DIFFERENTIAL SUBORDINATIONS, COEFFICIENTS ESTIMATE AND RADIUS CONSTANTS OF CERTAIN ANALYTIC FUNCTIONS

by

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CERTIFICATE

This is to certify that the thesis entitled "**Differential Subordinations, Coefficients Estimate and Radius Constants of Certain Analytic Functions**" submitted to the Delhi Technological University, Delhi for the award of Doctor of Philosophy is based on the original work carried out by me under the supervision of Dr. S. Sivaprasad Kumar, Department of Applied Mathematics, Delhi Technological University, Delhi. It is further certified that the work embodied in this thesis has neither partially nor fully submitted to any other university or institution for the award of any degree or diploma.

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Abstract

In this thesis, we have derived differential subordination, superordination and sandwich type results and also determined several coefficient estimates for certain classes of analytic functions. Further, we have investigated various radius problems associated with analytic functions with positive real part. The thesis comprises of seven chapters, which includes a chapter on introduction. In Chapter 2, we have obtained conditions on certain parameters so that the given differential subordination implication holds. In Chapter 3, we have discussed the properties of a class of linear operators which satisfy a common recurrence relation. Several sufficient conditions for Janowski, Sokół-Stankiewicz and strongly starlikeness are also determined. In addition to that we have given alternate proofs of certain results proved in [**17**]. In Chapter 4, we have considered a class of linear operators defined in terms of convolution which can be expressed as convex combination of two operators. For this class of operators, we have established the differential sandwich theorems. Several applications of these results are also discussed. In Chapter 5, we have derived estimate on the Fekete-Szegö functional for ceratin classes of functions and various special cases of our results are also pointed out. In Chapter 6, estimate on the initial coefficients of certain subclasses of bi-univalent functions are established. Further some of our results improve the known estimates, which are pointed out here along with some special cases to our results. In Chapter 7, radius of starlikeness such as starlikeness of order α, parabolic starlikeness and Sokół-Stankiewicz starlikeness for functions with fixed second coefficient are investigated when they satisfy certain conditions on the ratio f/g, where g is either starlike or convex function with fixed second coefficient.

Contents

List of Symbols

Symbol	**Description**	**Page**
$\mathcal{M}(\alpha)$	class of α–convex functions	11
$\mathcal{M}(\alpha,\varphi)$	α–convex functions with respect to φ	11
$\mathcal{CC}$	class of close-to-convex functions	13
$\mathcal{S}^*_L$	class of lemniscate starlike functions	13
$\mathcal{S}^*_P$	parabolic starlike function	13
σ	the class of normalized analytic bi-univalent functions	14
$\mathcal{S}^*_\sigma(\beta)$	class of bi-starlike functions of order β	14
$\mathcal{K}^*_\sigma(\beta)$	class of bi-convex functions of order β	14
$\mathcal{SS}^*_\sigma(\alpha)$	bi-strongly starlike functions of order α	14
$\mathcal{S}_b$	class of functions in $\mathcal{S}$ with fixed second coefficient	15
$\mathcal{P}_b$	class of functions in $\mathcal{P}$ with fixed second coefficient	15
$\mathcal{P}_b(\alpha)$	class of functions in $\mathcal{P}(\alpha)$ with fixed second coefficient	15
$\mathcal{P}_b[A,B]$	class of functions in $\mathcal{P}[A,B]$ with fixed second coefficient	15
$\mathcal{S}^*_b(\alpha)$	class of functions in $\mathcal{S}^*(\alpha)$ with fixed second coefficient	16
$\mathcal{K}^*_b(\alpha)$	class of functions in $\mathcal{K}(\alpha)$ with fixed second coefficient	16
$\mathcal{S}^*_b[A,B]$	class of functions in $\mathcal{S}^*[A,B]$ with fixed second coefficient	16
$\mathcal{K}_b[A,B]$	class of functions in $\mathcal{S}^*[A,B]$ with fixed second coefficient	16
$\mathcal{H}[a,n]$	class of analytic functions in $\mathbb{D}$ of the form	
	$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$	18
$\mathcal{A}(p,n)$	class of analytic functions in $\mathbb{D}$ of the form	
	$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k$ $(p,n \in \mathbb{N})$	18
$\mathcal{A}_p$	$\mathcal{A}(p,1)$	18
$\mathbb{N}$	set of all natural numbers	19
$(a)_n$	Pochhammer symbol	19
${}_lF_m$	generalized hypergeometric function	19
$H_p^{l,m}$	Dziok-Srivastava operator	19
$\mathcal{I}_p(r,\lambda)$	multiplier transform defined on the space $\mathcal{A}_p$	21
I^r_λ	multiplier transform defined on the space $\mathcal{A}$	21

Chapter 1

Introduction

The theory of univalent functions is a classical branch of complex analysis. It is classified under Geometric Function Theory (GFT) due to the fact that from simple geometrical considerations, many remarkable properties of univalent functions can be found. The celebrated Reimann mapping theorem gave rise to the birth of GFT in 1951 and the genesis of univalent function theory goes back to 1907 with a paper [**84**] by Koebe. The theory of univalent functions has rich and vast literature, the early development of the theory centered around the Bieberabach conjecture [**26**], which provides the coefficient estimate for univalent functions defined on the open unit disk. Apart from the classical papers [**84, 85**] by Koebe, several researchers enriched this area of research with their enormous contributions. The books by Pommerenke [**138**], Goodman [**62**], Duren [**50**], Goluzin [**60**], Graham and Kohr [**66**], Littlewood [**96**] and Schober [**163**] provide an introduction to the basics of univalent function theory. The proceeding [**49**], edited by Dold and Eckmann, is very useful as it consists of the lectures delivered by eminent researchers and mathematicians such as Ahlfors, Duren, Keogh, Goodman, Miller, Rudin, Suffridge and several others. The bibliography of schlicht functions [**31**] by Bernardi lists the related references of univalent functions from 1907 to 1981. Another useful proceeding [**92**] edited by Ławrynowicz contains selected papers related to GFT from those submitted by a part of mathematicians lecturing at the 8th conference on analytic functions held in Poland at Błażejewko in

1983. The book written by Hallenbeck and MacGregor [**70**] covers extreme points and support points theory. A brief literature of various books, monographs, lecture notes, survey articles published during that time on univalent function theory can be found at the end of the book [**70**]. The books written by Duren [**50**], Goluzin [**60**], Graham and Kohr [**66**] and Goodman [**62**] also enlist useful references on univalent function theory. Henrici [**72**] has also included two chapters of complex analysis in his book with one being on univalent function theory. The books [**47, 48, 169**] on complex analysis also provide some topics on univalent function theory. The book by Hayman [**71**] deals with the growth of univalent and multivalent functions and bound for the modulus and coefficients related quantities. The book [**190**] edited by Srivastava and Owa provides a collection of research-and-survey articles of recent times, related to the theory of analytic functions. However the book [**90**] edited by Kühnau contains some special topics contributed by eminent researchers such as Hayman (multivalent functions), Pommerenke (conformal maps), Prokhorov (bounded univalent functions), Aksent'ev and Shabalin (sufficient condition for univalent functions and quasiconformal extendibility of analytic functions). Bulboacă [**38**] has also provided a chronological order of books and monographs useful in this context. Goodman [**63**] has elaborated basics and a brief look into the literature (upto 1979) of univalent function theory. For survey on radius problems one may consult the book entitled 'Univalent Functions–II' by Goodman [**62**].

The concept of differential subordination was introduced in 1981 by Miller and Mocanu [**107**]. They were first to replace the real differential inequality with a true complex analogue. The monograph written by Miller and Mocanu [**108**] is a collection of results from more than 400 papers and provides a systematic study of differential subordination. Miller and Mocanu [**108**] provided very simple proofs of the various results in GFT which were proved earlier using lengthy and tedious techniques. In 2003, Miller and Mocanu [**105**] introduced the notion of differential superordination as a dual concept of differential subordination. The concepts of differential subordination and superordination together lead to sandwich result.

The theory of differential subordination extensively used to prove several interesting results for functions defined by linear operators. Bulboacă [**36–39**] used the theory of differential subordination to discuss various properties of functions defined by linear operators. Kumar et al. [**180, 181**] have also done extensive study of applications of differential subordination and superordination techniques to univalent functions and linear operators. For several other applications of differential subordination techniques one can refer the works of Ali et al. [**8, 20**], Ravichandran et al. [**145, 146, 148, 149**], Obradović et al. [**124**], Patel and Mishra [**134**], Srivastava and Attiya [**187**], Nunokawa et al. [**119**], Jinlin [**77**] and Ponnusamy [**139–141**], see also the references cited therein. Sokół et al. [**133, 184, 185**] have considered several classes of functions defined in terms of subordination and derived several properties of functions in those classes.

Recently, Ali et al. [**14**] extended the concept of second order differential subordination to the analytic functions with fixed second coefficient and derived several interesting results. Using the results developed in [**14**], Nagpal and Ravichandran [**113**] obtained sufficient conditions for starlikeness, close-to-convexity and several other interesting results. For several other applications of differential subordination for functions with fixed second coefficient in univalent function theory, see [**91**].

1.1 Univalent Functions

A function f defined on the open set $D \subset \mathbb{C}$ is said to be *univalent* if it is one-to-one that is if $f(z_1) = f(z_2)$, then $z_1 = z_2 (z_1, z_2 \in D)$. It is known that analytic functions which are univalent in the whole complex plane are of the form $az + b, a \neq 0$ only. Further the Riemann mapping theorem says "every proper simply connected domain in $\mathbb{C}$ is conformally equivalent to the unit disk", which ensures that properties of univalent functions defined on a simply connected domain correspond to properties of univalent function defined on the unit disk. We therefore restrict our study to analytic univalent functions defined on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Suppose g is an analytic univalent function defined on $\mathbb{D}$, whose Maclaurin expansion is given by

$$g(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \quad (b_1 \neq 0),$$

then clearly the function defined by

$$f(z) = \frac{g(z) - g(0)}{g'(0)}$$

is also an analytic univalent function in $\mathbb{D}$. Since properties of the functions f and g correspond to each other, we consider the functions with the normalizations

$$f(0) = 0 = f'(0) - 1$$

and such functions are of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}. \tag{1.1.1}$$

Let $\mathcal{A}$ denote the class of all normalized analytic functions defined on $\mathbb{D}$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Thus $\mathcal{S}$ is the class of normalized univalent functions. The mapping $k(z) = z/(1-z)^2$ is called Koebe function, which maps $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/4]$. It is an extremal function in $\mathcal{S}$ due to the very fact that it is impossible to add to the image domain any open set of points without affecting the univalence. There exist several sufficient conditions for the univalence of functions belonging to $\mathcal{A}$ in literature. Among those, a result of Alexander [**6**] proved in 1915, which states that *"if* $\mathrm{Re}(f'(z)) > 0$ *in any convex domain, then* f *is univalent"* is the most simple one and it was later generalized by Noshiro–Warschawski [**62**, Theorem 13, pp-88]. Thereafter several simple sufficient conditions are derived in terms of the quantities either $f'(z)$ or $f''(z)$, or in terms of their ratios or involving their general higher order derivatives (see, [**54, 55, 196, 200, 205**] and the references cited therein).

For functions $f \in \mathcal{S}$, given by (1.1.1), there are two questions that may be asked here, one is how does a given sequence of coefficients $\{a_n\}$ influence the geometric property of f? and secondly, if some property of f is known, how does this property influences the coefficients $\{a_n\}$ in (1.1.1)? To answer the second question, in 1916, Bieberbach [**26**] proved that the second Taylor's coefficient of each function in the class $\mathcal{S}$ is bounded by 2 i.e. $|a_2| \leq 2$ and equality holds if and only if f is either Koebe function $k(z) = z/(1-z)^2$ or one of its rotations. Observing the extremal nature of the Koebe function $k(z) = z/(1-z)^2 = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots$, Bieberbach conjectured that *if* $f \in \mathcal{S}$ *given by (1.1.1), then* $|a_n| \leq n$.. This conjecture was unresolved for a quite long time, the reason being that the methods known at that time were not substantial enough to show this statement in its completeness. Several researchers tried to figure out this conjecture under certain geometrical conditions on the image domains of normalized analytic univalent functions and some were able to prove it for some specific values of n. Consequently, several subclasses of $\mathcal{S}$ came into existence. Some of these subclasses and their properties are dealt in Section 1.1. This conjecture was successfully verified by Rogosinski [**153**] in 1931 for functions with real coefficients. The long awaited proof of the Bieberbach conjecture finally came when de Branges [**52**] in 1985, proved this conjecture affirmatively. For an insight into the concept of the proof, see [**53**]. An alternate but simpler proof of the Bieberbach conjecture is available in [**206**]. The books by Conway [**48**] and Gong [**61**] also contain proofs of the Bieberbach conjecture. For the historical development of the conjecture and the main ideas that led to the proof of the Bieberbach conjecture, one may refer to the article by Koepf [**86**]. The Bieberbach conjecture and its proof made the coefficient problems in general very interesting. Finding the estimate on coefficients is the most fascinating area of research in univalent function theory as it provides several geometric properties of the functions under study. The Bieberbach theorem, namely $|a_2| \leq 2$ for $f \in \mathcal{S}$, immediately implies the growth and distortion estimates respectively

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \quad \text{and} \quad \frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$

with equality in case of the Koebe function or one of its rotations. Consequent upon growth theorem, we have the Koebe one-quarter theorem [**50**], which says that *the* *range of every function in the class $\mathcal{S}$ contains the disk* $|w| < 1/4$.

Another type of coefficient problem is to find the sharp estimate on the functional $|a_2^2 - \mu a_3|$ for functions belonging to a particular class. Bieberbach, in 1916, proved that if $f \in \mathcal{S}$, then $|a_2^2 - a_3| \leq 1$. In 1933, Fekete and Szegö [**58**] proved the following inequality:

$$|\mu a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & (\mu \leq 0), \\ 1 + 2\exp\left(-\frac{2\mu}{1-\mu}\right) & (0 \leq \mu \leq 1), \\ 4\mu - 3 & (\mu \geq 1) \end{cases}$$

for functions in the class $\mathcal{S}$ and the result is sharp. The problem of finding sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ for any compact family of functions is later popularly known as the Fekete-Szegö problem. Ma and Minda [**101**] solved the Fekete-Szegö problem for functions in the class $\mathcal{S}$ such that the quantities zf'/f or $1 + zf''/f'$ maps the unit disk $\mathbb{D}$ onto a region in the right half-plane lying in a domain which is starlike with respect to 1, and symmetric with respect to the real axis. Recently, Ali et al. [**18**], gave a reformulation of a result given by Ma and Minda [**101**]. For more literature on Fekete-Szegö problem one can refer [**18, 21, 25, 123, 145, 151, 202**] and the references cited therein.

The subclasses of univalent functions are closely associated with functions having positive real part. For example, if $f \in \mathcal{A}$ satisfies $\operatorname{Re} f'(z) > 0$, then $f \in \mathcal{S}$, which holds for any convex domain D. This in fact, leads us to introduce now the class of functions with positive real part.

Carathéodory Class and Subordination

Let $\mathcal{P}$ be the class of analytic functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$. This class is known as the *Carathéodory class* or the *class of functions with positive real part* [**40, 41**]. The function $p(z) = (1 + z)/(1 - z)$ belongs to the class $\mathcal{P}$ and plays a vital role similar to the Koebe function for the class $\mathcal{S}$. Let $\mathbf{B}$ be the class of *Schwarz functions* that is $w \in \mathbf{B}$ if and only if w is an analytic function with

$w(0) = 0$ and $|w(z)| < 1$ on $\mathbb{D}$. The following correspondence between the classes **B** and $\mathcal{P}$ holds:

$$p \in \mathcal{P} \text{ if and only if } w(z) = \frac{p(z)-1}{p(z)+1} \in \mathbf{B}.$$

Thus, the properties of functions in the class $\mathcal{P}$ can be inferred from those of the class **B** and conversely. The Herglotz's representation formula for a function $p \in \mathcal{P}$ is given by

$$p(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t) \quad (z \in \mathbb{D}), \tag{1.1.2}$$

where μ is a non-decreasing function on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$. The Equation (1.1.2) immediately gives the growth and distortion estimates respectively

$$\frac{1+|z|}{1-|z|} \leq \operatorname{Re} p(z) \leq \frac{1+|z|}{1-|z|} \text{ and } |p'(z)| \leq \frac{2\operatorname{Re} p(z)}{1-|z|^2} \leq \frac{2}{(1-|z|)^2}$$

with equality holds for the function $p(z) = (1+z)/(1-z)$. We shall discuss about the various generalizations and related results of this class in Section 1.2 and in Chapter 5.

Herglotz's representation formula (1.1.2) immediately shows that $|c_n| \leq 2$ for $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$. Another useful coefficient inequality [**66**] in this context is provided below:

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Ma and Minda [**101**], in 1982, proved the following result:

Lemma 1.1.1. [**101**] *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots \in \mathcal{P}$, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2 & (v \leq 0), \\ 2 & (0 \leq v \leq 1), \\ 4v-2 & (v \geq 1). \end{cases}$$

When $v < 0$ or $v > 1$, equality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, equality holds if and only if

$$p(z) = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1, z \in \mathbb{D}) \tag{1.1.3}$$

or one of its rotations. For $v = 1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that equality holds in case of $v = 0$. Also for $0 < v < 1$, the following improved estimate holds:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \quad (1/2 \leq v < 1).$$

For a complex number v, the above inequality was proved by Koegh and Merkes [**83**]:

Lemma 1.1.2. [**83**]*(see also* [**149**]*) If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots \in \mathcal{P}$, then, for any complex number v,*

$$|c_2 - vc_1^2| \leq 2 \max\{1; |2v - 1|\}$$

and the equality holds for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2} \ \text{ and } \ p(z) = \frac{1 + z}{1 - z}.$$

The Lemmas 1.1.1 and 1.1.2 have been extensively used by several researchers to solve the Fekete-Szegö problem and several other coefficient problems. In Chapter 5 we shall provide an elaborated detail on this topic.

In general we express the analytic conditions associated with various subclasses of $\mathcal{S}$ either in terms of functions with positive real part or subordination. For two analytic functions f and g, we say that f is *subordinate* to g or g is *superordinate* to f, denoted by $f \prec g$, if there is an analytic function $w \in \mathbf{B}$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. By Schwarz lemma [**62**], it follows that $|f'(0)| \leq |g'(0)|$ and the image of each disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r, 0 \leq r < 1\}$ under g contains the image of the same disk under f, i.e. $f(\mathbb{D}_r) \subset g(\mathbb{D}_r)$. This fact is known as *subordination principle* or *Lindelöf principle*. The concept of subordination was given by Lindelöf [**63**], while Littlewood [**95, 96**] and Rogosinski [**154, 155**] investigated it further. In terms of subordination, $p \in \mathcal{P}$ if and only if $p(z) \prec (1 + z)/(1 - z)$. Consequently if $f \prec g$ in $\mathbb{D}_r$, we have $\max_{|z| \leq r} |f(z)| \leq \max_{|z| \leq r} |g(z)|$. For some other direct implications one may refer [**138, 138**].

