right| \\
&= \|g'\|_{C[0,1]}|\alpha_n(x) - x| + \|g''\|_{C[0,1]}\left|(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|.
\end{aligned}$$

For each $f \in C[0,1]$, we can write

$$\begin{aligned}
&\left|\hat{\beta}_n(f,x) - f(x)\right| \\
&\leq \left|\hat{\beta}_n(f,x) - \hat{\beta}_n(g,x)\right| + \left|\hat{\beta}_n(g,x) - g(x)\right| + |g - f| \\
&\leq 2\|g - f\|_{C[0,1]} + \left|\hat{\beta}_n(g;x) - g(x)\right| \\
&\leq 2\|g - f\|_{C[0,1]} + \left(|\alpha_n(x) - x| + \left|(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\right)\|g\|_{C[0,1]} \\
&\leq 2\left(\|g - f\|_{C[0,1]} + |\alpha_n(x) - x| + \left|(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\|g\|_{C[0,1]}\right).
\end{aligned}$$

Taking infimum over $g \in C^2[0,1]$, we get

$$\left|\hat{\beta}_n(f,x) - f(x)\right| \leq K\left(f; \left(|\alpha_n(x) - x| + \left|(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\right)\right).$$

□

For the King type Beta operators we immediately have the following Corollary:

Corollary 6.2.9. *For each* $f \in C[0,1]$

$$\left|\hat{\beta}_n\left(f;x\right) - f(x)\right| \leq K\left(f;\gamma_{n,x}\right),$$

where $\gamma_{n,x} = \frac{1}{2n}\left(2x-1\right)\left(2nx - \sqrt{4n^2x^2+4nx^2+1}+1\right).$

Assume that for each $x \in [0,1]$,$(\alpha_n(x))_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ satisfying

$$st - \lim_n \alpha_n(x) = x. \tag{6.2.4}$$

Then we have

$$st - \lim_n |x - \alpha_n(x)| = 0 \ \text{ and} \tag{6.2.5}$$

$$st - \lim_n \left|\frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right| = 0. \tag{6.2.6}$$

Such a sequence $(\alpha_n(x))_{n\in\mathbb{N}}$ can be constructed as follows. Choose

$$\alpha_n(x) = \begin{cases} 2 & \text{, if } n = m^2 \ (m \in \mathbb{N}) \\ \alpha_n^*(x) & \text{, otherwise} \end{cases}$$

where

$$\alpha_n^*(x) = \frac{-1+\sqrt{1+4n(n+1)x^2}}{2n}, \quad x \in [0,1] \text{ and } n \in \mathbb{N}.$$

It is clear that (6.2.4) is satisfied.

Theorem 6.2.10. *For each* $x \in [0,1]$ *and for every* $f \in C[0,1]$*, we have*

$$st - \lim_n \left|\hat{\beta}_n\left(f;x\right) - f(x)\right| = 0.$$

Proof. For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Now define the sets:

$$U = \left\{n : \delta_{n,x}^2 \geq r\right\},$$
$$U_1 = \left\{n : |x - \alpha_n(x)| \geq \sqrt{\frac{r-\varepsilon}{2}}\right\},$$
$$U_2 = \left\{n : \left|\frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right| \geq \frac{r-\varepsilon}{2}\right\},$$

where $\delta_{n,x} = \sqrt{(\alpha_n(x) - x)^2 + \dfrac{\alpha_n(x)(1-\alpha_n(x))}{n+1}}$. Then it follows that $U \subseteq U_1 \cup U_2$, which gives

$$\sum_{j=1}^{n} \chi_U(j) \leq \sum_{j=1}^{n} \chi_{U_1}(j) + \sum_{j=1}^{n} \chi_{U_2}(j). \qquad (6.2.7)$$

Multiplying both sides of (6.2.7) by $\dfrac{1}{n}$ and letting $n \to \infty$, we get using (6.2.5) and (6.2.6) that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \chi_U(j) = 0.$$

This guarantees that $st - \lim_n \delta_{n,x}^2 = 0$ which implies $st - \lim_n \omega(f, \delta_{n,x}) = 0$ .Using Theorem 6.2.6 completes the proof. □

Remark 6.2.11. *If we choose the sequence $(\alpha_n(x))_{n\in\mathbb{N}}$ as in (6.2.4), then our statistical approximation result Theorem 6.2.10 works; however its classical version does not work since*

$$\alpha_n(x) \nrightarrow x$$

in the usual sense.