1.2 Subclasses of Univalent Functions

If a function $f \in \mathcal{A}$ satisfies $\operatorname{Re} f'(z) > 0$ in $\mathbb{D}$, then the estimate $|c_n| \leq 2$ for functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ immediately yields $|a_n| \leq 2/n, n \geq 1$. Therefore one can expect an easy proof of the coefficient estimate by imposing additional geometric conditions on functions belonging to $\mathcal{S}$. For this purpose we shall here define certain geometric domains and the related classes. A domain $D \subset \mathbb{C}$ is said to be *starlike with respect to a point* $z_0 \in D$ if the line segments joining z_0 to other points $w \in D$ lie entirely in D. The domain D is said to be *convex* if it is starlike with respect to all its points; that is the line segment joining any two points of D lies entirely in D. A domain which is starlike with respect to the origin is called starlike domain. A function is said to be *starlike* if it maps $\mathbb{D}$ onto a starlike domain whereas a convex function is one which maps $\mathbb{D}$ onto a *convex* domain. The subclasses of $\mathcal{S}$ consisting of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively. Obviously, each convex function is starlike but not conversely and the inclusion $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$ holds. For example, the Koebe function $k(z) = z(1-z)^{-2}$ is starlike but not convex. Thus, the problem of computing the radius of convexity of starlike functions arises here, which is to find the largest disk $\mathbb{D}_r \subset \mathbb{D}$ such that $f(\mathbb{D}_r)$ is convex whenever f is starlike on the unit disk $\mathbb{D}$. In general, for two sub-families T_1 and T_2 of $\mathcal{A}$, the T_1-radius of T_2 is the largest number ρ such that $r^{-1}f(rz) \in T_1$ for all $f \in T_2$ and $0 < r \leq \rho$. The number ρ is called the T_1 radius of the class T_2. Results related to radius problems are provided in [**62**].

Analytically, $f \in \mathcal{S}^*$ iff $zf'/f \in \mathcal{P}$ and $f \in \mathcal{K}$ iff $1 + zf''/f' \in \mathcal{P}$. In terms of subordination, the conditions for starlikeness and convexity of functions in $\mathcal{S}$ can be written as

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}$$

respectively. The classes of starlike and convex functions were generalized by Robertson [**152**] in 1936. The classes of starlike and convex functions of order α $(0 \leq \alpha < 1)$

are defined by the following expressions respectively:

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\}$$

or equivalently

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\}$$

or equivalently

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \right\}.$$

The functions $f(z) = z/(1-z)^{2-2\alpha}$ and

$$f(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq 1/2; \\ -\log(1-z), & \alpha = 1/2 \end{cases}$$

play the role of extremal functions for the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. The class of *strongly starlike functions of order* η $(0 < \eta \leq 1)$ is defined by

$$\mathcal{SS}^*(\eta) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^{\eta} \right\} = \left\{ f \in \mathcal{S} : \left| \arg \frac{zf'(z)}{f(z)} \right| \leq \frac{\eta\pi}{2} \right\}.$$

This class was introduced by Brannan and Kirwan [**28**] and Stankiewicz [**191**]. Janowski [**76**] (see also [**137**]) generalized the classes of starlike and convex functions of order α by replacing the superordinate function $(1+z)/(1-z)$ with $(1+Az)/(1+Bz)$. For $-1 \leq B < A \leq 1$, the class $\mathcal{S}^*[A,B]$ of Janowski starlike functions and the class $\mathcal{K}[A,B]$ of Janowski convex functions are defined respectively by

$$\mathcal{S}^*[A,B] := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\mathcal{K}[A,B] := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\}.$$

In all the classes defined above, functions are characterized by the quantity either $zf'(z)/f(z)$ or $1 + (zf''(z)/f'(z))$ lying in a convex region on the right half-plane and this motivated Ma and Minda [**101**] to give a general and unified presentation of various subclasses of starlike and convex functions. For this purpose, they have considered the function $\varphi \in \mathcal{P}$ such that $\varphi'(0) > 0$ and $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$, which is symmetric with respect to the real axis. The class $\mathcal{S}^*(\varphi)$ consists of functions $f \in \mathcal{S}$ such that $zf'(z)/f(z) \prec \varphi(z)$ and $\mathcal{K}(\varphi)$ be the class of functions $f \in \mathcal{S}$ such that $1+zf''(z)/f'(z) \prec \varphi(z)$. Ma and Minda [**101**] proved growth, distortion, covering and coefficient estimates for functions in these classes. The Hadamard product (or convolution), which is defined by $(f*g)(z) = z+\sum_{n=2}^{\infty} a_n b_n z^n$ for analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is an important tool used to generalize and unify results in univalent function theory. Padmanabhan [**132**] used convolution and subordination to unify these classes and proved convolution theorems. Another generalization of starlike and convex functions was considered by Mocanu [**110**]. The notion of $\alpha-$*convex* functions was introduced by Mocanu [**110**], in 1963, with the aim to construct one parameter family of subclasses of $\mathcal{S}$ which provides a continuous passage from the starlike functions to convex functions. A function $f \in \mathcal{A}$ is said to be $\alpha-$*convex* if

$$\operatorname{Re}\left((1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right) > 0.$$

The class of all such functions is denoted by $\mathcal{M}(\alpha)$. Miller et al. [**106**], in 1973, proved that functions belonging to $\mathcal{M}(\alpha)$ are starlike for all real α and convex for $\alpha \geq 1$. For more detail, about this class, see [**62, 66**]. The class $\mathcal{M}(\alpha)$ was further generalized by Ali et al. [**21**], by introducing the class $\mathcal{M}(\alpha, \varphi)$ of α-convex functions with respect to φ consisting of functions $f \in \mathcal{A}$ satisfying

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \varphi(z).$$

The class $\mathcal{M}(\alpha, \varphi)$ includes the classes $\mathcal{M}(\alpha) := \mathcal{M}(\alpha, (1+(1-2\alpha)z)/(1-z))$, $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$.

There is a two way bridge between the class of starlike functions and the class of convex functions, namely $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$, which is known as Alexander theorem [**6**]. Since $zf'(z) = f(z) * z/(1-z)^2$, it follows that f is convex if and only if $f(z) * z/(1-z)^2$ is starlike. Further if take $g(z) = z/(1-z)^2$ and $h(z) = z/(1-z)$, then $(f*g)(z)/(f*h)(z) = zf'(z)/f(z)$. For the functions $g(z) = (z+z^2)/(1-z)^3$ and $h(z) = z/(1-z)^2$, it can be easily verified that

$$\frac{(f*g)(z)}{(f*h)(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

This very idea paved the path for researchers to consider the subordinations of the form either $(f*g)(z)/(f*h)(z) \prec (1+z)/(1+z)$ or even a more general representation $(f*g)(z)/(f*h)(z) \prec \varphi(z)$, where $\varphi \in \mathcal{P}$ is an analytic function in $\mathbb{D}$. For a given function $g \in \mathcal{A}$, Shanmugam [**164**] introduced the class $\mathcal{K}_g^\alpha(h)$ consisting of functions $f \in \mathcal{A}$ such that

$$\alpha\left(1 + \frac{z(g*f)''(z)}{(g*f)'(z)}\right) + (1-\alpha)\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z),$$

where $h(0) = 1$ and h is a convex univalent function with positive real part and the definition explicitly assumes that $(g*f)(z)/z \neq 0 \neq (g*f)'(z)$ in $\mathbb{D}$. The class $\mathcal{M}(\alpha, \varphi)$, introduced by Ali et al. [**21**], is more general than the class $\mathcal{K}_g^\alpha(h)$ in the sense that φ is starlike whereas h is convex. Shanmugam [**164**] also introduced some more classes and discussed convolution and inclusion properties of all those classes. Supramaniam et al. [**194**], in 2009, generalized the classes introduced by Shanmugam [**164**] and obtained the inclusion and convolution properties using the methods of convex hull and subordination. Note that starlikeness of φ, in the classes defined above, is required to prove the distortion and growth estimates whereas φ need to be convex in order to get the convolution theorem. Furthermore there is no such requirement if one is interested to obtain the coefficient estimate, see [**18**]. We shall use this fact in Chapter 5 to derive estimate on Fekete-Szegö functional. For several applications and open questions related to convolution one may refer [**75,159**].

Apart from the above stated classes, the other prominent subclasses of $\mathcal{S}$ include the class of close-to-convex functions introduced by Kaplan [**80**] and the class of ϕ-like functions introduced by Brickman [**33**]. A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there exists a convex function g (not necessarily normalized) such that $\operatorname{Re}(f'(z)/g'(z)) > 0$. The class of such functions is denoted by $\mathcal{CC}$. In view of the

Alexander theorem [**6**], the function $F(z) = zg'(z)$ is starlike, it follows that the condition in the definition of the class $\mathcal{CC}$ becomes $\mathrm{Re}(zf'(z)/F(z)) > 0$. Thus every starlike function is close-to-convex and the inclusion relation $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{CC} \subset \mathcal{S}$ holds. The class of ϕ-like functions is a generalization of the class of close-to-convex and starlike functions. Let ϕ be an analytic function in a domain containing $f(\mathbb{D})$ such that $\phi(0) = 0$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{D}) \setminus \{0\}$. Then the function $f \in \mathcal{A}$ is said be *$\phi-$like function* if $\mathrm{Re}(zf'(z)/\phi(f(z))) > 0$. The class of $\phi-$like function was further generalized by Ruscheweyh [**161**]. For a given univalent function q with $q(0) = 1$, a function $f \in \mathcal{A}$ is said to be *$\phi-$like with respect to q*, if the subordination $zf'(z)/\phi(f(z)) \prec q(z)$ holds. The class of $\phi-$like functions is extensively studied by several researchers including [**68, 148, 166**]. Inspired by several geometrically defined classes of functions in the univalent function theory, Sokół and Stankiewicz [**185**] introduced the class of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. This class is denoted by $\mathcal{S}^*_L = \{f \in \mathcal{A} : |(zf'(z)/f(z))^2 - 1| < 1\}$. Functions in the class $\mathcal{S}^*_L$ are called Sokół-Stankiewicz starlike functions. Rønning [**156**] introduced the class $\mathcal{S}^*_P$ of parabolic starlike functions which is given by

$$\mathcal{S}^*_P := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \right\}.$$

Obviously $\mathcal{S}^*_P = \mathcal{S}^*(\varphi)$ when $\varphi(z) = 1 + (2/\pi^2)(\log(1 - \sqrt{z})/(1 + \sqrt{z}))^2$. For more insight into the results related to parabolic starlike functions one may refer to [**16, 64, 157**].

Bi–Univalent Functions

Since univalent functions are one-to-one, they are invertible but their inverse functions need not be defined on the entire unit disk $\mathbb{D}$. However, the famous Koebe's one-quarter theorem ensures that the image of $\mathbb{D}$ under every function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus the inverse of every function $f \in \mathcal{S}$ will be defined on the disk $|z| < 1/4$. It can be easily verified that

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + \cdots \tag{1.2.1}$$

is defined on the disk $|w| < 1/4$. A *bi-univalent* function is a univalent function defined on the unit disk $\mathbb{D}$ for which the inverse function has a univalent extension to the unit disk $\mathbb{D}$.

A function in $\mathcal{S}$ is said to be in σ, the class of bi-univalent functions iff its inverse has a unique extension to the unit disk $\mathbb{D}$ i.e. if $f \in \mathcal{S}$ is said to be in σ if and only if $f^{-1} \in \mathcal{S}$. In 1967, Lewin [**93**] introduced the class σ of bi-univalent functions and showed that the second coefficient of every function $f \in \sigma$ satisfies the non-sharp inequality $|a_2| \leq 1.51$. Some examples of functions which belong to the class σ are $z/(1-z)$, $-\log(1-z)$, and $(1/2)(\log(1+z)/(1-z))$. However the functions $k(z) = z/(1-z)^2$, $z - z^2/2$ and $z/(1-z^2)$ are not members of the class σ. We now enlist a few subclasses of σ. For $0 \leq \beta < 1$, a function $f \in \sigma$ is said to be respectively in the class $\mathcal{S}_\sigma^*(\beta)$ of *bi-starlike functions of order* β and $\mathcal{K}_\sigma(\beta)$ of *bi-convex functions of order* β whenever both f and f^{-1} are respectively starlike and convex functions of order β. For $0 < \alpha \leq 1$, the function $f \in \sigma$ is *strongly bi-starlike function of order* α if both the functions f and f^{-1} are strongly starlike functions of order α, the class of all such functions is denoted by $\mathcal{SS}_\sigma^*(\alpha)$. The classes $\mathcal{S}_\sigma^*(\beta)$, $\mathcal{K}_\sigma(\beta)$ and $\mathcal{SS}_\sigma^*(\alpha)$ were introduced by Brannan and Taha [**30**] in 1985 (see also [**29**]), they obtained estimate on the initial coefficients a_2 and a_3 for functions belong to these classes. Smith [**182**] proved that if $f(z) = z + a_2z^2 + a_3z^2$ with $a_2, a_3 \in \mathbb{R}$, is bi-univalent, then $|a_2| \leq 2/\sqrt{27}$ and $|a_3| \leq 4/27$. He also conjectured that for V_n, the set of all bi-univalent polynomials of the form $f(z) = z + a_2z^2 + a_3z^2 + \cdots + a_nz^n$, the following estimate must hold:

$$\max_{V_n} |a_n| \leq \frac{(n-1)^{n-1}}{n^n}.$$

Kedzierawski and Waniurski [**82**] validated this conjecture for $n = 3, 4$. Kedzierawski [**81**] considered the cases when f and f^{-1} belong to different subclasses of univalent functions, and determined the estimates on a_2 and a_3. Similar problems for functions in certain classes defined by subordination are studied in Chapter 6.

Functions with Fixed Second Coefficient

In GFT, finding the estimate for coefficients of functions in a specific class plays an important role, as it reveals the geometric nature of the function. For example, the bound for second coefficient of functions in the class $\mathcal{S}$ gives the growth, distortion and covering theorems. These applications of estimate on second coefficient in case of univalent functions attracted several researcher to explore the properties of functions with fixed second coefficient. Here below we shall give some classes of functions with fixed second coefficient. It is well known that if $p(z) = 1 + b_1 z + \cdots \in \mathcal{P}$, then $|b_1| \leq 2$. Further if $\tau = e^{-i \arg b_1}$, then $p(\tau z) = 1 + |b_1| z + \cdots \in \mathcal{P}$. Thus, there is no loss of generality in taking b_1 to be non-negative, see [**103, 104**]. The class of univalent functions of the form $f(z) = z + a_2 z^2 + \cdots$ with fixed second coefficient $a_2 = 2b$ $(|b| \leq 1)$ is denoted by $\mathcal{S}_b$. The investigation of properties of functions with fixed second coefficients started in 1920 with Gronwall's [**67**] growth and distortion theorems for functions in $\mathcal{S}_b$. The class $\mathcal{P}_b(\alpha)$ is the collection of functions $p(z) = 1 + 2b(1-\alpha)z + \cdots$ $(|b| \leq 1)$, which are analytic and satisfying $\operatorname{Re}(p(z)) > \alpha$ $(|b| \leq 1)$ for $z \in \mathbb{D}$. Assume $\mathcal{P}_b := \mathcal{P}_b(0)$. Tepper [**197**] obtained the sharp estimate on $|p(z)|$ and sharp lower bound on $\operatorname{Re}(p(z))$ for the class $\mathcal{P}_b$. These results were further generalized by McCarty [**103**] in 1972, for functions in the class $\mathcal{P}_b(\alpha)$. For $p \in \mathcal{P}_b(\alpha)$ the sharp lower bound on $\operatorname{Re}(zp'(z)/p(z))$ was obtained by McCarty [**104**]. For $-1 \leq B < A \leq 1$ and $|b| \leq 1$, let $\mathcal{P}_{b,n}[A,B]$ denote the class of functions defined by

$$\mathcal{P}_{b,n}[A,B] := \left\{ p : p(z) = 1 + b(A-B)z^n + \cdots \prec \frac{1+Az}{1+Bz} \right\}.$$

For functions $p \in \mathcal{P}_{b,n}[A,B]$, Padmanabhan and Ganesan [**131**] obtained the sharp estimates for $|p(z)|, |p'(z)|$ and $|zp'(z)/p(z)|$ under the conditions $A + B \geq 0$ and $AB < 0$. Further they utilized these results to obtain the sharp radius of convexity for starlike functions of the form $f(z) = z + a_{n+1} z^{n+1} + \cdots$. For the functions $p \in \mathcal{P}_b[A,B] := \mathcal{P}_{b,1}[A,B]$, Tuan and Anh [**198**] obtained a sharp lower bound for $\operatorname{Re}\left(\mu p(z) + \nu z p'(z)/p(z)\right)$ $(\mu, \nu \geq 0)$ and also obtained the radius of convexity for functions in the class defined in terms of the ratio of normalized analytic functions satisfying certain geometric conditions, see [**199**].

For $|b| \leq 1$, the classes defined by

$$\mathcal{S}_b^*(\alpha) := \left\{ f(z) = z + 2b(1-\alpha)z^2 + \cdots : \frac{zf'(z)}{f(z)} \in \mathcal{P}_b(\alpha) \right\}$$

and

$$\mathcal{K}_b(\alpha) := \left\{ f(z) = z + b(1-\alpha)z^2 + \cdots : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_b(\alpha) \right\}$$

are called the classes of starlike and convex functions of order α $(0 \leq \alpha < 1)$ with fixed second coefficient respectively. For $|b| \leq 1$, Tuan and Anh [**198**] also considered more general classes

$$\mathcal{S}_b^*[A, B] = \left\{ f(z) = z + b(A-B)z^2 + \cdots : \frac{zf'(z)}{f(z)} \in \mathcal{P}_b \right\}$$

and

$$\mathcal{K}_b[A, B] = \left\{ f(z) = z + \frac{b(A-B)}{2}z^2 + \cdots : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_b \right\}$$

and obtained the growth and distortion estimates for functions in these classes, in fact these results generalize the results (growth and distortion estimates) of Tepper [**197**]. Further contents related to this topic will be covered in Chapter 7.

1.3 Differential Subordination

The concept of differential subordination in the complex-plane is a generalization of differential inequality on the real line. In fact a differential inequality or a set of differential inequalities of a real function depicts the characterization or bound for a real function. In GFT, there are several differential subordination implications, which lead to the characterization of a function under consideration. The notations and definitions related to differential subordination and differential superordination are provided below:

Let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be analytic and ϕ be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the *first-order differential subordination*

$$\psi(p(z), zp'(z); z) \prec \phi(z), \tag{1.3.1}$$

then p is called a *solution* of the differential subordination. The univalent function q is called a *dominant of the solutions of the differential subordination*, or more simply a dominant, if $p \prec q$ for all p satisfying (1.3.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.3.1), is called the *best dominant* of (1.3.1). The best dominant is unique up to a rotation of $\mathbb{D}$.

Let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ and ϕ be analytic in $\mathbb{D}$. If p and $\psi(p(z), zp'(z); z)$ are univalent in $\mathbb{D}$ and satisfy the *first-order differential superordination*

$$\phi(z) \prec \psi(p(z), zp'(z); z). \tag{1.3.2}$$

Then p is called a *solution* of the differential superordination. An analytic function q is called a *subordinant of the solutions of the differential superordination*, or more simply subordinant if $q \prec p$ for all p satisfying (1.3.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.3.2) is said to be the *best subordinant* of (1.3.2). The best subordinant is unique up to a rotation of $\mathbb{D}$.

Definition 1.3.1. [**105**, Definition 2, p. 817] Denote by $\mathcal{Q}$, the set of all functions f that are analytic and injective on $\overline{\mathbb{D}} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(f)$.

Lemma 1.3.1. [**108**, Theorem 3.4h, p. 132] *Let q be univalent in $\mathbb{D}$ and let θ and ϕ be analytic in a domain $D \supset q(\mathbb{D})$ with $\phi(z) \neq 0$, when $z \in q(\mathbb{D})$.*

Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$ and suppose that either

(i) h is convex or Q is starlike univalent in $\mathbb{D}$.

In addition, assume that

(ii) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0$ *for* $z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1.3.3}$$

then $p \prec q$ and q is the best dominant.

Lemma 1.3.2. [**37**, Corollary 3.2, p. 289] *Let q be univalent in the unit disk $\mathbb{D}$ and θ and q be analytic in a domain $D \supset q(\mathbb{D})$. Suppose that* $\operatorname{Re}(\theta'(q(z))/\phi(q(z))) > 0$, *and $Q(z) := zq'(z)\phi(q(z))$ is starlike univalent in $\mathbb{D}$.*

If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, with $p(\mathbb{D}) \subseteq D$, and $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathbb{D}$, then

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{1.3.4}$$

implies $q \prec p$ and q is the best subordinant.

The above mentioned results are required to obtain several sandwich theorems.

1.4 Linear Operators

Let us recall some definitions that are needed in sequel. A function f is called p-valent (or multivalent of order p) in $\mathbb{D}$ if the equation $f(z) = w_0$ has at most p roots in $\mathbb{D}$ where the roots are counted with their multiplicities and for some w_1 the equation $f(z) = w_1$ has exactly p roots in $\mathbb{D}$. Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$, consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}(p, n)$ be the class of analytic functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (z \in \mathbb{D}; n, p \in \{1, 2, 3, \ldots\}). \tag{1.4.1}$$

Clearly $\mathcal{A}(1, 1) =: \mathcal{A}$ and let $\mathcal{A}_p := \mathcal{A}(p, 1)$. Recall that the symbol $*$ denotes the Hadamard product (or convolution), the convolution of two p-valent functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$ is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k.$$

For complex numbers a and c ($c \neq 0, -1, -2, \ldots$), the *confluent (or Kummer) hypergeometric function* ${}_1F_1(a, c; z)$ is defined by

$${}_1F_1(a, c; z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots \quad (z \in \mathbb{D}). \tag{1.4.2}$$

In terms of the Pochhammer symbol $(a)_n$, which is defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\dots(a+n-1), & (n \in \mathbb{N} := \{1,2,3\dots\}), \end{cases}$$

the function defined in (1.4.2) can be written as

$$_1F_1(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \tag{1.4.3}$$

Let a, b and c $(c \neq 0, -1, -2, \dots)$ are any complex numbers. Then the function defined by the following series

$$_2F_1(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}, \quad (z \in \mathbb{D}) \tag{1.4.4}$$

is called the *Gaussian hypergeometric function* and it satisfies the hypergeometric differential equation $z(1-z)w''(z) + (c-(a+b+1))w'(z) - abw(z) = 0$.

For $\alpha_j \in \mathbb{C}$ $(j = 1,2,\dots,l)$, $\beta_j \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$ $(j = 1,2,\dots m)$ and $l \leq m+1; l,m \in \mathbb{N}_0 := \{0,1,2,\dots\}$ the *generalized hypergeometric function* is defined by the infinite series

$$_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!}.$$

Corresponding to the function

$$h_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z) := z^p \ {}_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z), \tag{1.4.5}$$

the Dziok-Srivastava operator [**51**] (see also [**186**]) $H_p^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)$ is defined by the Hadamard product as follows:

$$\begin{aligned} H_p^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z) &:= h_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z) * f(z) \\ &= z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \dots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} \dots (\beta_m)_{n-p}} \frac{a_n z^n}{(n-p)!}. \end{aligned}$$

For brevity, we let $H_p^{l,m}[\alpha_1] = H_p^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)$. The Dziok-Srivastava operator satisfies the following recurrence relation:

$$z(H_p^{l,m}[\alpha_1]f(z))' = \alpha_1 H_p^{l,m}[\alpha_1+1]f(z) - (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z).$$

The operator $H_p^{l,m}[\alpha_1]$ generalizes several known operators. A few of them are listed below:

1. The Hohlov operator [**74**] $\mathcal{F}(\alpha,\beta,\gamma)f(z) := H_1^{2,1}(\alpha,\beta;\gamma)f(z)$.

2. The Carlson-Shaffer linear operator [**42**]

$$\mathcal{L}(\alpha,\gamma)f(z) := H_1^{2,1}(\alpha,1;\gamma)f(z) =: \mathcal{F}(\alpha,1,\gamma)f(z).$$

3. The Ruscheweyh operator [**160**]

$$\mathcal{D}^\lambda f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) = H_1^{2,1}(\lambda+1,1;1)f(z) \quad (\lambda \geq 1).$$

4. The operator

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt$$

is called the generalized Bernardi-Libera-Livingston linear integral operator (see [**32, 94, 100**]). For $c = 1$ this operator reduces to the operator

$$F(z) = \frac{2}{z}\int_0^z f(t)dt,$$

introduced by Libera [**94**]. Clearly the generalized Bernardi-Libera-Livingston operator can be written as $F(z) = H_1^{2,1}(c+1,1;c+2)f(z)$. Note that the classes of starlike, convex and close-to-convex functions are closed under the generalized Bernardi-Libera-Livingston operator [**108**].

5. The Srivastava-Owa fractional derivative operator (*cf.* [**126**], [**130**]) defined by

$$\Omega^\lambda f(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z) = H_1^{(2,1)}(2,1;2-\lambda)f(z),$$

where D_z^λ is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda}d\zeta.$$

Here f is an analytic function defined on a simply connected domain of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{1-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$. The operator D_z^λ is called fractional derivative operator of order $\lambda, 0 \leq \lambda < 1$.

Prompted by the Komatu integral operator [**87**] and the differential and integral operators defined by Sălăgean [**162**], recently Cho and Kim [**45**] introduced a more general linear operator called the multiplier transform defined as follows: For $\lambda \geq 0$ and any integer n, the multiplier transform $\mathcal{I}_\lambda^n : \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathcal{I}_\lambda^r f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^r a_k z^k.$$

The operator $\mathcal{I}_1^n$ was studied by Uralegaddi and Somantha [**204**]. The $p-$valent analogue of multiplier transform defined by Cho and Kim [**45**], was given by Kumar et al. [**181**] as follows:

$$\mathcal{I}_p(r,\lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda} \right)^r a_k z^k \ \ (\lambda \geq 0, r \in \mathbb{Z}).$$

The following recurrence relation satisfied by multiplier transform:

$$z(I_p(r,\lambda)f(z))' = (p+\lambda)I_p(r+1,\lambda)f(z) - \lambda I_p(r,\lambda)f(z).$$

The operator

$$I_\lambda^r := I_1(r,\lambda) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n a_k z^k \ \ (\lambda \geq 0, r \in \mathbb{Z})$$

was studied by Cho and Srivastava [**46**] and Cho and Kim [**45**]. Uralegaddi and Somanatha [**204**] studied the operator $I_r := I_1(r, 1)$. Another special case of this is the multiplier transform introduced by Al-Kharasani and Al-Areefi [**23**] which includes operators defined in [**89, 118**] and [**117**] as well as the Jung-Kim-Srivastava operator [**79**] and its p-valent analogue of Liu [**97**]. The book written by Bulboacă [**39**] offers an extensive collection of various results involving linear operators. For a survey on linear operators, see [**167**]. In Chapter 3, we discuss the properties of functions defined by certain linear operators.

1.5 Synopsis of the Thesis

The thesis is comprised of 7 chapters. It begins with an introductory chapter which incorporates the basic definitions, terminologies and concepts that are required in the sequel. The next three chapters focus on subordination theorems, the subsequent couple of chapters deals with coefficient estimates and the concluding chapter handles radius problems for analytic functions with fixed second coefficient. We now enlist below a chapter wise brief of the research study:

In **Chapter 2**, motivated by the works in [**7, 9, 11, 133, 184**], we shall establish certain differential subordination implications. For analytic function $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$, the conditions on $\beta \neq 0, A$ and B are determined in the following cases:

1. $1 + \beta \frac{zp'(z)}{p^k(z)} \prec \frac{1+Az}{1+Bz}$ implies $p(z) \prec \sqrt{1+z}$ $\quad (-1 < k \leq 3)$
2. $1 + \beta \frac{zp'(z)}{p^n(z)} \prec \sqrt{1+z}$ implies $p(z) \prec \frac{1+Az}{1+Bz}$ $\quad (n = 0, 1, 2)$
3. $p(z) + \beta \frac{zp'(z)}{p^n(z)} \prec \sqrt{1+z}$ implies $p(z) \prec \sqrt{1+z}$ $\quad (n = 0, 1, 2)$
4. $p(z) + \beta \frac{zp'(z)}{p(z)} \prec \sqrt{1+z}$ implies $p(z) \prec \frac{1+Az}{1+Bz}$ $\quad (-1 \leq B < A \leq 1)$.

Recently Ali et al. [**17**] obtained condition on the constants $A, B, D, E \in [-1, 1]$ and β so that

$$1 + \beta \frac{zp'(z)}{p^n(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz} \quad (n = 0, 1).$$

Alternate proofs of these results are also provided in this chapter. Further we concluded with the condition on $A, B, D, E \in [-1, 1]$ and β such that

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}.$$

In recent times, numerous linear operators were introduced in GFT, many of these linear operators are unified in this chapter by defining the class $\mathcal{O}_p$ of all linear operators which satisfy either of the recurrence relation

$$z[L_p^a f(z)]' = \alpha_a L_p^{a+1} f(z) - (\alpha_a - p) L_p^a f(z)$$

or

$$z[L_p^b f(z)]' = \alpha_b L_p^{b-1} f(z) - (\alpha_b - p) L_p^b f(z).$$

In **Chapter 3**, the differential subordination, superordination and corresponding sandwich results of p-valent analytic functions defined using the general linear operator as well as a related integral transform are investigated. Further, these results are applied to obtain sufficient conditions for functions $f \in \mathcal{A}$ to be Janowski starlike, strongly starlike of order η and lemniscate starlike. In addition to that several other interesting applications are given. Our main results generalize several existing known results in the literature. For example, Obradović [**121**, Theorem 2], in 1997, proved that if $f \in \mathcal{A}$ satisfies $\operatorname{Re} f'(z) > 0$, then $\operatorname{Re}(f(z)/z) > 0$. This result was generalized as an application of our results. We have shown that: If $\operatorname{Re} f'(z) > (3\alpha - 1)/2$ $(0 \leq \alpha < 1)$, then $\operatorname{Re}(f(z)/z) > \alpha$. This result reduces to [**121**, Theorem 2] when $\alpha = 1/3$. In addition to that several other interesting applications are given which deal with the sufficient conditions for starlikeness.

Lupaş in two separate papers [**4**] and [**5**], introduced a new operator $RI^\alpha(n, \lambda, l)$ for functions $f \in \mathcal{A}$ and $f \in \mathcal{A}_n$ respectively as follows:

$$RI^\alpha(n, \lambda, l)f(z) = (1 - \alpha)R^n f(z) + \alpha I(n, \lambda, l)f(z) \quad (\alpha \geq 0),$$

where $R^n f(z)$ and $I(n, \lambda, l)f(z)$ are respectively the Ruscheweyh derivative and the generalized multiplier transform. In **Chapter 4**, inspired by the operators considered by Lupaş [**2, 4, 5**], a generalized linear operator

$$\mathcal{O}_{g,h}(\alpha)f(z) = (1 - \alpha)(f * g)(z) + \alpha(f * h)(z) \ (\alpha \in \mathbb{C})$$

is defined on the space of normalized analytic functions for each pair (g, h) of normalized analytic functions. In this chapter differential subordination, differential superordination and corresponding sandwich results involving the generalized linear operator $\mathcal{O}_{g,h}(\alpha)$ are obtained. Some relevant connections of our results with earlier work are also pointed out. Some special cases of our main results are:

a.) Let $f \in \mathcal{A}$ and F be defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt.$$

If $\operatorname{Re} c > -1$ and

$$f'(z) + (1-\alpha)zf''(z) \prec \frac{1-(1-2\beta)z}{1-z} \quad (\beta < 1),$$

then

$$F'(z) + (1-\alpha)zF''(z) \prec 2(1-\beta)\, {}_2F_1(1, c+1; c+2; -z) + 2\beta - 1. \quad (1.5.1)$$

The function on the right of (1.5.1) is convex and is the best dominant. When $c = 0$ and $\alpha = 1$ the above result reduces to the result [**108**, Lemma 5.5k] of Miller and Mocanu.

b.) Let $f \in \mathcal{A}$. If $\operatorname{Re}(f'(z) + (1-\alpha)zf''(z)) > \beta$, then

$$\operatorname{Re}\left(\frac{\alpha f(z) + (1-\alpha)zf'(z)}{z}\right) > 2(\beta-1)\ln 2 + 2\beta - 1.$$

The above result generalizes the result of Owa et al. [**99**, Corollary 1] and for $\alpha = 0$, the result reduces to the result [**69**, Theorem 6] of Hallenbeck.

c.) If $f \in \mathcal{A}$ satisfies the inequality

$$\operatorname{Re}(f'(z) + (1-\alpha)zf''(z)) > \frac{3\beta - 1}{2} \quad (0 \leq \beta < 1),$$

then

$$\operatorname{Re}\left(\frac{\alpha f(z) + (1-\alpha)zf'(z)}{z}\right) > \beta.$$

The later result coincides with the results [**174**, Example 3.5] of Kumar et al. and [**121**, Theorem 2] of Obradović, for the choice of $\alpha = 0$ and 1 respectively.

In the preceding chapters, we have discussed various differential subordination theorems and derived several sufficient conditions for starlikeness and convexity. In the subsequent couple of chapters, we shall discuss the problems related to the coefficient estimates. In general this problem can be stated as "if a function $f \in \mathcal{A}$ satisfies certain geometric property, then how does this fact affects its initial coefficients?"

In **Chapter 5**, estimate on the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized analytic function $f \in \mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$ is obtained. As a special case an alternate and comparatively easy proof of result [**202**, Theorem 1] proved by Tuneski and Darus is also provided. Using our main results we have proved the following:

Let $f \in \mathcal{A}$ and satisfies

$$f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1} \prec \frac{1+Cz}{1+Dz},$$

then, for any complex number μ, we have

$$|a_3 - \mu a_2^2| \leq \frac{C-D}{2-\lambda} \max\left\{1; \left|D + \frac{(1+\lambda-2\mu)(\lambda-2)(C-D)}{(1-\lambda)^2}\right|\right\}.$$

For $C = 1-2a, 0 \leq a < 1$, $0 < \lambda < 1$ and $D = -1$, this result reduces to [**202**, Theorem 1] of Tuneski and Darus. Note that our proof is quite different from that one given by Tuneski and Darus [**202**]. For $a = 0$, the above result reduces to [**202**, Corollary 1] due to Tuneski and Darus. Setting $C = k$ $(0 < k \leq 1)$ and $D = 0$ in the above result, we obtain the result [**202**, Theorem 2] of Tuneski and Darus.

Similarly the Fekete-Szegö problems for two more classes $\mathcal{N}_{g,h}(\alpha, \varphi)$ and $\mathcal{S}_g^\alpha(\varphi)$ defined using Hadamard product and subordination are determined. Further, some special cases of the main results are also discussed. Our result generalize several results proved in [**83, 102, 112, 151, 188, 202**].

For example, let $\alpha \geq 0$ and $g(z) = z + \sum_{n=2}^\infty g_n z^n$ with g_2, g_3 non zero real numbers. If $f \in \mathcal{S}_g^\alpha((1+z)/(1-z))$, then, for any real number μ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(1+\alpha)^2|g_3|}\left(\frac{3+10\alpha-\alpha^2}{2\alpha+1} - \frac{4\mu g_3}{g_2^2}\right) & \text{if } \mu \leq \sigma_1; \\ \frac{1}{(2\alpha+1)|g_3|} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{1}{(1+\alpha)^2|g_3|}\left(\frac{\alpha^2-10\alpha-3}{2\alpha+1} + \frac{4\mu g_3}{g_2^2}\right) & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(1+4\alpha-\alpha^2)g_2^2}{2(2\alpha+1)g_3} \text{ and } \sigma_2 := \frac{(3\alpha+1)g_2^2}{(2\alpha+1)g_3}.$$

If we take $g(z) = z + \sum_{n=2}^\infty n^m z^n, m = 0, 1, 2, 3, \ldots$, then this result reduces to [**151**, Theorem 2] of Răducanu.

Chapter 6 deals with the estimates on the initial coefficients of bi-univalent functions belonging to certain classes defined by subordination. We obtained the estimate on initial coefficient a_2 of bi-univalent functions belonging to the class $\mathcal{R}_\sigma(\lambda, \varphi)$ as well as estimates on a_2 and a_3 for functions in the classes $\mathcal{S}_\sigma^*(\varphi)$, the class of bi-starlike functions and $K_\sigma(\varphi)$, the class of bi-convex functions. Some special cases of our main results are also provided, which in fact reveals that our estimates are better than the earlier existing estimates. For example, Brannan and Taha [**29**, Theorem 4.1] proved the estimates $|a_2| \leq \sqrt{1-\beta}$ $(0 \leq \beta < 1)$ and $|a_3| \leq 1-\beta$ for functions $f \in \mathcal{K}_\sigma(\beta)$. This result is improved in the following results:

a.) If $f \in \mathcal{K}_\sigma[\beta]$ $(0 \leq \beta < 1)$, then

$$|a_2| \leq 1-\beta \text{ and } |a_3| \leq \frac{(1-\beta)(3-2\beta)}{3}.$$

b.) If $f \in \mathcal{S}^*_\sigma[\beta]$ $(0 \leq \beta < 1)$, then

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)}, & 0 \leq \beta \leq 1/2; \\ \sqrt{(1-\beta)(3-2\beta)}, & 1/2 \leq \beta < 1. \end{cases} \tag{1.5.2}$$

For functions in the class $\mathcal{S}^*_\sigma(\beta)$ Brannan and Taha [**29**, Theorem 3.1] proved that $|a_2| \leq \sqrt{2(1-\beta)}$. If we compare the result [**29**, Theorem 3.1] with the result $|a_2| \leq 2(1-\beta)$ for function $f \in \mathcal{S}^*(\beta)$, given by Robertson [**152**], we see that Brannan and Taha's estimate is better over the Robertson's result only when $0 \leq \beta \leq 1/2$. Also it may be noted that our estimate for a_2 given in (1.5.2) improves the estimate given by Brannan and Taha [**29**, Theorem 3.1]. Further actuated by the work of Kedzierawski [**81**], we obtained the estimates on initial coefficients a_2 and a_3, when f and f^{-1} belong to different subclasses of univalent functions. For example, if function f is $\varphi-$starlike ($\varphi-$convex) and f^{-1} is $\varphi-$convex ($\varphi-$starlike), the estimates on a_2 and a_3 are obtained.

Chapter 7 handles the radius problems for analytic functions with fixed second coefficient. Motivated by the works in [**10, 59, 131, 172, 197, 199**], in this chapter, we consider the functions with fixed second coefficient in the following cases:

1. $\operatorname{Re}(f(z)/g(z)) > 0$, where $\operatorname{Re}(g(z)/z) > 0$.

2. $\operatorname{Re}(f(z)/g(z)) > 0$, where $\operatorname{Re}(g(z)/z) > 1/2$.

3. $|f(z)/g(z) - 1| < 1$, where $\operatorname{Re}(g(z)/z) > 0$ or g is convex.

4. $|f'(z)/g'(z) - 1| < 1$, where g is univalent or starlike or convex.