6.2.3 Best Error Estimation

Let ψ_x be the first central moment function defined by $\psi_x(y) = y - x$. In order to get a better error estimation on a subinterval I of $[0,1]$, in the approximation by means of the operators β_n , we are aimed to find a functional sequence (s_n), $s_n : I \to A$, satisfying

$$\delta_{n,x}^* = \sqrt{\hat{\beta}_n(\psi_x^2; s_n(x))} \leq \sqrt{\beta_n(\psi_x^2; x)} = \delta_{n,x} \quad \text{for } x \in I. \qquad (6.2.8)$$

By Lemmas 6.3.2 and 6.2.3(iv), 6.2.8 takes the form

$$\frac{n}{n+1}s_n^2(x)+\left(\frac{1}{n+1}-2x\right)s_n(x)-(\frac{n}{n+1}-2)x^2-\frac{2}{n+1}x\leq 0. \qquad (6.2.9)$$

Let

$$\Delta_n(x):=\left(\frac{1}{n+1}-2x\right)^2+4\frac{n}{n+1}\left\{(\frac{n}{n+1}-2)x^2+\frac{2}{n+1}x\right\}.$$

Then it is clear that

$$\Delta_n(x)\geq 0 \qquad (6.2.10)$$

and

$$x+\frac{x}{n}-\frac{1}{2n}\in[0,1] \qquad (6.2.11)$$

hold for every $x\in I=[\frac{1}{4},\frac{3}{4}]$ and for every $n\geq 1$. Therefore, from (6.2.9), (6.2.10) and (6.2.11), we get

$$\frac{2x-\frac{1}{n+1}-\sqrt{\Delta_n(x)}}{2\frac{n}{n+1}}\leq s_n(x)\leq\frac{2x-\frac{1}{n+1}+\sqrt{\Delta_n(x)}}{2\frac{n}{n+1}}.$$

Then $s_n(x)$ takes its minimum when

$$s_n(x)=x+\frac{x}{n}-\frac{1}{2n}.$$

Therefore, for all $x\in[\frac{1}{4},\frac{3}{4}]$, we define a new Beta type operator by

$$\beta_n^s(f;x)=\beta_n(f;s_n(x))=\frac{1}{B\left(ns_n(x),n(1-s_n(x))\right)}\int_0^1 t^{ns_n(x)-1}(1-t)^{n(1-s_n(x))-1}f(t)dt.$$

Then, for all $x\in[\frac{1}{4},\frac{3}{4}]$ and $n\geq 1$, we have

$$\beta_n^s(\psi_x^2;x)=\frac{x(1-x)}{n}-\frac{1}{4n(n+1)}\leq\frac{x(1-x)}{n+1}=\beta_n(\psi_x^2;x)$$

which shows that the operators $\beta_n^s(f;x)$ provides the better estimation than the operators $\beta_n(f;x)$.

6.3 Approximation of Jain Operators

Several extension and generalization of Bernstein polynomials have been given by various mathematician like Szâsz [124], Meyer-König and Zeller [102], Meir and Sharma [101], Stancu [123] and Balázs [17]. Mirakyan [104] has also given another modification with the help of the Poisson distribution. Recently, Acar, Gupta & Aral [3] and Aral & Gupta [16] have investigated the generalized Szâsz operators and its Durrmeyer form, respectively.

Later on in the same way with the help of a Poisson type distribution,

$$w_\beta(k,\alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1}e^{-(\alpha+k\beta)},\ k \in \mathbb{N}^0 = \{0\} \cup \mathbb{N},$$

for $0 < \alpha < \infty$ and $|\beta| < 1$, Jain [83] defined the following class of positive linear operators,

$$P_n^{[\beta]}(f,x) = \sum_{k=0}^{\infty} w_\beta(k;nx) f\left(\frac{k}{n}\right),\ x \geq 0, \tag{6.3.1}$$

where $\beta \in [0,1]$ and $f \in C(\mathbb{R}_+)$, the space of all real valued continuous functions defined on $\mathbb{R}_+$. Original Szâsz-Mirakyan operator can easily be obtained for $\beta = 0$. Recently Deo et al. [37, 39] studied another modification of Bernstein operators and Gupta [68] introduced q analogue of Bernstein operator. Now the following lemmas follow from [83], for the operators $P_n^{[\beta]}$ mentioned by (6.3.1).