We generalize the results proved by Ali et al. [**10**]. We have obtained the sharp radii of starlikeness of order α, parabolic starlikeness and Sokół-Stankiewicz starlikeness for functions with fixed second coefficient.

Chapter 2

Sufficient Conditions for Starlike Functions Associated with the Lemniscate of Bernoulli

2.1 Introduction

Motivated by the fact that each convex function is starlike of order half, in 1969, Mocanu [**63**] introduced the concept of α-convex function, which provides a continuous passage from the starlike functions to convex functions. In 1973, Miller et al. [**106**] proved that α–convex functions are starlike for all real α and convex for $\alpha \geq 1$. Inspired by the paper [**106**], in 1999, Silverman [**170**] considered a class of normalized analytic functions involving an expression which is the quotient of the analytic representations of convex and starlike functions. For $0 < b \leq 1$, he introduced the class $\mathcal{G}_b$ defined by

$$\mathcal{G}_b := \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b \right\}$$

and proved that the functions in the class $\mathcal{G}_b$ are starlike of order $2/(1+\sqrt{1+8b})$, that is the inclusion $\mathcal{G}_b \subset \mathcal{S}^*(2/(1+\sqrt{1+8b})$ holds. Further, Obradović and Tuneski [**125**]

The contents of this chapter appeared in [**173**].

improved this result by proving $\mathcal{G}_b \subset \mathcal{S}^*[0,-b] \subset \mathcal{S}^*(2/(1+\sqrt{1+8b})$. Silverman proved that $\mathcal{G}_1 \not\subset \mathcal{K}$, $\mathcal{G}_1 \subset \mathcal{S}^*(1/2)\setminus\mathcal{K}$ and $\mathcal{G}_b \subset \mathcal{K}$ $(b \leq \sqrt{2}/2)$. Further he discussed the radius of convexity problem for the class $\mathcal{G}_b$. Tuneski [**201**] obtained conditions on A, B and b so that $\mathcal{G}_b \subset \mathcal{S}^*[A,B]$ $(-1 \leq B < A \leq 1)$ holds. Earlier, in 1999, inspired by the work of Silverman [**170**], Nunokawa et al. [**120**] obtained sufficient conditions for functions in the class $\mathcal{G}_b$ to be strongly starlike, strongly convex and starlike. We notice that by setting $p(z) = zf'(z)/f(z)$ $(f \in \mathcal{A})$, the inclusion $\mathcal{G}_b \subset \mathcal{S}^*[A,B]$ can be written as

$$1+\frac{zp'(z)}{p^2(z)} \prec 1+bz \quad \Rightarrow \quad p(z) \prec \frac{1+Az}{1+Bz}.$$

Indeed, the above implication is a special case of the following

$$1+\frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez} \quad \Rightarrow \quad p(z) \prec \frac{1+Az}{1+Bz}.$$

Consider another result proved by Frasin and Darus [**57**]

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{2-\alpha} \Rightarrow \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1-\alpha \quad (f \in \mathcal{A}, 0 \leq \alpha < 1).$$

By setting $p(z) = z^2 f'(z)/(f(z))^2$ $(f \in \mathcal{A})$, it can be easily seen that the above result is a particular case of the implication

$$1+\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \quad \Rightarrow \quad p(z) \prec \frac{1+Az}{1+Bz}.$$

The above results motivated Ali et al. [**17**] to formulate a general approach to discuss differential subordination results. For a function p analytic in $\mathbb{D}$ with $p(0)=1$, they obtained the conditions on $A, B, D, E \in [-1,1]$ and β so that

$$1+\beta\frac{zp'(z)}{p^n(z)} \prec \frac{1+Dz}{1+Ez} \quad \Rightarrow \quad p(z) \prec \frac{1+Az}{1+Bz} \quad (n=0,1).$$

In the similar direction, in 2012, Ali et al. [**9**] determined sufficient conditions for $p(z) \prec \sqrt{1+z}$, whenever either the following

$$1+\beta\frac{zp'(z)}{p^n(z)} \prec \sqrt{1+z} \quad (n=0,1,2) \ \text{ or } \ (1-\beta)p(z)+\beta p^2(z)+\beta zp'(z) \prec \sqrt{1+z}$$

holds.

Motivated by the works in [**7, 9, 11, 133, 184**], in Section 2.2, we have determined condition on β so that $p(z) \prec \sqrt{1+z}$, whenever either of the following subordinations

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec \frac{1+Az}{1+Bz} \quad (-1<k\leq 3) \text{ and } p(z)+\beta\frac{zp'(z)}{p^n(z)} \prec \sqrt{1+z} \quad (n=0,1,2)$$

hold. Similarly, condition on β is determined so that $p(z) \prec (1+Az)/(1+Bz)$, whenever $1+\beta zp'(z)/p^n(z) \prec \sqrt{1+z}$ $(n=0,1,2)$. At the end of Section 2.2 the implication $p(z)+\beta zp'(z)/p(z) \prec \sqrt{1+z}$ implies $p(z) \prec (1+Az)/(1+Bz)$ is also considered. In Section 2.3, we have given alternative proofs of the results [**17**, Lemma 2.1, 2.10]. Further, this section is concluded with some conditions on the parameters $A, B, D, E \in [-1,1]$ and β such that

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez} \quad \Rightarrow \quad p(z) \prec \frac{1+Az}{1+Bz}.$$

To prove our main results of this chapter, we require the following results in addition to Lemma 1.3.1.

Lemma 2.1.1. [**108**, Corollary 3.4h1, p. 135] *Let q be univalent in $\mathbb{D}$, and let ϕ be analytic in a domain D containing $q(\mathbb{D})$. Let $zq'(z)\phi(q(z))$ be starlike in $\mathbb{D}$. If p is analytic in $\mathbb{D}$, $p(0)=q(0)$ and satisfies $zp'(z)\phi(p(z)) \prec zq'(z)\phi(q(z))$, then $p \prec q$ and q is the best dominant.*

Lemma 2.1.2. [**108**, Corollary 3.4a, p. 120] *Let q be analytic in $\mathbb{D}$, and let ϕ be analytic in a domain D containing $q(\mathbb{D})$ and suppose $\operatorname{Re}\phi(q(z))>0$ and either q is convex, or $Q(z)=zq'(z)\phi(q(z))$ is starlike in $\mathbb{D}$. If p is analytic in $\mathbb{D}$, with $p(0)=q(0)$, $p(\mathbb{D})\subset D$ and $p(z)+zp'(z)\phi(p(z)) \prec q(z)$, then $p \prec q$.*

2.2 Conditions for Sokół-Stankiewicz Starlikeness

Throughout this chapter, we shall assume that β is a non-zero real number until further specified. In the first result, condition on β is obtained so that the subordination

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec \frac{1+Az}{1+Bz} \text{ implies } p(z) \prec \sqrt{1+z} \ (-1<B<A\leq 1).$$

Theorem 2.2.1. *Assume that* $-1 < B < A \leq 1, |\beta| \geq 2^{(k+3)/2}(A-B) + |B\beta|$ *and* $-1 < k \leq 3$. *If* p *is an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$, *and satisfying the subordination*

$$1 + \beta \frac{zp'(z)}{p^k(z)} \prec \frac{1+Az}{1+Bz}, \tag{2.2.1}$$

then $p(z) \prec \sqrt{1+z}$.

Proof. Let $q(z) = \sqrt{1+z}$, and consider the function $Q : \mathbb{D} \to \mathbb{C}$ defined by

$$Q(z) := \beta \frac{zq'(z)}{q^k(z)} = \frac{\beta z}{2(1+z)^{(k+1)/2}}.$$

From the definition of Q, we have

$$\frac{zQ'(z)}{Q(z)} := 1 - \frac{k+1}{2} \frac{z}{1+z}.$$

Since the function $z/(1+z)$ maps the unit disk onto the plane $\operatorname{Re} w < 1/2$, it follows that

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 1 - \frac{k+1}{4} \geq 0,$$

for $-1 < k \leq 3$, and hence Q is starlike in $\mathbb{D}$. Consider the subordination

$$1 + \beta \frac{zp'(z)}{p^k(z)} \prec 1 + \beta \frac{zq'(z)}{q^k(z)}.$$

The above subordination can be written as $1 + zp'(z)\phi(p(z)) \prec 1 + zq'(z)\phi(q(z))$ by defining $\phi(w) = \beta/w^k$. Thus all conditions of Lemma 2.1.1 are satisfied and hence $p(z) \prec q(z)$. In order to prove our result, we need to show the following:

$$\frac{1+Az}{1+Bz} \prec 1 + \frac{\beta z q'(z)}{q^k(z)} = 1 + \frac{\beta z}{2(1+z)^{(k+1)/2}} := h(z).$$

To prove this, let us consider the function

$$w = \Phi(z) = \frac{1+Az}{1+Bz}.$$

Then, we have

$$\Phi^{-1}(w) = \frac{w-1}{A-Bw}.$$

Since the subordination $\Phi(z) \prec h(z)$ is equivalent to $z \prec \Phi^{-1}(h(z))$, now it is enough to show that $|\Phi^{-1}(h(e^{it}))| \geq 1, -\pi \leq t \leq \pi$.

Note that

$$\begin{aligned}\Phi^{-1}(h(z)) &= \frac{h(z)-1}{A-Bh(z)}\\ &= \frac{\beta z}{2(A-B)(1+z)^{(k+1)/2}-\beta Bz}.\end{aligned}$$

For $z=e^{it}$, $-\pi \le t \le \pi$, we have

$$|\Phi^{-1}(h(e^{it}))| \ge \frac{|\beta|}{2(A-B)(2\cos(t/2))^{(k+1)/2}+|B\beta|} =: g(t).$$

Now $g'(t)=0$ implies $t=0$. Since

$$g''(0) = \frac{2^{\frac{k-5}{2}}(A-B)(1+k)|\beta|\left(2^{\frac{5+k}{2}}(A-B)+2|\beta B|\right)}{\left(2^{\frac{k+3}{2}}(A-B)+|\beta B|\right)^3} > 0,$$

it follows by second derivative test that $g(t)$ attains its minimum at $t=0$ and

$$g(0) = \frac{|\beta|}{2^{(k+3)/2}(A-B)+|B\beta|}.$$

Also $g(\pi)=g(-\pi)=1/|B|$. Now it is easy to see that

$$\min_{|t|\le\pi} g(t) = \{g(\pi), g(-\pi), g(0)\} = g(0),$$

and $g(0)\ge 1$ for $|\beta| \ge 2^{(k+3)/2}(A-B)+|B\beta|$. This completes the proof. □

Theorem 2.2.2. *Assume that* $(A-B)\beta \ge \sqrt{2}(1+|B|)^2+(1-B)^2$. *If* p *is an analytic function defined on* $\mathbb{D}$ *with* $p(0)=1$, *and satisfying the subordination*

$$1+\beta z p'(z) \prec \sqrt{1+z},$$

then

$$p(z) \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1).$$

Proof. Let $q:\mathbb{D}\to\mathbb{C}$ be defined by

$$q(z) = \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1),$$

and consider the function

$$Q(z) = \beta z q'(z) = \frac{\beta(A-B)z}{(1+Bz)^2}.$$

Then the function q is univalent in the unit disk $\mathbb{D}$ and from the definition of Q, we have

$$\frac{zQ'(z)}{Q(z)} = \frac{1-Bz}{1+Bz}.$$

Since

$$\begin{aligned}\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) &= \operatorname{Re}\left(\frac{(1-Bz)(1+B\bar{z})}{|1+Bz|^2}\right)\\ &= \frac{1-B^2|z|^2}{|1+Bz|^2} > 0 \quad (-1 \leq B < 1,\ z \in \mathbb{D}),\end{aligned}$$

it follows that the function Q is starlike in $\mathbb{D}$. Now it is easy to see that the subordination $1+\beta zp'(z) \prec 1+\beta zq'(z)$ can be written as $1+zp'(z)\phi(p(z)) \prec 1+zq'(z)\phi(q(z))$ by defining $\phi(w) = \beta$. Thus all conditions of Lemma 2.1.1 are fulfilled, and that the subordination

$$1+\beta zp'(z) \prec 1+\beta zq'(z)$$

implies $p(z) \prec q(z)$. Now in order to prove our theorem we need to show that the following subordination must holds:

$$\sqrt{1+z} \prec 1+\beta zq'(z) = 1+\beta\frac{(A-B)z}{(1+Bz)^2} =: h(z).$$

For this purpose, let $w = \Phi(z) = \sqrt{1+z}$. Then $\Phi^{-1}(w) = w^2-1$. Since the subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$, it follows that in order to prove the result, it is enough to show that the inequality $|\Phi^{-1}(h(e^{it}))| \geq 1, -\pi \leq t \leq \pi$ holds.

For $z = e^{it}$ $(-\pi \leq t \leq \pi)$, we have

$$\begin{aligned}|\Phi^{-1}(h(e^{it}))| &= \left|\left(1+\beta\frac{(A-B)e^{it}}{(1+Be^{it})^2}\right)^2 - 1\right|\\ &\geq \left|1+\beta\frac{(A-B)e^{it}}{(1+Be^{it})^2}\right|^2 - 1 \geq 1\end{aligned}$$

provided the following inequality holds:

$$\left|1+\beta\frac{(A-B)e^{it}}{(1+Be^{it})^2}\right| \geq \sqrt{2}. \tag{2.2.2}$$

For the above inequality to hold we should show that the minimum of the left side expression of the above inequality must be greater than or equal to $\sqrt{2}$, and for this we consider the expression

$$\begin{aligned}\left|1+\beta\frac{(A-B)e^{it}}{(1+Be^{it})^2}\right| =&\frac{|1+(2B+\beta(A-B))e^{it}+B^2e^{2it}|}{|1+2Be^{it}+B^2e^{2it}|}\\ \geq&\frac{\mathrm{Re}(2B+\beta(A-B)+B^2e^{it}+e^{-it})}{1+2|B|+B^2}\\ =&\frac{2B+\beta(A-B)+(1+B^2)x}{(1+|B|)^2}=g(x),\end{aligned}$$

where $x=\cos t, -1\leq x\leq 1$. It is easy to verify that $g(x)\geq g(-1)$ for $-1\leq x\leq 1$. Thus the inequality in (2.2.2) holds if $g(-1)\geq\sqrt{2}$, that is, if the following inequality holds:

$$\frac{2B+\beta(A-B)-(1+B^2)}{(1+|B|)^2}\geq\sqrt{2}$$

or equivalently, if the inequality $(A-B)\beta\geq\sqrt{2}(1+|B|)^2+(1-B)^2$ holds. This establishes the theorem. □

Theorem 2.2.3. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0)=1$, and satisfying the subordination*

$$1+\beta\frac{zp'(z)}{p(z)}\prec\sqrt{1+z}. \tag{2.2.3}$$

If the conditions $0\leq B<A<1$ and $\beta\leq(1-\sqrt{2})(1-A)(1-B)$ hold, then

$$p(z)\prec\frac{1+Az}{1+Bz}.$$

Proof. Define the function $q:\mathbb{D}\to\mathbb{C}$ by $q(z)=(1+Az)/(1+Bz)$, and consider the function

$$Q(z):=\frac{\beta zq'(z)}{q(z)}=\frac{\beta(A-B)z}{(1+Az)(1+Bz)}.$$

From the definition of Q, differentiating logarithmically, we have

$$\frac{zQ'(z)}{Q(z)}=\frac{1-ABz^2}{(1+Az)(1+Bz)}.$$

Now in order to prove that Q is starlike, we have to show that $\mathrm{Re}(zQ'(z)/Q(z))>0$ in the unit disk $\mathbb{D}$. For this, we consider

$$\mathrm{Re}\left(\frac{1-ABz^2}{(1+Az)(1+Bz)}\right)=\mathrm{Re}\left(\frac{(1-ABz^2)(1+A\bar{z})(1+B\bar{z})}{|(1+Az)(1+Bz)|^2}\right).$$

Since denominator of the expression in the right hand side is always positive, it remains only to show that $\operatorname{Re}((1-ABz^2)(1+A\bar{z})(1+B\bar{z}))>0$, and this is evident from

$$\begin{aligned}\operatorname{Re}(1-ABz^2)(1+A\bar{z})(1+B\bar{z}) &= 1-A^2B^2|z|^4+(A+B)(1-AB|z|^2)\operatorname{Re} z)\\ &= (1-AB|z|^2)(1+AB|z|^2+(A+B)\operatorname{Re} z)\\ &\geq (1-AB|z|^2)(1+AB|z|^2-|A+B||z|).\end{aligned}$$

Since $(1-AB|z|^2)(1+AB|z|^2-|A+B||z|)>0$ in all the cases whether $A+B$ is positive or negative or zero, it follows that Q is starlike in $\mathbb{D}$. Since the subordination

$$1+\beta\frac{zp'(z)}{p(z)}\prec 1+\beta\frac{zq'(z)}{q(z)}$$

can be written as $1+zp'(z)\phi(p(z))\prec 1+zq'(z)\phi(q(z))$ by defining $\phi(w)=\beta/w$, and $Q(z)=zq'(z)/q(z)$ is starlike in $\mathbb{D}$, it follows from Lemma 2.1.1 that $p(z)\prec q(z)$.

Now in order to prove our theorem we need to show the following subordination:

$$\sqrt{1+z}\prec 1+\beta\frac{zq'(z)}{q(z)}=1+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}=:h(z).$$

For this purpose, let $w=\Phi(z)=\sqrt{1+z}$. Then $\Phi^{-1}(w)=w^2-1$. Since the subordination $\Phi(z)\prec h(z)$ is equivalent to the subordination $z\prec\Phi^{-1}(h(z))$, it is enough to show $|\Phi^{-1}(h(e^{it}))|\geq 1,\ -\pi\leq t\leq\pi$.

For $z=e^{it}$, $-\pi\leq t\leq\pi$, we have

$$\left|\Phi^{-1}(h(e^{it}))\right|=\left|\left(1+\frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right)^2-1\right|\geq 1$$

provided that the following inequality holds:

$$\left|1+\frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right|\geq\sqrt{2}.$$

Now consider the expression in the left hand side of the above inequality

$$\begin{aligned}\left|1+\frac{\beta(A-B)e^{it}}{(1+Ae^{it})(1+Be^{it})}\right| \geq& 1+(A-B)\beta\operatorname{Re}\left(\frac{e^{it}}{(1+Ae^{it})(1+Be^{it})}\right)\\ =& 1+(A-B)\beta\frac{A+B+(1+AB)x}{(1+A^2+2Ax)(1+B^2+2Bx)}=g(x),\end{aligned}$$

where $x = \cos t$. Since $-1 \leq x \leq 1$, it follows that $g(x) \geq g(-1)$ for $0 \leq B < A < 1$. Now by a simple computation, we obtain $g(-1) \geq \sqrt{2}$ for $\beta \leq (1-\sqrt{2})(1-A)(1-B)$. Thus $\Phi(z) \prec h(z)$ and this completes the proof. □

Theorem 2.2.4. *Assume that* $-1 \leq B < A \leq 1$ *and*

$$(A-B)\beta \geq (\sqrt{2}-1)(1+|A|)^2 + (1-A)^2.$$

If p *is an analytic function defined on* $\mathbb{D}$ *with* $p(0) = 1$, *and satisfying the subordination*

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec \sqrt{1+z},$$

then

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1),$$

and consider the function

$$Q(z) = \frac{\beta z q'(z)}{q^2(z)} = \frac{\beta(A-B)z}{(1+Az)^2}.$$

By a logarithmic differentiation of Q, we have

$$\frac{zQ'(z)}{Q(z)} = \frac{1-Az}{1+Az}.$$

Since

$$\operatorname{Re}\left(\frac{1-Az}{1+Az}\right) = \frac{1-A^2|z|^2}{|1+Az|^2} > 0 \quad (-1 < A \leq 1 \text{ and } z \in \mathbb{D}),$$

it follows that $\operatorname{Re}(zQ'(z))/Q(z) > 0$ in $\mathbb{D}$ and hence Q is starlike therein. Now it is easy to see that the subordination

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{zq'(z)}{q^2(z)}$$

can be written as $1 + zp'(z)\phi(p(z)) \prec 1 + zq'(z)\phi(q(z))$ by defining $\phi(w) = \beta/w^2$. Thus all the conditions of Lemma 2.1.1 are satisfied and an application of the same leads to

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. Now in order to prove our theorem we need to show the following:

$$\sqrt{1+z} \prec 1+\beta\frac{zq'(z)}{q^2(z)} = 1+\beta\frac{(A-B)z}{(1+Az)^2} =: h(z).$$

For this, let $w = \Phi(z) = \sqrt{1+z}$, and so $\Phi^{-1}(w) = w^2-1$. Since the subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$, it is enough to show $|\Phi^{-1}(h(e^{it}))| \geq 1,\ -\pi \leq t \leq \pi$. Now, we have

$$|\Phi^{-1}(h(e^{it}))| = \left|\left(1+\beta\frac{(A-B)e^{it}}{(1+Ae^{it})^2}\right)^2 - 1\right| \geq 1$$

provided the following inequality holds:

$$\left|1+\beta\frac{(A-B)e^{it}}{(1+Ae^{it})^2}\right| \geq \sqrt{2}. \tag{2.2.4}$$

Consider the left hand side of the above inequality

$$\begin{aligned}\left|1+\beta\frac{(A-B)e^{it}}{(1+Ae^{it})^2}\right| &= \frac{|1+(2A+\beta(A-B))e^{it}+A^2e^{2it}|}{|1+2Ae^{it}+A^2e^{2it}|} \\ &\geq \frac{\mathrm{Re}(2A+\beta(A-B)+A^2e^{it}+e^{-it})}{1+2|A|+A^2} \\ &= \frac{2A+\beta(A-B)+(1+A^2)x}{(1+|A|)^2} = g(x),\end{aligned}$$

where $x = \cos t, -1 \leq x \leq 1$, and of course $g(x) \geq g(-1)$. The inequality in (2.2.4) holds if $g(-1) \geq \sqrt{2}$, that is if

$$\frac{2B+\beta(A-B)-(1+A^2)}{(1+|A|)^2} \geq \sqrt{2}$$

or equivalently if $(A-B)\beta \geq (\sqrt{2}-1)(1+|A|)^2+(1-A)^2$ holds. Thus $\Phi(z) \prec h(z)$, and now the theorem follows at once. □

Theorem 2.2.5. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$, and satisfying the subordination $p(z)+\beta zp'(z) \prec \sqrt{1+z}$ $(\beta > 0)$. Then $p(z) \prec \sqrt{1+z}$.*

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$ with $q(0) = 1$. Since $q(\mathbb{D}) = \{w : |w^2-1| < 1\}$ is the right-half of the lemniscate of Bernoulli, $q(\mathbb{D})$ is a convex domain, and hence q is a convex function. Consider the function ϕ defined by $\phi(w) = \beta$. Since by assumption $\beta > 0$, it follows that

$$\mathrm{Re}\,\phi(q(z)) = \mathrm{Re}\,\phi(\sqrt{1+z}) = \beta > 0.$$

Consider the function Q defined by

$$Q(z) := zq'(z)\phi(q(z)) = \beta\frac{z}{2\sqrt{1+z}}.$$

From the definition of Q, we have

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = 1 - \operatorname{Re}\left(\frac{z}{2(1+z)}\right).$$

Since the function $z/(1+z)$ maps the unit disk $\mathbb{D}$ on to the region $\operatorname{Re} w < 1/2$, it follows that

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) \geq \frac{3}{4} > 0$$

and hence the function Q is starlike. Thus all the conditions of Lemma 2.1.2 are fulfilled and hence the result follows at once. □

Theorem 2.2.6. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$, and satisfying*

$$p(z) + \beta\frac{zp'(z)}{p(z)} \prec \sqrt{1+z},\ \beta > 0.$$

Then $p(z) \prec \sqrt{1+z}$.

Proof. Let q be given by $q(z) = \sqrt{1+z}$. Then as before in the proof of Theorem 2.2.5, the function is convex in the unit disk $\mathbb{D}$. Define the function $\phi(w) = \beta/w$. Since q maps the unit disk $\mathbb{D}$ onto $q(\mathbb{D}) = \{w : |w^2 - 1| < 1\}$, the right half of the lemniscate of Bernoulli and by assumption $\beta > 0$, it follows that

$$\operatorname{Re}\phi(q(z)) = \frac{\beta}{|\sqrt{1+z}|^2}\operatorname{Re}\left(\sqrt{1+z}\right) > 0.$$

Consider the function Q defined by

$$Q(z) := \beta\frac{zq'(z)}{q(z)} = \frac{\beta z}{2(1+z)}.$$

From the definition of Q, we have

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = 1 - \operatorname{Re}\left(\frac{z}{1+z}\right).$$

Since the function $z/(1+z)$ maps the unit disk $\mathbb{D}$ onto the region $\operatorname{Re} w < 1/2$, it follows that

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) \geq \frac{1}{2} > 0$$

and hence the function Q is starlike in the unit disk $\mathbb{D}$, and hence the result follows from Lemma 2.1.2. $\square$

Theorem 2.2.7. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$p(z) + \beta \frac{zp'(z)}{p^2(z)} \prec \sqrt{1+z}, \quad \beta > 0.$$

Then $p(z) \prec \sqrt{1+z}$.

Proof. Let q be given by $q(z) = \sqrt{1+z}$. Then, as before, q is a convex function in the unit disk $\mathbb{D}$. Let us define $\phi(w) = \beta/w^2$ and therefore

$$\operatorname{Re} \phi(q(z)) = \operatorname{Re} \left(\frac{\beta}{1+z} \right).$$

Since the function $1/(1+z)$ maps the unit disk $\mathbb{D}$ onto the region $\operatorname{Re} w > 1/2$, and by assumption $\beta > 0$, it follows that $\operatorname{Re} \phi(q(z)) > \beta/2 > 0$. Consider the function Q defined by

$$Q(z) = \beta \frac{zq'(z)}{q^2(z)} = \beta \frac{z}{2(1+z)^{\frac{3}{2}}}.$$

From the definition of Q, by logarithmic differentiation, we have

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) = 1 - \frac{3}{2} \operatorname{Re} \left(\frac{z}{1+z} \right).$$

Since the function $w = z/(1+z)$ maps the unit disk $\mathbb{D}$ onto the region $\operatorname{Re} w < 1/2$, it follows that

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) > \frac{1}{4} > 0$$

and hence the function Q is starlike and the result now follows by an application of Lemma 2.1.2. $\square$

Theorem 2.2.8. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$, and satisfying the subordination*

$$p(z) + \beta \frac{zp'(z)}{p(z)} \prec \sqrt{1+z}.$$

If $-1 < B < A < 1$ and the inequalities

$$(1 + |B|)(A - B)\beta \leq (1 - |A|)(1 - |B|)(1 + |A| - \sqrt{2}(1 + |B|))$$

and

$$\beta > \max\left\{0; \frac{(1-|A|-|B|)(1+|A|)(1-|B|)}{(2|A|+B^2+1)}\right\} \tag{2.2.5}$$

hold, then

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (1+Az)/(1+Bz)$ and consider the functions Q and h given as follows:

$$Q(z) := zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = q(z) + \beta\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{q(z)}{\beta} + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}.$$

From the definition of Q and q, we have

$$\frac{zQ'(z)}{Q(z)} = \frac{1-ABz^2}{(1+Az)(1+Bz)}.$$

Now in order to prove that Q is starlike we have to show that $\operatorname{Re}(zQ'(z)/Q(z)) > 0$ in the unit disk $\mathbb{D}$. For this, we consider

$$\operatorname{Re}\left(\frac{1-ABz^2}{(1+Az)(1+Bz)}\right) = \operatorname{Re}\left(\frac{(1-ABz^2)(1+A\bar{z})(1+B\bar{z})}{|(1+Az)(1+Bz)|^2}\right).$$

Since denominator of the expression in the right hand side is always positive, it remains only to show that $\operatorname{Re}((1-ABz^2)(1+A\bar{z})(1+B\bar{z})) > 0$, and this is evident from the following

$$\begin{aligned}\operatorname{Re}(1-ABz^2)(1+A\bar{z})(1+B\bar{z}) &= 1-A^2B^2|z|^4 + (A+B)(1-AB|z|^2)\operatorname{Re} z \\ &= (1-AB|z|^2)(1+AB|z|^2+(A+B)\operatorname{Re} z) \\ &\geq (1-AB|z|^2)(1+AB|z|^2-|A+B||z|).\end{aligned}$$

Since $(1-AB|z|^2)(1+AB|z|^2-|A+B||z|) > 0$, for both the cases whether $A+B$ is positive or negative or zero, it follows that Q is starlike in $\mathbb{D}$.

Now we shall show that $\operatorname{Re}(zh'(z)/Q(z)) > 0$ $(z \in \mathbb{D})$. For this purpose, let us consider

$$\begin{aligned}\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) &= \frac{1}{\beta}\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right)+\operatorname{Re}\left(\frac{1-Bz}{1+Bz}-\frac{(A-B)z}{(1+Az)(1+Bz)}\right)\\ &= \frac{1}{\beta}\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right)+\operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right)+\operatorname{Re}\left(\frac{1}{1+Az}\right)-\operatorname{Re}\left(\frac{1}{1+Bz}\right).\end{aligned}$$

In view of the inequalities:

$$\frac{1-|B|}{1+|B|}\leq \operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right)\leq\frac{1+|B|}{1-|B|},\ \frac{1}{1+|A|}\leq\operatorname{Re}\left(\frac{1}{1+Az}\right)\leq\frac{1}{1-|A|},$$

we have the following inequality

$$\begin{aligned}&\frac{1}{\beta}\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right)+\operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right)+\operatorname{Re}\left(\frac{1}{1+Az}\right)-\operatorname{Re}\left(\frac{1}{1+Bz}\right)\\ &\qquad\geq\frac{1-|A|}{(1+|B|)\beta}+\frac{1-|B|}{1+|B|}+\frac{|A|+|B|}{(1+|A|)(1-|B|)}\\ &\qquad\qquad=\frac{1+\beta-(|A|+|B|)}{(1+|B|)\beta}+\frac{|A|+|B|}{(1+|A|)(1-|B|)}.\end{aligned}$$

Thus in view of the assumption (2.2.5) of theorem, we have $\operatorname{Re}(zh'(z)/Q(z)) > 0$.

The subordination

$$p(z)+\beta\frac{zp'(z)}{p(z)}\prec q(z)+\beta\frac{zq'(z)}{q(z)}$$

can be written as (1.3.3) by defining $\theta(w) := w$ and $\phi(w) := \beta/w$ $(\beta \neq 0)$. Clearly the functions θ and ϕ are analytic in $\mathbb{C}$ and $\phi(w) \neq 0$. Thus all the conditions of Lemma 1.3.1 are fulfilled and hence it follows that $p(z) \prec q(z)$.

In order to prove our result, we need to show

$$\Phi(z):=\sqrt{1+z}\prec q(z)+\beta\frac{zq'(z)}{q(z)}=\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}:=h(z).$$

Since the subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$ it is sufficient to show $|\Phi^{-1}(h(z))| \geq 1$ on $|z| = 1$. Now the inequality

$$\left|\Phi^{-1}(h(z))\right|=\left|\left(\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}\right)^2-1\right|\geq 1$$

holds provided

$$\left|\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}\right|\geq\sqrt{2}. \tag{2.2.6}$$

Further when $|z|=1$, we have

$$\begin{aligned}\left|\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}\right| &\geq \operatorname{Re}\left(\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}\right)\\ &\geq \operatorname{Re}\left(\frac{1+Az}{1+Bz}\right)-\left|\frac{\beta(A-B)z}{(1+Az)(1+Bz)}\right|\\ &\geq \frac{1-|A|}{1+|B|}-\frac{(A-B)\beta}{(1-|A|)(1-|B|)}.\end{aligned}$$

Thus the inequality in (2.2.6) holds if the quantity in the right side of the above inequality is greater than or equal to $\sqrt{2}$, that is when

$$(1+|B|)(A-B)\beta \leq (1-|A|)(1-|B|)(1+|A|-\sqrt{2}(1+|B|))$$

holds. This completes the proof. □

2.3 Conditions for Janowski Starlikeness

We shall begin with Theorems 2.3.1, 2.3.2 in which we have given alternate proofs of the results [**17**, Lemma 2.1, 2.10] due to Ali et al.

Theorem 2.3.1. *Assume that $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ and let*

$$|\beta|(A-B) \geq (D-E)(1+B^2)+|2B(D-E)-E\beta(A-B)|.$$

Let p be an analytic function defined on $\mathbb{D}$ with $p(0)=1$, and satisfying

$$1+\beta zp'(z) \prec \frac{1+Dz}{1+Ez}.$$

Then

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Define the function $q:\mathbb{D}\to\mathbb{C}$ by

$$q(z)=\frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1).$$

Then q is convex in $\mathbb{D}$ with $q(0)=1$, and let us consider the function Q given by

$$Q(z)=\beta zq'(z)=\frac{\beta(A-B)z}{(1+Bz)^2}$$

and, as shown in the proof of Theorem 2.2.2, it follows that Q is starlike in the unit disk $\mathbb{D}$. Now it is easy to see that the subordination $1+\beta zp'(z) \prec 1+\beta zq'(z)$ can be written as $1+zp'(z)\phi(p(z)) \prec 1+zq'(z)\phi(q(z))$ by defining $\phi(w)=\beta$. Thus all conditions of Lemma 2.1.1 are fulfilled, and that the subordination

$$1+\beta zp'(z) \prec 1+\beta zq'(z) \text{ implies } p(z) \prec q(z).$$

In view of the above implication, we need to show

$$\frac{1+Dz}{1+Ez} \prec 1+\beta zq'(z) = 1+\beta\frac{(A-B)z}{(1+Bz)^2} = h(z).$$

For this purpose let us define $w=\Phi(z)=(1+Dz)/(1+Ez)$, and therefore its inverse function is given by $\Phi^{-1}(w)=(w-1)/(D-Ew)$. Since the subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$, it follows that to prove the result it is sufficient to show $|\Phi^{-1}(h(z))| \geq 1$ on $|z|=1$. Since the inequality

$$\begin{aligned} |\Phi^{-1}(h(z))| &= \left|\frac{\beta(A-B)z}{(D-E)(1+B^2z^2)+(2B(D-E)-\beta E(A-B))z}\right| \\ &\geq \frac{(A-B)|\beta||z|}{((D-E)(1+B^2|z|^2)+|(2B(D-E)-\beta E(A-B))|)|z|} \\ &\geq \frac{(A-B)|\beta|}{((D-E)(1+B^2)+|(2B(D-E)-\beta E(A-B))|)} \geq 1 \end{aligned}$$

holds on $|z|=1$, for $|\beta|(A-B) \geq (D-E)(1+B^2)+|(2B(D-E)-E\beta(A-B))|$, it follows that $q(\mathbb{D}) \subset h(\mathbb{D})$, that is, $q(z) \prec h(z)$, and this completes the proof. □

It should be noted that Ali et al. [**17**] made the assumption $AB>0$ in order to prove their result [**17**, Lemma 2.10], whereas in our proof (of the same result) this assumption is not required.

Theorem 2.3.2. *Assume that $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ and let*

$$|\beta|(A-B) \geq (D-E)(1+|AB|)+|(A+B)(D-E)-E\beta(A-B)|.$$

Let p be an analytic function defined on $\mathbb{D}$ with $p(0)=1$, and satisfying

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez}.$$

Then

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1).$$

Consider the function Q given by

$$Q(z) = \frac{\beta z q'(z)}{q(z)} = \frac{\beta(A-B)z}{(1+Az)(1+Bz)}$$

and, as shown in the proof of Theorem 2.2.3, it follows that Q is starlike in the unit disk $\mathbb{D}$. The subordination $1 + \beta z p'(z)/q(z) \prec 1 + \beta z q'(z)/q(z)$ can be written as $1 + zp'(z)\phi(p(z)) \prec 1 + zq'(z)\phi(q(z))$ by defining $\phi(w) = \beta/w$. Thus all conditions of Lemma 2.1.1 are satisfied, and hence the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)} \text{ implies } p(z) \prec q(z).$$

It follows from Lemma 2.1.1 that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)}$$

implies $p(z) \prec q(z)$. In view of the above implication, to prove the result, we need to show

$$\frac{1+Dz}{1+Ez} \prec 1 + \beta\frac{zq'(z)}{q(z)} = 1 + \beta\frac{(A-B)z}{(1+Bz)^2} = h(z).$$

For this purpose let us define $w = \Phi(z) = (1+Dz)/(1+Ez)$, and therefore its inverse function is given by $\Phi^{-1}(w) = (w-1)/(D-Ew)$. Since the subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$, it is sufficient to show $|\Phi^{-1}(h(z))| \ge 1$ on $|z| = 1$. Now the function $\Phi^{-1}(h(z))$ is given by

$$\begin{aligned}\Phi^{-1}(h(z)) &= \frac{\beta(A-B)z}{(D-E)(1+Az)(1+Bz) - \beta E(A-B)z} \\ &= \frac{\beta(A-B)z}{(D-E)(1+ABz^2) + ((A+B)(D-E) - \beta E(A-B))z}.\end{aligned}$$

For $|z| = 1$, we have

$$|\Phi^{-1}(h(z))| \ge \frac{|\beta|(A-B)}{(D-E)(1+|AB|) + |(A+B)(D-E) - \beta E(A-B)|} \ge 1,$$

whenever the inequality $|\beta|(A-B) \ge (D-E)(1+|AB|) + |(A+B)(D-E) - E\beta(A-B)|$ holds, and hence $q(\mathbb{D}) \subset h(\mathbb{D})$ or equivalently $q \prec h$, this completes the proof. □

Theorem 2.3.3. *Assume that* $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ *and let*

$$|\beta|(A-B) \geq (D-E)(1+A^2) + |2A(D-E) - E\beta(A-B)|.$$

Let p *be an analytic function defined on* $\mathbb{D}$ *with* $p(0)=1$, *and satisfying the subordination*

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez}.$$

Then

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (1+Az)/(1+Bz)$ $(-1 \leq B < A \leq 1)$, and consider the function Q defined by

$$Q(z) = \frac{\beta z q'(z)}{q^2(z)} = \frac{\beta(A-B)z}{(1+Az)^2}$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{1-Az}{1+Az}.$$

As before, a computation shows Q is starlike in $\mathbb{D}$. It follows from Lemma 2.1.1, that the subordination

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\beta\frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. To prove the result, it is enough to show that

$$\frac{1+Dz}{1+Ez} \prec 1+\beta\frac{zq'(z)}{q^2(z)} = 1+\beta\frac{(A-B)z}{(1+Az)^2} = h(z).$$

For this purpose let us define $w = \Phi(z) = (1+Dz)/(1+Ez)$ its inverse function is given by $\Phi^{-1}(w) = (w-1)/(D-Ew)$. Since the subordination $\Phi(z) \prec h(z)$ is equivalent to the subordination $z \prec \Phi^{-1}(h(z))$ in order to prove the result, it is sufficient to show $|\Phi^{-1}(h(z))| \geq 1$ on $|z|=1$. Since the inequality

$$\begin{aligned}
|\Phi^{-1}(h(z))| &= \left|\frac{\beta(A-B)z}{(D-E)(1+A^2z^2) + (2A(D-E) - \beta E(A-B))z}\right| \\
&\geq \frac{(A-B)|\beta||z|}{(D-E)(1+A^2|z|^2) + |2A(D-E) - \beta E(A-B)||z|} \\
&\geq \frac{(A-B)|\beta|}{(D-E)(1+A^2) + |2A(D-E) - \beta E(A-B)|} \\
&\geq 1,
\end{aligned}$$

holds on $|z|=1$, for $|\beta|(A-B) \geq (D-E)(1+A^2)+|2A(D-E)-E\beta(A-B)|$, it follows that $q(\mathbb{D}) \subset h(\mathbb{D})$ or $q \prec h$, and this completes the proof. □

For $\beta = 1$, Theorem 2.3.3 reduces to the result [**17**, Lemma 2.6] of Ali et al.