Lemma 6.3.1. *[83] Let $e_i(x) = x^i,\ i = 0,1,2,$ then for $x \in [0,\infty)$, $n \in \mathbb{N}$ and $\beta \neq 1$, we have*

(i) $P_n^{[\beta]}(e_0;x) = 1,$

(ii) $P_n^{[\beta]}(e_1;x) = \frac{1}{1-\beta}x,$

(iii) $P_n^{[\beta]}(e_2;x) = \frac{1}{(1-\beta)^2}x^2 + \frac{1}{n(1-\beta)^3}x.$

Lemma 6.3.2. *For $x \in [0,\infty)$, $n \in \mathbb{N}$, $\beta \neq 1$ and $\varphi_x(t) = t - x$, we have*

(i) $P_n^{[\beta]}(\varphi_x;x) = \frac{\beta}{1-\beta}x,$

(ii) $P_n^{[\beta]}(\varphi_x^2;x) = \frac{\beta^2}{(1-\beta)^2}x^2 + \frac{1}{n(1-\beta)^3}x.$

6.3.1 Voronovskaya Type Results & Error Estimation

In this section we compute the Voronovskaya type results of these operators $P_n^{[\beta]}$ given by (6.3.1).

Let $f \in C_B[0,\infty)$ be the space of all real valued continuous bounded functions on $[0,\infty)$, equipped with the norm $\|f\| = \sup\limits_{t\in[0,\infty)} |f(t)|$. The Peetre's K_2-functional is defined by (0.2.2) and modulus of continuity by (0.2.1) with norm

$$\|f\|_{C_B^2} = \|f\|_{C_B} + \|f'\|_{C_B} + \|f''\|_{C_B}. \tag{6.3.2}$$

From [42], there exists a positive constant C such that

$$K_2(f,\delta) \le C\omega_2\left(f,\sqrt{\delta}\right) \tag{6.3.3}$$

and

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\delta}\sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|.$$

Theorem 6.3.3. *Let $f \in C_B[0,\infty)$, then for every $x \in [0,\infty)$ and for $C > 0$, we have*

$$\left|\left(P_n^{[\beta]}f\right)(x) - f(x)\right| \le C\omega_2\left(f,\sqrt{\frac{\beta}{1-\beta}x}\right),\ \beta \ne 1. \tag{6.3.4}$$

Proof. Let $g \in W_\infty^2$. Using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

From Lemma 6.3.2, we have

$$\left(P_n^{[\beta]}g\right)(x) - g(x) = \left(P_n^{[\beta]}g'(x)(t-x)\right)(x) + \left(P_n^{[\beta]}\int_x^t (t-u)g''(u)du\right)(x).$$

We know that

$$\left|\int_x^t (t-u)g''(u)du\right| \le (t-u)^2\|g''\|.$$

Therefore

$$\begin{aligned}\left|\left(P_n^{[\beta]}g\right)(x)-g(x)\right| &\leq \left(P_n^{[\beta]}\left(t-x\right)\right)(x)\left\|g'\right\|+\left(P_n^{[\beta]}(t-u)^2\right)(x)\left\|g''\right\| \\ &= \frac{\beta}{1-\beta}x\left\|g'\right\|+\left(\frac{\beta^2x^2}{(1-\beta)^2}+\frac{x}{n(1-\beta)^3}\right)\left\|g''\right\| \\ &\leq \frac{\beta}{1-\beta}x\left\{\left\|g'\right\|+\left\|g''\right\|\right\} \\ &\leq \frac{\beta}{1-\beta}x\left\|g''\right\|.\end{aligned}$$

By Lemma 6.3.1, we have

$$\left|\left(P_n^{[\beta]}g\right)(x)\right| \leq \sum_{k=0}^{\infty}\frac{\alpha}{k!}(\alpha+k\beta)^{k-1}e^{-(\alpha+k\beta)}g\left(\frac{k}{n}\right) \leq \|g\|.$$

Hence

$$\begin{aligned}\left|\left(P_n^{[\beta]}g\right)(x)-g(x)\right| &\leq \left|\left(P_n^{[\beta]}(g-f)\right)(x)-(g-f)(x)\right|+\left|\left(P_n^{[\beta]}g\right)(x)-g(x)\right| \\ &\leq 2\|g-f\|+\left(\frac{\beta}{1-\beta}x\right)\left\|g''\right\|.\end{aligned}$$

Taking the infimum on the right side over all $g \in W_\infty^2$ and using (6.3.3), we get the required result. □

6.3.2 A-statistical Convergence

In this section of the chapter, we use concept of A-statistical convergence and study the Korovkin type approximation theorem for the the operators $P_n^{[\beta]}$.