Theorem 2.3.4. [**17**, Lemma 2.6] *Assume that* $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ *and* $(A-B) \geq (D-E)(1+A^2)+|2A(D-E)-E(A-B)|$. *Let* p *be an analytic function defined on* $\mathbb{D}$ *with* $p(0)=1$ *satisfying the subordination*

$$1+\frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez}.$$

Then

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

2.4 Applications

In this section, we shall discuss some applications of the results obtained in previous sections. These results provide sufficient conditions for functions to be Sokół-Stankiewicz and Janowski starlike.

Corollary 2.4.1. *Let* $f \in \mathcal{A}$ *satisfies the subordination*

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz} \quad (-1<B<A\leq 1).$$

Further if the inequality $(1-|B|)|\beta| \geq 2\sqrt{2}(A-B)$ *holds, then* $f \in \mathcal{S}^*_L$.

Proof. For a function $f \in \mathcal{A}$, define a function $p : \mathbb{D} \to \mathbb{C}$ by $p(z) = zf'(z)/f(z)$. Then p is analytic on $\mathbb{D}$ and $p(0)=1$. From the definition of p, we have

$$1+\beta zp'(z) = 1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right).$$

The result follows from Theorem 2.2.1. □

Taking $A=1, B=0$ and $\beta = 2\sqrt{2}$ in Corollary 2.4.1, we have the following result which provides a sufficient condition for Sokół-Stankiewicz starlikeness:

Corollary 2.4.2. *Let $f \in \mathcal{A}$ and satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \frac{1}{2\sqrt{2}}.$$

Then $f \in \mathcal{S}_L^*$.

The following corollary provides a sufficient condition for Janowski starlike function:

Corollary 2.4.3. *Assume that $-1 \leq B < A \leq 1$ and*

$$(A-B)\beta \geq \sqrt{2}(1+|B|)^2 + (1-B)^2.$$

If $f \in \mathcal{A}$ and satisfies

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z},$$

then $f \in \mathcal{S}^*[A,B]$.

Proof. Let $p(z) = zf'(z)/f(z), f \in \mathcal{A}$. Then p is analytic in $\mathbb{D}$ and $p(0) = 1$. Using the definition of function p, we have

$$zp'(z) = \frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right).$$

Now the result follows from Theorem 2.2.2. □

Now taking $A = 1$ and $B = -1$ in Corollary 2.4.3, we have the following sufficient condition for starlike functions.

Corollary 2.4.4. *Let $\beta \geq 2(\sqrt{2}+1)$. If $f \in \mathcal{A}$ and satisfies*

$$\left|1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \sqrt{2},$$

then $f \in \mathcal{S}^*$.

Corollary 2.4.5. *Let $0 \leq B < A < 1$ and $0 \neq \beta \leq (1-\sqrt{2})(1-A)(1-B)$. If $f \in \mathcal{A}$ and satisfies the following subordination*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z},$$

then $f \in \mathcal{S}^*[A,B]$.

Proof. Let $p(z) = zf'(z)/f(z)$. Then p is analytic in $\mathbb{D}$ and $p(0) = 1$. Further a computation gives

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}.$$

In view of the subordination (2.2.3) of Theorem 2.2.3, we get the required result. □

Corollary 2.4.6. *Let* $-1 \leq B < A \leq 1$ *and* $(A-B)\beta \geq (\sqrt{2}-1)(1+|A|)^2+(1-A)^2$. *If* $f \in \mathcal{A}$ *and satisfies*

$$1 + \beta \frac{f(z)}{zf'(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z},$$

then $f \in \mathcal{S}^*[A, B]$.

Proof. Let $p(z) = zf'(z)/f(z)$. Then p is analytic in $\mathbb{D}$ and $p(0) = 1$. Further computation gives

$$1 + \beta \frac{zp'(z)}{p^2(z)} = 1 + \beta \frac{f(z)}{zf'(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Now from Theorem 2.2.4, the required result. □

Corollary 2.4.7. *If* $f \in \mathcal{A}$ *and satisfies*

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \sqrt{1+z}, \quad \beta > 0,$$

then $f \in \mathcal{S}^*_L$.

Proof. Let $p(z) = zf'(z)/f(z)$. Then we have

$$\begin{aligned} p(z) + \beta \frac{zp'(z)}{p(z)} &= \frac{zf'(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \\ &= (1-\beta)\frac{zf'(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)}\right). \end{aligned}$$

Now from Theorem 2.2.6, we obtain the desired result. □

Corollary 2.4.8. *If* $f \in \mathcal{A}$ *and satisfies the subordination*

$$\frac{zf'(z)}{f(z)} + \beta \frac{f(z)}{zf'(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} \quad (\beta > 0),$$

then $f \in \mathcal{S}^*_L$.

Proof. Taking $p(z) = zf'(z)/f(z)$ in Theorem 2.2.7, we have

$$p(z) + \beta\frac{zp'(z)}{p^2(z)} = \frac{zf'(z)}{f(z)} + \beta\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Thus in view of Theorem 2.2.7, the result follows immediately. □

Corollary 2.4.9. *Let* $|\beta|(A - B) \geq 2(1-\alpha)(1+B^2) + |4B(1-\alpha) + \beta(A-B)|$. *If* $f \in \mathcal{A}$ *and satisfies the subordination*

$$1 + \beta\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\alpha)z}{1-z} \quad (0 \leq \alpha < 1),$$

then $f \in \mathcal{S}^*[A, B]$.

Proof. Taking $p(z) = zf'(z)/f(z)$, we have

$$1 + \beta zp'(z) = 1 + \beta\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Now setting $E = -1$ and $D = 1 - 2\alpha$ in Theorem 2.3.1, we have

$$1 + \beta\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\alpha)z}{1-z}.$$

Thus the result follows at once from Theorem 2.3.1. □

By taking $A = 1, B = -1$ and $\alpha = 0$ the condition

$$|\beta|(A - B) \geq 2(1-\alpha)(1+B^2) + |4B(1-\alpha) + \beta(A-B)|$$

reduces to $|\beta| \geq 2 + |\beta - 2|$. It is easy to verify that this condition is true for $\beta \geq 2$. Thus we have the following sufficient condition for starlikeness from Corollary 2.4.9.

Corollary 2.4.10. *If* $f \in \mathcal{A}$ *and satisfies*

$$1 + \beta\,\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right) > 0 \quad (\beta \geq 2),$$

then $f \in \mathcal{S}^*$.

Chapter 3

Sandwich Theorems for Multivalent Functions Involving a Unified Linear Operator

3.1 Introduction

In the literature, the properties of functions defined by special cases of convolution operators have widely been studied using a recurrence relation satisfied by them, see [**32,42,45,51,74,79,87,94,97,100,130,159,167,204**]. The Dziok-Srivastava operator (see page 19) is one among the special cases of the convolution operator. Several interesting properties such as inclusion relationship, subordination properties, differential sandwich results etc. of the classes of functions defined by Dziok-Srivastava operator or its special cases including the Hohlov operator [**74**], the Carlson-Shaffer operator [**42, 99**], the Ruscheweyh derivatives [**160**], the generalized Bernardi-Libera-Livingston integral operator [**32,94,100**] and the Srivastava-Owa fractional derivative operators [**126, 130**], rests on the following identity:

$$z(H_p^{l,m}[\alpha_1]f(z))' = \alpha_1 H_p^{l,m}[\alpha_1+1]f(z) - (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z). \tag{3.1.1}$$

Most of the results of this chapter appeared in [**174**].

The multiplier transform $I_p(r,\lambda)$ on $\mathcal{A}_p$, which is also a special case of convolution operator, introduced by Kumar et al. [**180**] and investigated in [**1, 23, 181**] satisfies the following identity:

$$z(I_p(r,\lambda)f(z))' = (p+\lambda)I_p(r+1,\lambda)f(z) - \lambda I_p(r,\lambda)f(z). \tag{3.1.2}$$

The operator $I_p(r,\lambda)$ is closely related to the Sălăgean operator [**162**]. The operator $I_\lambda^r := I_1(r,\lambda)$ was studied by Cho and Srivastava [**46**] and Cho and Kim [**45**]. For many special cases of this operator we refer to Section 1.4. There are enough literature in which the properties of functions/class of functions defined by linear operators were investigated using a recurrence relation satisfied by operators under consideration. The recent work (among others) of Ravichandran et al. [**146**], Kumar et al. [**180,181**], Al-Kharsani and Al-Areefi [**23**], Ali et al. [**15**], Cho and Kim [**45**] and Kwon and Cho [**89**] and the references given therein may also be cited in this connection.

Recently Kumar et al. [**180, 181**] derived differential sandwich results for Dziok-Srivastava operator and multiplier transform. They also discussed some inclusion relations using the theory of differential subordination. Inspired by these, in 2008, Al-Kharsani and Al-Areefi [**23**] also established differential sandwich results for Dziok-Srivastava operator and multiplier transform. These results were established using a recurrence relation satisfied by Dziok-Srivastava operator and multiplier transform. In 2011, Ali et al. [**15**] defined a class of operators which satisfy a common recurrence relation and they derived several differential subordination results. In the following definition, all the operators which satisfy a common recurrence relation are unified.

Definition 3.1.1. [**15**] Let $\mathcal{O}_p$ be the class of all linear operators L_p^a defined on $\mathcal{A}_p$ satisfying

$$z[L_p^a f(z)]' = \alpha_a L_p^{a+1} f(z) - (\alpha_a - p)L_p^a f(z). \tag{3.1.3}$$

One may also consider the operators satisfying the following recurrence relation

$$z[L_p^b f(z)]' = \alpha_b L_p^{b-1} f(z) - (\alpha_b - p)L_p^b f(z)$$

but their properties are very similar to the operators satisfying the recurrence relation in (3.1.3). In fact our results are motivated by Ali [**15, 23, 180, 181**] and the recent results of Miller and Mocanu [**105**] on differential superordination. The results of Miller and Mocanu were later used extensively by Bulboacă [**36, 37**] to investigate superordination-preserving integral operators as well as by several others [**1, 23, 51, 136, 150, 180, 181, 186**].

In the following sections, several subordination and superordination theorems and their corresponding sandwich theorems are obtained. Further several sufficient conditions for normalized analytic functions to be in the classes $\mathcal{S}$, $\mathcal{S}^*(\alpha)$, $\mathcal{SS}^*(\eta)$ and $\mathcal{S}_L^*$ are established. In this chapter many existing results are generalized. For example, the results of Kumar et al. [**181**] are special cases of our results for the choice of $\mu = 1 = \nu$ and L is the Dziok-Srivastava operator and the multiplier transform. Similarly when $\mu = 1, \nu = 0$ and L is the Dziok-Srivastava operator and the multiplier transform our results reduce to the results proved by Al-Kharsani and Al-Areefi [**23**]. Some results proved by Obradović [**121**], Chichra [**43**], Owa and Obradović [**127**] are also shown to the special cases of our results.

We shall use the Definition 1.3.1 and Lemmas 1.3.1 and 1.3.2 to establish our main results.

3.2 Sandwich Theorems

In this section, we shall discuss some differential subordination, superordination and corresponding sandwich results. For brevity, we shall use the following notations:

$$\Omega^a_{L,\mu,\nu}(f(z)) := \left(\frac{L_p^{a+1}f(z)}{z^p}\right)^{\mu}\left(\frac{z^p}{L_p^a f(z)}\right)^{\nu} \text{ and } \Omega^a_{L,\mu,\nu}(f(z),F(z)) := \frac{\Omega^a_{L,\mu,\nu}(f(z))}{\Omega^a_{L,\mu,\nu}(F(z))},$$

where $f, F \in \mathcal{A}_p$ and the powers are principal one, μ and ν are chosen to be real numbers such that they do not assume the value zero simultaneously.

Theorem 3.2.1. *Let q be convex univalent in $\mathbb{D}$ with $q(0) = 1$ and $f \in \mathcal{A}_p$. Let $\alpha_{a+1} \neq 0$, $\operatorname{Re}(\alpha_{a+1}\mu - \alpha_a\nu) \geq 0$. Assume that χ and Φ are respectively defined by*

$$\chi(z) := \frac{1}{\alpha_{a+1}}\left[(\alpha_{a+1}\mu - \alpha_a\nu)q(z) + zq'(z)\right] \tag{3.2.1}$$

and

$$\Phi(z) := \Omega^a_{L,\mu,\nu}(f(z))\Upsilon_L(z), \tag{3.2.2}$$

where

$$\Upsilon_L(z) := \mu\Omega^{a+1}_{L,1,1}(f(z)) - \frac{\alpha_a\nu}{\alpha_{a+1}}\Omega^a_{L,1,1}(f(z)).$$

1. *If* $\Phi(z) \prec \chi(z)$, *then*

$$\Omega^a_{L,\mu,\nu}(f(z)) \prec q(z)$$

   and q *is the best dominant.*

2. *If* $\chi(z) \prec \Phi(z)$,

$$0 \neq \Omega^a_{L,\mu,\nu}(f(z)) \in \mathcal{H}[1,1] \cap \mathcal{Q} \text{ and } \Phi(z) \text{ is univalent in } \mathbb{D}, \tag{3.2.3}$$

then

$$q(z) \prec \Omega^a_{L,\mu,\nu}(f(z))$$

and q *is the best subordinant.*

Proof. Define the function $P : \mathbb{D} \to \mathbb{C}$ by

$$P(z) = \Omega^a_{L,\mu,\nu}(f(z)), \tag{3.2.4}$$

where the branch of P is so chosen such that $P(0) = 1$. Then P is analytic in $\mathbb{D}$. From the definition of P, from (3.2.4), we have

$$\begin{aligned} \frac{zP'(z)}{P(z)} &= \frac{z[\Omega^a_{L,\mu,\nu}(f(z))]'}{\Omega^a_{L,\mu,\nu}(f(z))} \\ &= \mu\frac{z(L^{a+1}_p f(z))'}{L^{a+1}_p f(z)} - \nu\frac{z(L^a_p f(z))'}{L^a_p f(z)} + p(\nu-\mu). \end{aligned} \tag{3.2.5}$$

Using the identity

$$z(L^a_p f(z))' = \alpha_a L^{a+1}_p f(z) - (\alpha_a - p)L^a_p f(z), \tag{3.2.6}$$

in (3.2.5), we have

$$\begin{aligned} &\Omega^a_{L,\mu,\nu}(f(z))\left(\mu\Omega^{a+1}_{L,1,1}(f(z)) - \frac{\alpha_a\nu}{\alpha_{a+1}}\Omega^a_{L,1,1}(f(z))\right) \\ &\qquad = \frac{1}{\alpha_{a+1}}[(\alpha_{a+1}\mu - \alpha_a\nu)P(z) + zP'(z)]. \end{aligned} \tag{3.2.7}$$

In view of (3.2.7), the subordination $\Phi(z) \prec \chi(z)$ becomes

$$(\alpha_{a+1}\mu - \alpha_a\nu)P(z) + zP'(z) \prec (\alpha_{a+1}\mu - \alpha_a\nu)q(z) + zq'(z)$$

and this can be written as (1.3.3), by defining

$$\theta(w) = (\alpha_{a+1}\mu - \alpha_a\nu)w \ \text{ and } \ \phi(w) = 1.$$

Note that $0 \neq \phi(w)$ and $\theta(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Setting

$$Q(z) := zq'(z) \ \text{ and } \ h(z) := \theta(q(z)) + Q(z) = (\alpha_{a+1}\mu - \alpha_a\nu)q(z) + zq'(z).$$

Since q is convex, it follows that $Q(z) = zq'(z)$ is starlike and in light of the hypothesis of Theorem 3.2.1, we see that

$$\begin{aligned} \operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) &= \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) \\ &= \operatorname{Re}\left(\alpha_{a+1}\mu - \alpha_a\nu + 1 + \frac{zq''(z)}{q'(z)}\right) > 0. \end{aligned}$$

By an application of Lemma 1.3.1, we obtain that $P \prec q$ or equivalently we can write $\Omega^a_{L,\mu,\nu}(f(z)) \prec q(z)$. The second half of Theorem 3.2.1 follows by a similar application of Lemma 1.3.2. □

Using Theorem 3.2.1, we obtain the following "sandwich result".

Corollary 3.2.2. *Let q_j $(j = 1, 2)$ be convex univalent in $\mathbb{D}$ with $q_j(0) = 1$. Assume that $\operatorname{Re}(\alpha_{a+1}\mu - \alpha_a\nu) \geq 0$ and Φ be as defined in (3.2.2). Further assume that*

$$\chi_j(z) := \frac{1}{\alpha_{a+1}}\left[(\alpha_{a+1}\mu - \alpha_a\nu)q_j(z) + zq_j'(z)\right].$$

If (3.2.3) holds and $\chi_1(z) \prec \Phi(z) \prec \chi_2(z)$, then $q_1(z) \prec \Omega^a_{L,\mu,\nu}(f(z)) \prec q_2(z)$.

Theorem 3.2.3. *Let q be convex univalent in $\mathbb{D}$ with $q(0) = 1$ and α_a be a complex number. Assume that $\operatorname{Re}(\mu\alpha_{a+1} - \nu\alpha_a) \geq 0$ and $f \in \mathcal{A}_p$. Define the functions F, χ and Ψ respectively by*

$$F(z) := \frac{\alpha_a}{z^{\alpha_a - p}} \int_0^z t^{\alpha_a - p - 1} f(t)dt, \tag{3.2.8}$$

$$\chi(z) := (\mu\alpha_{a+1} - \nu\alpha_a)q(z) + zq'(z) \tag{3.2.9}$$

and

$$\Psi(z) := \Omega^a_{L,\mu,\nu}(F(z))\left[\mu\alpha_{a+1}\Omega^a_{L,1,0}(f(z), F(z)) - \nu\alpha_a\Omega^a_{L,0,-1}(f(z), F(z))\right]. \tag{3.2.10}$$

1. *If* $\Psi(z) \prec \chi(z)$, *then* $\Omega^a_{L,\mu,\nu}(F(z)) \prec q(z)$ *and* q *is the best dominant.*

2. *If* $\chi(z) \prec \Psi(z)$,

$$0 \neq \Omega^a_{L,\mu,\nu}(F(z)) \in \mathcal{H}[1,1] \cap \mathcal{Q} \text{ and } \Psi(z) \text{ is univalent in } \mathbb{D}, \tag{3.2.11}$$

then $q(z) \prec \Omega^a_{L,\mu,\nu}(F(z))$ *and* q *is the best subordinant.*

Proof. From the definition of F, given in (3.2.8), we obtain that

$$\alpha_a f(z) = (\alpha_a - p)F(z) + zF'(z). \tag{3.2.12}$$

By convoluting (3.2.12) with $\mathscr{L}_a(z) \in \mathcal{A}_p$, where $L^a_p(f(z)) = \mathscr{L}_a(z) * f(z)$ and using the fact that $z(f * g)'(z) = f(z) * zg'(z)$, we obtain

$$\alpha_a L^a_p(f(z)) = (\alpha_a - p)L^a_p(F(z)) + z(L^a_p(F(z)))'. \tag{3.2.13}$$

Define the function $P : \mathbb{D} \to \mathbb{C}$ by

$$P(z) = \Omega^a_{L,\mu,\nu}(F(z)), \tag{3.2.14}$$

where the branch of P is so chosen such that $P(0) = 1$. Clearly P is analytic in $\mathbb{D}$. Using (3.2.13) and (3.2.14), we have

$$\begin{aligned} \Omega^a_{L,\mu,\nu}(F(z)) \left(\mu\alpha_{a+1}\Omega^a_{L,1,0}(f(z), F(z)) - \nu\alpha_a\Omega^a_{L,0,-1}(f(z), F(z))\right) \\ = (\mu\alpha_{a+1} - \nu\alpha_a)P(z) + zP'(z). \end{aligned} \tag{3.2.15}$$

Using (3.2.15), the subordination $\Psi(z) \prec \chi(z)$ becomes

$$(\mu\alpha_{a+1} - \nu\alpha_a)P(z) + zP'(z) \prec (\mu\alpha_{a+1} - \nu\alpha_a)q(z) + zq'(z)$$

and this can be written as (1.3.3), by defining the functions θ and ϕ as follows:

$$\theta(w) = (\mu\alpha_{a+1} - \nu\alpha_a)q(z) \text{ and } \phi(w) = 1.$$

Note that $0 \neq \phi(w)$ and $\theta(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Setting

$$Q(z) := zq'(z) \text{ and } h(z) := \theta(q(z)) + Q(z) = (\mu\alpha_{a+1} - \nu\alpha_a)q(z) + zq'(z).$$

In light of the assumption of our Theorem 3.2.3, we see that Q is starlike and

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\mu\alpha_{a+1} - \nu\alpha_a + 1 + \frac{zq''(z)}{q'(z)}\right) > 0.$$

An application of Lemma 1.3.1, gives $P \prec q$ or $\Omega^a_{L,\mu,\nu}(F(z)) \prec q(z)$. By an application of Lemma 1.3.2 the proof of the second half of Theorem 3.2.3 follows at once. □

Corollary 3.2.4. *Let q_j $(j = 1, 2)$ be convex univalent in $\mathbb{D}$ with $q_j(0) = 1$ and α_a be a complex number. Further assume that $\operatorname{Re}(\mu\alpha_{a+1} - \nu\alpha_a) \geq 0$ and Ψ be as defined in (3.2.10). If (3.2.11) holds and $\chi_1(z) \prec \Psi(z) \prec \chi_2(z)$, then*

$$q_1(z) \prec \Omega^a_{L,\mu,\nu}(F(z)) \prec q_2(z),$$

where $\chi_j(z) := (\mu\alpha_{a+1} - \nu\alpha_a)q_j(z) + zq_j'(z)$ $(j = 1, 2)$ and F is defined by (3.2.8).

Theorem 3.2.5. *Let q be analytic in $\mathbb{D}$ with $q(0) = 1$ and $\alpha_{a+1} = \alpha_a$. If $f \in \mathcal{A}_p$, then*

$$\Omega^a_{L,\mu,\nu}(f(z)) \prec q(z) \Leftrightarrow \Omega^{a+1}_{L,\mu,\nu}(F(z)) \prec q(z).$$

Further

$$q(z) \prec \Omega^a_{L,\mu,\nu}(f(z)) \Leftrightarrow q(z) \prec \Omega^{a+1}_{L,\mu,\nu}(F(z)),$$

where F is defined by (3.2.8).

Proof. Using the identity

$$z[L^a_p(f(z))]' = \alpha_a L^{a+1}_p(f(z)) - (\alpha_a - p)L^a_p(f(z))$$

in (3.2.13), we get

$$L^a_p(f(z)) = L^{a+1}_p(F(z)). \tag{3.2.16}$$

Since $\alpha_{a+1} = \alpha_a$, we have

$$\begin{aligned} \alpha_a L^{a+1}_p(f(z)) &= z(L^a_p(f(z)))' + (\alpha_a - p)L^a_p(f(z)) \\ &= z(L^{a+1}_p(F(z)))' + (\alpha_a - p)L^{a+1}_p(F(z)) \\ &= \alpha_{a+1}L^{a+2}_p(F(z)). \end{aligned} \tag{3.2.17}$$

Therefore, from (3.2.16) and (3.2.17), we have $\Omega^{a+1}_{L,\mu,\nu}(F(z)) = \Omega^a_{L,\mu,\nu}(f(z))$ and hence the result follows at once. □

Now we will use Theorem 3.2.5 to state the following "sandwich result".

Corollary 3.2.6. *Let $f \in \mathcal{A}_p$ and α_a is independent of a. Let ϕ_i $(i = 1, 2)$ be analytic in $\mathbb{D}$ with $\phi_i(0) = 1$ and F is defined by (3.2.8). Then*

$$\phi_1(z) \prec \Omega^{a}_{L,\mu,\nu}(f(z)) \prec \phi_2(z)$$

if and only if

$$\phi_1(z) \prec \Omega^{a+1}_{L,\mu,\nu}(F(z)) \prec \phi_2(z).$$

3.3 Applications

Several applications of the results proved in Section 3.2 are discussed in this section. Mainly the results are applied to the Dziok-Srivastava operator and the Multiplier transform. Further, using these results, sufficient conditions for Janowski, strongly and Sokół-Stankiewicz starlikeness are derived.

Applications Involving the Dziok-Srivastava Operator

We begin with some interesting applications of Theorem 3.2.1 for the Dziok-Srivastava operator. Note that the first part of Theorem 3.2.1 holds even if we assume

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, \operatorname{Re}(\alpha_1(\nu - \mu) - \mu)\}$$

instead of "q is convex and $\operatorname{Re}(\alpha_1(\mu-\nu)+\mu) \geq 0$" and leads to the following corollary.

Corollary 3.3.1. *Let $\operatorname{Re}(u - vB) \geq |v - \bar{u}B|$ where $u = \alpha_1(\mu - \nu) + \mu + 1$ and $v = [\alpha_1(\mu - \nu) + \mu - 1]B$. If $f \in \mathcal{A}_p$ satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \left(\mu\Omega^{\alpha_1+1}_{H,1,1}(f(z)) - \frac{\alpha_1\nu}{\alpha_1+1}\Omega^{\alpha_1}_{H,1,1}(f(z))\right)$$
$$\prec \frac{1}{\alpha_1+1}\left([\alpha_1(\mu-\nu)+\mu]\frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2}\right) \quad (\alpha_1 \neq -1),$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \prec \frac{1+Az}{1+Bz} \quad (-1 < B < A \leq 1)$$

and $(1 + Az)/(1 + Bz)$ is the best dominant.

Proof. Let us define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1+Az}{1+Bz} \quad (-1 < B < A \leq 1). \tag{3.3.1}$$

Then q is univalent with $q(0) = 1$ and for $z = re^{i\theta}, 0 \leq r < 1$, it is easy to see that

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) = \frac{1 - B^2r^2}{1 + B^2r^2 + 2Br\cos\theta} > 0.$$

Hence q is convex in $\mathbb{D}$. Setting $u = \alpha_1(\mu - \nu) + \mu + 1$ and $v = [\alpha_1(\mu - \nu) + \mu - 1]\, B$, we have

$$\begin{aligned} \alpha_1(\mu - \nu) + \mu + 1 + \frac{zq''(z)}{q'(z)} &= \frac{[\alpha_1(\mu - \nu) + \mu + 1] + [\alpha_1(\mu - \nu) + \mu - 1]Bz}{1 + Bz} \\ &= \frac{u + vz}{1 + Bz} =: w(z). \end{aligned}$$

The function w maps $\mathbb{D}$ into the disk

$$\left| w - \frac{\bar{u} - \bar{v}B}{1 - B^2} \right| \leq \frac{|v - \bar{u}B|}{1 - B^2}.$$

From this we see that

$$\mathrm{Re}\left(\alpha_1(\mu - \nu) + \mu + 1 + \frac{zq''(z)}{q'(z)}\right) \geq \frac{\mathrm{Re}(\bar{u} - \bar{v}B) - |v - \bar{u}B|}{1 - B^2} \geq 0$$

provided $\mathrm{Re}(\bar{u} - \bar{v}B) \geq |v - \bar{u}B|$ or $\mathrm{Re}(u - vB) \geq |v - \bar{u}B|$. Thus the result follows at once by an application of Theorem 3.2.1. □

The Dominant: $q(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$.

The function q is convex univalent and maps the unit disk $\mathbb{D}$ onto a domain $\mathrm{Re}\, q(z) > \alpha$. In view of the subordination part of Theorem 3.2.1, for the Dziok-Srivastava operator, we have the following result:

Corollary 3.3.2. *Let* $0 \leq \alpha < 1$ *and* $\mathrm{Re}(\alpha_1(\mu - \nu) + \mu) \geq 0$. *If*

$$\begin{aligned} &\Omega_{H,\mu,\nu}^{\alpha_1}(f(z)) \left(\mu \Omega_{H,1,1}^{\alpha_1+1}(f(z)) - \frac{\alpha_1 \nu}{\alpha_1 + 1} \Omega_{H,1,1}^{\alpha_1}(f(z))\right) \\ &\qquad \prec \frac{1}{\alpha_1 + 1}\left((\alpha_1(\mu - \nu) + \mu)\frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{2(1 - \alpha)z}{(1 - z)^2}\right) \quad (\alpha_1 \neq -1), \end{aligned}$$

then $\mathrm{Re}\, \Omega_{H,\mu,\nu}^{\alpha_1}(f(z)) > \alpha$.

Proof. The function $q : \mathbb{D} \to \mathbb{C}$ defined by

$$q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1).$$

Let us denote $\beta = \alpha_1(\mu - \nu) + \mu$. Then by hypothesis of theorem, we have $\operatorname{Re} \beta \geq 0$. Now consider the function

$$w(z) = 1 + \beta + \frac{zq''(z)}{q'(z)} = \beta + \frac{1 + z}{1 - z}.$$

It is easy to see that w maps the unit disk $\mathbb{D}$ onto $\operatorname{Re} w > \operatorname{Re} \beta \geq 0$. The result now follows by an application of the subordination part of Theorem 3.2.1. □

Note that if $p = 1, l = m + 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ..., m)$, then we have $H_1[1]f(z) = f(z), H_1[2]f(z) = zf'(z)$ and $H_1[3]f(z) = \frac{1}{2}z^2 f''(z) + zf'(z)$. Putting $\alpha_1 = 1$, $p = 1, l = m + 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ..., m)$ in Corollary 3.3.2, we obtain the following.

Corollary 3.3.3. *Let* $0 \leq \alpha < 1$ *and* $2\mu \geq \nu$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\operatorname{Re}\left((f'(z))^\mu \left(\frac{z}{f(z)} \right)^\nu \left(\mu \left(2 + \frac{zf''(z)}{f'(z)} \right) - \nu \frac{zf'(z)}{f(z)} \right) \right) > \frac{2(2\mu - \nu)\alpha - (1 - \alpha)}{2},$$

then

$$\operatorname{Re}\left((f'(z))^\mu \left(\frac{z}{f(z)} \right)^\nu \right) > \alpha.$$

Proof. From Corollary 3.3.2, we see that

$$(f'(z))^\mu \left(\frac{z}{f(z)} \right)^\nu \left(\mu \left(2 + \frac{zf''(z)}{f'(z)} \right) - \nu \frac{zf'(z)}{f(z)} \right) \prec (2\mu - \nu) \frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{2(1 - \alpha)z}{(1 - z)^2} =: h(z).$$

We now investigate the image domain $h(\mathbb{D})$. Assuming $a = 1 - 2\alpha$ and $b = 2\mu - \nu$, we have

$$h(z) = \frac{b + (1 + a - b + ab)z - abz^2}{(1 - z)^2},$$

where $h(0) = b$ and $h(-1) = [2b(1 - a) - (1 + a)]/4$. The boundary curve of the image of $h(\mathbb{D})$ is given by $h(e^{i\theta}) = u(\theta) + iv(\theta)$, $-\pi < \theta < \pi$, where

$$u(\theta) = \frac{(1 + a - b + ab) + (1 - a)b \cos \theta}{2(\cos \theta - 1)} \quad \text{and} \quad v(\theta) = \frac{(1 + a)b \sin \theta}{2(1 - \cos \theta)}.$$

By eliminating θ, we obtain the equation of the boundary curve as

$$v^2 = -b^2(1+a)\left(u - \frac{2b(1-a)-(a+1)}{4}\right). \qquad (3.3.2)$$

Obviously (3.3.2) represents a parabola opening towards the left, with the vertex at the point $\left(\frac{2b(1-a)-(a+1)}{4}, 0\right)$ and negative real axis as its axis. Hence $h(\mathbb{D})$ is the exterior of the parabola (3.3.2) which includes the right half plane

$$u > \frac{2b(1-a)-(a+1)}{4}.$$

Hence the result follows at once. □

Taking $\mu = 1 = \nu$ in Corollary 3.3.3, we have the following result which provides a sufficient condition for starlike function of order α.

Example 3.3.4. If $f \in \mathcal{A}$ and satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right) > \frac{3\alpha - 1}{2},$$

then $f \in \mathcal{S}^*(\alpha)$.

Setting $\mu = 0$ and $\nu = -1$ in Corollary 3.3.3, we obtain the following result.

Example 3.3.5. If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} f'(z) > \frac{3\alpha - 1}{2} \quad (0 \le \alpha < 1),$$

then

$$\operatorname{Re}\frac{f(z)}{z} > \alpha.$$

Remark 3.3.1. When $\alpha = 1/3$, Example 3.3.5 reduces to a result of Obradović [**121**, Theorem 2].

The following corollary is a straight forward consequence of the first part of Theorem 3.2.3 for the case when $q(z) = (1 + (1 - 2\alpha)z)/(1 - z)$.

Corollary 3.3.6. *Let* $0 \leq \alpha < 1$ *and* $\mathrm{Re}((\mu-\nu)\alpha_1+\mu) \geq 0$. *If* $f \in \mathcal{A}_p$, F *as defined in (3.2.8) and satisfying the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z))(\mu(\alpha_1+1)\Omega^{\alpha_1}_{H,1,0}(f(z),F(z)) - \nu\alpha_1\Omega^{\alpha_1}_{H,0,-1}(f(z),F(z))) \prec ((\mu-\nu)\alpha_1+\mu)\frac{1+(1-2\alpha)z}{1-z} + \frac{2(1-\alpha)z}{(1-z)^2}$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z)) \prec \frac{1+(1-2\alpha)z}{1-z}$$

and $(1+(1-2\alpha)z)/(1-z)$ *is the best dominant.*

Setting $p=1, l=m+1, \alpha_1=1$ and $\alpha_{i+1}=\beta_i$ $(i=1,2,...m)$ in Corollary 3.3.6, we obtain the following result:

Corollary 3.3.7. *Let* $0 \leq \alpha < 1$ *and* $2\mu \geq \nu$. *If* $f \in \mathcal{A}$, F *is defined by*

$$F(z) = \int_0^z \frac{f(t)}{t}dt \tag{3.3.3}$$

and

$$\mathrm{Re}\left\{(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right)\right\} < \frac{2(2\mu-\nu)\alpha-(1-\alpha)}{2},$$

then

$$\mathrm{Re}\left((F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\right) > \alpha.$$

Proof. From Corollary 3.3.6, we see that the subordination

$$(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right) \prec (2\mu-\nu)\frac{1+(1-2\alpha)z}{1-z} + \frac{2(1-\alpha)z}{(1-z)^2} =: h(z)$$

implies that

$$\mathrm{Re}\left[(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\right] > \alpha.$$

Let $z = e^{i\theta}, -\pi \leq \theta \leq \pi$. Then, we have

$$\begin{aligned}\mathrm{Re}(h(e^{i\theta})) &= \mathrm{Re}\left\{(2\mu-\nu)\frac{1+(1-2\alpha)e^{i\theta}}{1-e^{i\theta}} + \frac{2(1-\alpha)e^{i\theta}}{(1-e^{i\theta})^2}\right\} \\ &= (2\mu-\nu)\alpha - \frac{(1-\alpha)}{2}\left(\frac{1}{\sin^2(\theta/2)}\right) =: k(\theta).\end{aligned}$$

Since $2\mu - \nu \geq 0$, it is easy to see that $k(\theta)$ attains its maximum at $\theta = \pi$ and

$$\max_{|\theta|\leq\pi} k(\theta) = \frac{2(2\mu-\nu)\alpha - (1-\alpha)}{2}.$$

Hence the result follows at once. □

The following result, provides a sufficient condition for starlikeness, is obtained from Corollary 3.3.7 by putting $\mu = 1 = \nu$.

Example 3.3.8. Let $0 \leq \alpha < 1$. If $f \in \mathcal{A}$, F as defined in (3.3.3) and

$$\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\left(2\frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)}\right)\right) < \frac{3\alpha-1}{2},$$

then $F \in \mathcal{S}^*(\alpha)$.

The Dominant: $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\eta}$ $(0 < \eta \leq 1)$.