Now let $A = [a_{jn}]\,(j,n \in \mathbb{N})$ be a non-negative regular summability matrix. Assume that for each $t \in [0,\infty)$,$(\alpha_n^*(t))_{n\in\mathbb{N}}$ is a sequence in $[0,\infty)$ satisfying

$$st_A - \lim_n \alpha_n^*(t) = t,$$

then we have

$$st_A - \lim_n\left(t-\alpha_n^*(t)\right) = 0. \tag{6.3.5}$$

Theorem 6.3.4. *$A = [a_{jn}]$ is a non-negative regular summability matrix. Then, for each $x \in [0, \infty)$ and for every $f \in C[0, \infty)$, we have*

$$st_A - \lim_n \left| P_n^{[\beta]}(e_i; x) - e_i \right| = 0;\ e_i(t) = t^i, i = 0, 1, 2.$$

Proof. From Lemma 6.3.1, obviously $st_A - \lim_n \left| P_n^{[\beta]}(e_0; x) - 1 \right| = 0.$

$$\left| P_n^{[\beta]}(e_1; x) - x \right| \leq \left| \frac{\beta x}{1 - \beta} \right| = S(n, x), \quad x \in [0, \infty), \tag{6.3.6}$$

where $S(n, x) = \left| \frac{\beta x}{1-\beta} \right|$.
Now, for a given $\varepsilon > 0$, we define

$$V = \left\{ n : S(n, x) \geq \varepsilon \right\}.$$

Therefore, by (6.3.6), we obtain

$$\sum_{n : \left| P_n^{[\beta]}(e_1, x) - x \right| \geq \varepsilon} a_{jn} \leq \sum_{n \in V} a_{jn}.$$

Taking limit as $j \to \infty$ and from Lemma 6.3.1 and (6.3.5) we get the result.
Similarly,

$$\left| P_n^{[\beta]}(e_2; x) - x^2 \right| = \left| \frac{x^2}{(1 - \beta)^2} + \frac{x}{n(1 - \beta)^3} - x^2 \right| \tag{6.3.7}$$

$$= \left| \frac{x^2 \beta (2 - \beta)}{(1 - \beta)^2} + \frac{x}{n(1 - \beta)^3} \right| \tag{6.3.8}$$

$$= S_1(n, x) + S_2(n, x), \quad x \in [0, \infty). \tag{6.3.9}$$

$$T = \left\{ n : S_1(n, x) + S_2(n, x) \geq \varepsilon \right\},$$

$$T_1 = \left\{ n : S_1(n, x) \geq \frac{\varepsilon}{2} \right\}$$

$$T_2 = \left\{ n : S_2(n, x) \geq \frac{\varepsilon}{2} \right\}$$

Now, we have $T \subseteq T_1 \cup T_2$. Therefore by (6.3.9), we obtain

$$\sum_{n:\left|P_n^{[\beta]}(e_2,x)-x^2\right|\geq\varepsilon} a_{jn} \leq \sum_{n\in T} a_{jn} \leq \sum_{n\in T_1} a_{jn} + \sum_{n\in T_2} a_{jn}.$$

Taking limit as $j \to \infty$ and (6.3.5) gives the result. □

Similarly, we have

$$st_A - \lim_n \left\| P_n^{[\beta]}(e_1 - xe_0)^j \right\|_{C[0,\infty)} = 0, j = 1, 2. \tag{6.3.10}$$

Now we give a Korovkin type theorem for the operators $P_n^{[\beta]}(f;x)$ via A-statistical convergence.

Theorem 6.3.5. *Let $A = [a_{jn}]\,(j, n \in N)$ is a non-negative regular summability matrix. Then, for each $x \in [0,\infty)$ and for every $f \in C[0,\infty)$, we have*

$$st_A - \lim_n \left| P_n^{[\beta]}(f;x) - f(x) \right| = 0.$$

Proof. For a given $\varepsilon > 0$, define the following sets,

$$R = \left\{ n : \delta_{n,x}^2 \geq \varepsilon \right\},$$

$$R_1 = \left\{ n : (x - \alpha_n^*(x)) \geq \varepsilon \right\},$$

where $\delta_{n,x} = \sqrt{\frac{\beta}{1-\beta}x}$. Now it is easy to see that $R \subseteq R_1$, which gives

$$\sum_{n\in R} a_{j,n} \leq \sum_{n\in R_1} a_{j,n}.$$

Taking limit as $j \to \infty$ and using (6.3.5), we have $\lim_j \sum_{n\in R} a_{jn} = 0$. This gives that $st_A - \lim_n \delta_{n,x}^2 = 0$ which follows $st_A - \lim_n \omega(f, \delta_{n,x}) = 0$. Using Theorem 6.3.3 completes the proof. □

Now we give the rate of A-statistical convergence for the operators $P_n^{[\beta]}(f;x)$ by using the Peetre's K-functional in the space $C_B^2[0,\infty)$.

Theorem 6.3.6. *For each* $f \in C_B[0,\infty)$

$$\left\|P_n^{[\beta]}(f,x)-f(x)\right\|_{C_B} \leq \kappa\left(f;\gamma_{n,x}\right),$$

where $\kappa\left(f;\gamma_{n,x}\right)$ *is the sequence of Peetre's* K*-functional and*

$$\gamma_{n,x}=\left\|P_n^{[\beta]}((e_1-x),x)\right\|_{C_B}+\left\|P_n^{[\beta]}((e_1-x)^2,x)\right\|_{C_B}$$

and $st_A-\lim_n \gamma_{n,x}=0$ *for each fixed* $x\in[0,\infty)$.

Proof. Applying Taylor expansion to the function $f \in C_B^2[0,\infty)$, we get

$$P_n^{[\beta]}(f,x)-f(x)=f'(x)P_n^{[\beta]}((e_1-x),x)+\frac{1}{2}f''(\xi)P_n^{[\beta]}((e_1-x)^2,x),\ \ \xi\in(t,x).$$

Hence

$$\begin{aligned}&\left\|P_n^{[\beta]}(f,x)-f(x)\right\|_{C_B}\\&\leq \|f'\|_{C_B}\left\|P_n^{[\beta]}((e_1-x),x)\right\|_{C[0,\infty)}+\|f''\|_{C_B}\left\|P_n^{[\beta]}((e_1-x)^2,x)\right\|_{C[0,\infty)}.\end{aligned} \tag{6.3.11}$$

Using (6.3.2) and (6.3.11), for each $g \in C_B^2[0,\infty)$,

$$\begin{aligned}\left\|P_n^{[\beta]}(g,x)-g(x)\right\|_{C_B^2}&=\left(\left\|P_n^{[\beta]}((e_1-x),x)\right\|_{C[0,\infty)}+\left\|P_n^{[\beta]}((e_1-x)^2,x)\right\|_{C[0,\infty)}\right)\|g\|_{C_B^2}\\&=\gamma_{n,x}\|g\|_{C_B^2}.\end{aligned}$$

For each $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$,

$$\begin{aligned}\left\|P_n^{[\beta]}(f,x)-f(x)\right\|_{C_B^2}&\leq\left\|P_n^{[\beta]}(f,x)-P_n^{[\beta]}(g,x)\right\|_{C_B}+\left\|P_n^{[\beta]}(g,x)-g(x)\right\|_{C_B^2}+\|g-f\|_{C_B}\\&\leq 2\|g-f\|_{C_B}+\left\|P_n^{[\beta]}(g,x)-g(x)\right\|_{C_B^2}\\&\leq 2\|g-f\|_{C_B}+\gamma_{n,x}\|g\|_{C_B^2}\leq 2\left(\|g-f\|_{C_B}+\gamma_{n,x}\|g\|_{C_B^2}\right).\end{aligned}$$

Taking infimum over $g \in C_B^2[0,\infty)$, we get

$$\left\|P_n^{[\beta]}(f,x)-f(x)\right\|_{C_B^2}\leq\kappa\left(f;\gamma_{n,x}\right).$$

From Theorem 6.3.5, we get $st_A - \lim_n \gamma_{n,x} = 0$, therefore $st_A - \lim_n \kappa\,(f; \gamma_{n,x}) = 0$. □

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List of Research Papers

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