The function q maps the unit disk $\mathbb{D}$ onto a sector $|\arg w| \leq \pi/2$ and is a convex function. Taking this function as a dominant in the first part of Theorem 3.2.1, for the Dziok–Srivastava operator, we have the following result:

Corollary 3.3.9. *Let* $0 < \eta \leq 1$, $\alpha_1 \neq -1$ *and* $\operatorname{Re}(\alpha_1(\mu-\nu)+\mu) \geq 0$. *If* $f \in \mathcal{A}_p$ *and satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z))\left(\mu\Omega^{\alpha_1+1}_{H,1,1}(f(z)) - \frac{\alpha_1\nu}{\alpha_1+1}\Omega^{\alpha_1}_{H,1,1}(f(z))\right)$$
$$\prec \frac{1}{\alpha_1+1}\left((\alpha_1(\mu-\nu)+\mu) + \frac{2\eta z}{1-z^2}\right)\left(\frac{1+z}{1-z}\right)^{\eta},$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \prec \left(\frac{1+z}{1-z}\right)^{\eta}$$

and the function $((1+z)/(1-z))^{\eta}$ *is the best dominant.*

By taking $p = 1, l = m+1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...m)$, in Corollary 3.3.9, we have the following:

Corollary 3.3.10. *Let* $0 < \eta \le 1$ *and* $2\mu \ge \nu$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\delta\pi}{2},$$

then

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2}$$

where

$$\delta = \eta + 1 - \frac{2}{\pi}\arctan\left(\frac{2\mu-\nu}{\eta}\right).$$

Proof. In view of Corollary 3.3.9, we see that the subordination

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right) \prec \left((2\mu-\nu)+\frac{2\eta z}{1-z^2}\right)\left(\frac{1+z}{1-z}\right)^{\eta} =: h(z)$$

implies that

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \text{ or equivalently } \left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2}.$$

Now we shall find the minimum value of $\arg h(z)$ over $|z| < 1$. For this purpose let $z = e^{i\theta}$, $-\pi \le \theta \le \pi$. Since $h(\mathbb{D})$ is symmetrical about the real axis, we shall restrict ourself to $0 < \theta \le \pi$. Setting $t = \cot\theta/2$, we have $t \ge 0$ and for $z = (it-1)/(it+1)$, we arrive at

$$\begin{aligned} h(e^{i\theta}) &= (it)^{\eta-1}\left[(2\mu-\nu)it - \frac{\eta(1+t^2)}{2}\right] \\ &= (it)^{\eta-1}G(t), \quad t = \cot\theta/2 \end{aligned}$$

where

$$G(t) = (2\mu-\nu)it - \frac{\eta(1+t^2)}{2}.$$

Let $G(t) = U(t) + iV(t)$, where $U(t) = -(\eta(1+t^2))/2$ and $V(t) = (2\mu-\nu)t$, there arises two cases namely $2\mu > \nu$ and $2\mu = \nu$. If $2\mu > \nu$, then a calculation shows that $\min_{t\ge 0}\arg G(t)$ occurs at $t = 1$ and

$$\min_{t\ge 0}\arg G(t) = \pi - \arctan\left(\frac{2\mu-\nu}{\eta}\right).$$

Thus

$$\min_{|z|<1} \arg h(z) \quad = \quad \frac{(\eta+1)\pi}{2} - \arctan\left(\frac{2\mu-\nu}{\eta}\right).$$

If $2\mu = \nu$, then $\arg G(t) = \pi$ and $\min_{|z|<1} \arg h(z) = (\eta+1)\pi/2$. Thus for $2\mu \geq \nu$, we have

$$\min_{|z|<1} \arg h(z) = \frac{(\eta+1)\pi}{2} - \arctan\left(\frac{2\mu-\nu}{\eta}\right).$$

This completes the proof. □

Letting $\mu = 1 = \nu$ in Corollary 3.3.10, we have the following result which gives a sufficient condition for strongly starlike functions of order η.

Example 3.3.11. Let $0 < \eta \leq 1$. If $f \in \mathcal{A}$ and satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right)\right| < \frac{\delta\pi}{2},$$

then $f \in \mathcal{SS}^*(\eta)$, where $\delta = \eta + 1 - \frac{2}{\pi}\arctan\frac{1}{\eta}$.

By taking $q(z) = ((1+z)/(1-z))^\eta$ in the subordination part of Theorem 3.2.3 for the Dzoik Srivastava operator, we have the following result:

Corollary 3.3.12. *Let* $0 < \eta \leq 1$ *and* $\operatorname{Re}[(\mu-\nu)\alpha_1 + \mu] \geq 0$. *If* $f \in \mathcal{A}_p$, F *as defined in (3.2.8) and satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z))\left((\alpha_1+1)\mu\Omega^{\alpha_1}_{H,1,0}(f(z),F(z)) - \nu\alpha_1\Omega^{\alpha_1}_{H,0,-1}(f(z),F(z))\right)$$
$$\prec \left(((\mu-\nu)\alpha_1+\mu) + \frac{2\eta z}{(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^\eta,$$

then

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z)) \prec \left(\frac{1+z}{1-z}\right)^\eta$$

and $((1+z)/(1-z))^\eta$ *is the best dominant.*

By putting $p = 1, l = m+1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...m)$ in Corollary 3.3.12, we obtain the following result.

Corollary 3.3.13. *Let $0 < \eta \leq 1$ and $2\mu \geq \nu$. If $f \in \mathcal{A}$, F as defined in (3.3.3) and*

$$\left|\arg\left\{(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right)\right\}\right| < \frac{(\eta+1)\pi}{2} - \arctan\frac{(2\mu-\nu)}{\eta},$$

then

$$\left|\arg\left\{(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2}.$$

Proof. From Corollary 3.3.12, we observe that the subordination

$$(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right) \prec \left(2\mu - \nu + \frac{2\eta z}{(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^{\eta} =: h(z)$$

implies that

$$\left|\arg\left\{(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2} \quad (z \in \mathbb{D}).$$

We shall find the minimum value of the $\arg h(z)$ over $|z| < 1$. Since $h(\mathbb{D})$ is symmetrical about the real axis, by taking $z = e^{i\theta}$, we need to consider only the case $0 < \theta \leq \pi$. Set $t = \cot\theta/2$, then $t \geq 0$. Also, for $z = (it-1)/(it+1)$, after a simple computation, we have

$$\begin{aligned} h(e^{i\theta}) &= (it)^{\eta-1}\left[(2\mu-\nu)it - \frac{\eta(1+t^2)}{2}\right] \\ &= (it)^{\eta-1}H(t), \end{aligned}$$

where

$$H(t) = (2\mu-\nu)it - \frac{\eta(1+t^2)}{2}.$$

Writing $H(t) := U(t) + iV(t)$, where $U(t) = -\eta(1+t^2)/2$ and $V(t) = (2\mu-\nu)t$. If $2\mu > \nu$, then a calculation shows that $\min_{t\geq 0}\arg H(t)$ occurs at $t = 1$ and

$$\min_{t\geq 0}\arg H(t) = \pi - \arctan\left(\frac{2\mu-\nu}{\eta}\right).$$

Thus, we have

$$\min_{|z|<1}\arg h(z) = \frac{(\eta+1)\pi}{2} - \arctan\left(\frac{2\mu-\nu}{\eta}\right).$$

If $2\mu = \nu$, then $\min\arg H(t)_{t\geq 0} = \pi$. Therefore

$$\min_{|z|<1}\arg h(z) = \frac{(\eta+1)\pi}{2}.$$

Thus for $2\mu \geq \nu$, we have

$$\min_{|z|<1} \arg h(z) = \frac{(\eta+1)\pi}{2} - \arctan\left(\frac{2\mu-\nu}{\eta}\right).$$

This completes the proof of corollary. □

Putting $\mu = 1 = \nu$ in Corollary 3.3.13, we have the following result providing a sufficient condition for strongly starlikeness.

Example 3.3.14. Let $0 < \eta \leq 1$. If $f \in \mathcal{A}$, F as defined in (3.3.3) and

$$\left|\arg\left(\frac{zF'(z)}{F(z)}\left(2\frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)}\right)\right)\right| < \frac{(\eta+1)\pi}{2} - \arctan\frac{1}{\eta},$$

then $f \in \mathcal{SS}^*(\eta)$.

The Dominant: $q(z) = \sqrt{1+z}$.

The function q is a convex function which maps the unit disk $\mathbb{D}$ onto the interior of the right-half of the lemniscate Bernoulli. The following result is a consequence of the first part of Theorem 3.2.1 for the Dziok-Srivastava operator for the dominant $q(z) = \sqrt{1+z}$.

Corollary 3.3.15. *Let $\alpha_1 \neq -1$ and $\operatorname{Re}[\alpha_1(\mu-\nu)+\mu] \geq 0$. If $f \in \mathcal{A}_p$ and satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(f(z))\left(\mu\Omega^{\alpha_1+1}_{H,1,1}(f(z)) - \frac{\alpha_1\nu}{\alpha_1+1}\Omega^{\alpha_1}_{H,1,1}(f(z))\right) \prec \frac{1}{\alpha_1+1}\left([\alpha_1(\mu-\nu)+\mu]\sqrt{1+z} + \frac{z}{2\sqrt{1+z}}\right),$$

then $\Omega^{\alpha_1}_{H,\mu,\nu}(f(z)) \prec \sqrt{1+z}$ and $\sqrt{1+z}$ is the best dominant.

By taking $p = 1, l = m+1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...m)$ in Corollary 3.3.15, we obtain the following result.

Corollary 3.3.16. *Let $2\mu \geq \nu$. If $f \in \mathcal{A}$ and satisfies the subordination*

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(2+\frac{zf''(z)}{f'(z)}\right) - \nu\frac{zf'(z)}{f(z)}\right) \prec (2\mu-\nu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}},$$

then

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu} \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ *is the best dominant.*

We obtain the following example from Corollary 3.3.16.

Example 3.3.17. If $f \in \mathcal{A}$ and satisfies

$$\left|\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \sqrt{1.22} \approx 1.10,$$

then $f \in \mathcal{S}^*_L$.

Proof. Putting $\mu = \nu = 1$ in Corollary 3.3.16, we see that the subordination

$$\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z}+\frac{z}{2\sqrt{1+z}} =: h(z),$$

implies that

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}.$$

The dominant $h(z)$ can be written as

$$h(z) = \frac{3z+2}{2\sqrt{1+z}}.$$

Writing $h(e^{i\theta}) = u(\theta) + iv(\theta)$, $-\pi < \theta < \pi$, we have

$$u(\theta) = \frac{3\cos(3\theta/4)+2\cos(\theta/4)}{2\sqrt{2\cos(\theta/2)}}$$

and

$$v(\theta) = \frac{3\sin(3\theta/4)-2\sin(\theta/4)}{2\sqrt{2\cos(\theta/2)}}.$$

Squaring and adding, we have

$$u^2(\theta)+v^2(\theta) = \frac{13+12\cos\theta}{8\cos(\theta/2)} =: k(\theta).$$

It is easy to see that $k(\theta)$ attains its minimum at $\theta = \arccos(\sqrt{1/24}) \approx 78.22^\circ$ and $k(\theta) \geq \sqrt{3/2} \approx 1.22$. In addition to that since $h(0) = 1$ and $h(-1) = -\infty$, it follows that the image domain $h(\mathbb{D})$ is the interior of a domain bounded by parabola opening towards left which contains the interior of the circle $u^2 + v^2 = 1.22$. □

Taking the dominant $q(z) = \sqrt{1+z}$ in the first part of Theorem 3.2.3, we have the following corollary for the Dzoik-Srivastava operator.

Corollary 3.3.18. *Let* $0 < \eta \leq 1$ *and* $\operatorname{Re}(\alpha_1(\mu-\nu)+\mu) \geq 0$. *If* $f \in \mathcal{A}_p$, F *as defined in (3.2.8) and satisfies the subordination*

$$\Omega^{\alpha_1}_{H,\mu,\nu}(F(z))\left((\alpha_1+1)\mu\Omega^{\alpha_1}_{H,1,0}(f(z),F(z)) - \alpha_1\nu\Omega^{\alpha_1}_{H,0,-1}(f(z),F(z))\right) \\ \prec (\alpha_1(\mu-\nu)+\mu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}},$$

then $\Omega^{\alpha_1}_{H,\mu,\nu}(F(z)) \prec \sqrt{1+z}$ *and* $\sqrt{1+z}$ *is the best dominant.*

Putting $p = 1, l = m+1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ $(i = 1, 2, ...m)$ in Corollary 3.3.18, we obtain the following result.

Corollary 3.3.19. *Let* $2\mu \geq \nu$. *If* $f \in \mathcal{A}$, F *as defined in (3.3.3) and*

$$(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu}\left(2\mu\frac{f'(z)}{F'(z)} - \nu\frac{f(z)}{F(z)}\right) \prec (2\mu-\nu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}},$$

then $(F'(z))^{\mu}\left(\frac{z}{F(z)}\right)^{\nu} \prec \sqrt{1+z}$ *and* $\sqrt{1+z}$ *is the best dominant.*

Putting $\mu = \nu = 1$ in the above Corollary 3.3.19, we have the following example.

Example 3.3.20. If $f \in \mathcal{A}$, F as defined in (3.3.3) and

$$\left|\frac{zF'(z)}{F(z)}\left(2\frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)}\right)\right| < \sqrt{1.22} \approx 1.10,$$

then $F \in \mathcal{S}^*_L$.

Proof. Putting $\mu = \nu = 1$ in Corollary 3.3.16, we see that the subordination

$$\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z} + \frac{z}{2\sqrt{1+z}} =: h(z),$$

implies that

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}.$$

The dominant $h(z)$ can be written as

$$h(z) = \frac{3z+2}{2\sqrt{1+z}}.$$

Writing $h(e^{i\theta}) = u(e^{i\theta}) + iv(e^{i\theta}), -\pi < \theta < \pi$, we have

$$u(\theta) = \frac{3\cos(3\theta/4) + 2\cos(\theta/4)}{2\sqrt{2\cos(\theta/2)}} \quad \text{and} \quad v(\theta) = \frac{3\sin(3\theta/4) - 2\sin(\theta/4)}{2\sqrt{2\cos(\theta/2)}}.$$

A simple calculation gives

$$u^2(\theta) + v^2(\theta) = \frac{13 + 12\cos\theta}{8\cos(\theta/2)} =: k(\theta).$$

A computation shows that $k(\theta)$ has minimum at $\theta = \arccos(\sqrt{1/24}) \approx 78.22°$ and $k(\theta) \geq \sqrt{3/2} \approx 1.22$. In view of this and since $h(0) = 1$ and $h(-1) = -\infty$, it follows that the image domain $h(\mathbb{D})$ is the interior of a domain bounded by parabola opening towards left which contains the interior of the circle $u^2 + v^2 = 1.22$. This completes the proof. $\square$

Applications Involving the Multiplier Transform

Next we discuss some applications related to the multiplier transform.

The Dominant: $q(z) = \dfrac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1)$.

For this q the first part of Theorem 3.2.1 yields the following:

Corollary 3.3.21. *Let $0 \leq \alpha < 1$, $\lambda \neq -p$ be any complex number and suppose that* $\mathrm{Re}((\nu - \mu)(\lambda + p)) \geq 0$. *If $f \in \mathcal{A}_p$ and satisfies the following subordination*

$$\Omega^r_{I,\mu,\nu}(f(z))\left(\mu\Omega^{gr+1}_{I,1,1}(f(z)) - \nu\Omega^r_{I,1,1}(f(z))\right) \prec (\mu - \nu)\frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{1}{\lambda + p}\frac{2(1 - \alpha)z}{(1 - z)^2},$$

then

$$\Omega^r_{I,\mu,\nu}(f(z)) \prec \frac{1 + (1 - 2\alpha)z}{1 - z}$$

and $(1 + (1 - 2\alpha)z)/(1 - z)$ *is the best dominant.*

Note that for $p = 1, \lambda = 0$ and $r = 0$, we have $I_1(0,0)f(z) = f(z)$, $I_1(1,0)f(z) = zf'(z)$, $I_1(2,0)f(z) = z(zf''(z) + f'(z))$. Putting these values in Corollary 3.3.21, we have the following result.

Corollary 3.3.22. *Let* $0 \leq \alpha < 1$ *and* $\mu \geq \nu$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\operatorname{Re}\left((f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(1+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right)\right) > \frac{2(\mu-\nu)\alpha-(1-\alpha)}{2},$$

then

$$\operatorname{Re}\left((f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right) > \alpha.$$

Proof. In view of Corollary 3.3.21, it is observed that the subordination

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(1+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right) \prec (\mu-\nu)\frac{1+(1-2\alpha)z}{1-z}+\frac{2(1-\alpha)z}{(1-z)^2} =: h(z),$$

implies that

$$\operatorname{Re}\left((f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right) > \alpha.$$

We now investigate the image domain $h(\mathbb{D})$. Let $a = 1-2\alpha$ and $b = \mu-\nu$. We can rewrite the function h as

$$h(z) = \frac{b+(1+a-b+ab)z-abz^2}{(1-z)^2},$$

so that $h(0) = b$ and $h(-1) = [2b(1-a)-(1+a)]/4$. The boundary curve of the image domain is given by $h(e^{i\theta}) = u(\theta)+iv(\theta)$, $-\pi \leq \theta \leq \pi$, where

$$u(\theta) = \frac{(1+a-b+ab)+(1-a)b\cos\theta}{2(\cos\theta-1)}$$

and

$$v(\theta) = \frac{(1+a)b\sin\theta}{2(1-\cos\theta)}.$$

Eliminating θ, we obtain the equation of the boundary curve as

$$v^2 = -b^2(1+a)\left(u-\frac{2b(1-a)-(1+a)}{4}\right). \tag{3.3.4}$$

Since $b = \mu-\nu \geq 0$ and $a+1 > 0$, it follows that (3.3.4) represents a parabola opening towards the left, with the vertex at the point $((2b(1-a)-(1+a))/4, 0)$ with negative real axis as its axis. Thus $h(\mathbb{D})$ is the exterior of the parabola (3.3.4) includes the right half-plane

$$u > \frac{2b(1-a)-(1+a)}{4}.$$

This completes the proof. □

Setting $\mu = \nu = 1$ in Corollary 3.3.22, we have the following result which provides a sufficient condition for starlikeness of order α.

Example 3.3.23. Let $0 \le \alpha < 1$. If $f \in \mathcal{A}$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \in \mathbb{C} \setminus \left\{w : w \le \frac{\alpha - 1}{2}\right\},$$

then $f \in S^*(\alpha)$.

Remark 3.3.2. For $\alpha = 0$, Example 3.3.23 reduces to a result obtained by Owa and Obradović [**127**, Corollary 2].

Putting $\mu = 1$ and $\nu = 0$ in Corollary 3.3.22, we have the following result:

Example 3.3.24. Let $0 \le \alpha < 1$. If $f \in \mathcal{A}$ and satisfies

$$\operatorname{Re}(f'(z) + zf''(z)) > \frac{3\alpha - 1}{2},$$

then $\operatorname{Re} f'(z) > \alpha$.

Remark 3.3.3. The above Example 3.3.24 generalize the result [**43**, Theorem 5] due to Chichra. Further Corollary 3.3.22 reduces to [**121**, Theorem 2] when $\mu = 0, \nu = -1$ and $\alpha = 1/3$.

The Dominant: $q(z) = \left(\frac{1+z}{1-z}\right)^\eta$ $(0 < \alpha \le 1)$.

From the first part of Theorem 3.2.1 for multiplier transform, we have the following:

Corollary 3.3.25. *Let* $0 < \eta \le 1$, $\lambda \ne -p$ *be any complex number and assume that* $\operatorname{Re}((\mu - \nu)(\lambda + p)) \ge 0$. *If* $f \in \mathcal{A}_p$, *and satisfies the subordination*

$$\Omega^r_{I,\mu,\nu}(f(z))\left(\mu\Omega^{r+1}_{I,1,1}(f(z)) - \nu\Omega^r_{I,1,1}(f(z))\right) \prec \left((\mu - \nu) + \frac{2\eta z}{(\lambda + p)(1 - z^2)}\right)\left(\frac{1+z}{1-z}\right)^\eta,$$

then

$$\Omega^r_{I,\mu,\nu}(f(z)) \prec \left(\frac{1+z}{1-z}\right)^\eta$$

and $((1+z)/(1-z))^\eta$ *is the best dominant.*

Putting $p = 1, \lambda = 0$ and $r = 0$ in Corollary 3.3.25, we obtain the following corollary.

Corollary 3.3.26. *Let* $0 < \eta \leq 1$ *and* $\mu \geq \nu$. *If* $f \in \mathcal{A}$ *and satisfies*

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(1+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\delta\pi}{2},$$

where

$$\delta = \eta + 1 - \frac{2}{\pi}\arctan\frac{\mu-\nu}{\eta},$$

then

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2}.$$

Proof. From Corollary 3.3.25, we see that the subordination

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\left(\mu\left(1+\frac{zf''(z)}{f'(z)}\right)-\nu\frac{zf'(z)}{f(z)}\right) \prec \left((\mu-\nu)+\frac{2\eta z}{(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^{\eta} =: h(z)$$

implies that

$$\left|\arg\left\{(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu}\right\}\right| < \frac{\eta\pi}{2}.$$

We need to find the minimum value of $\arg h(z)$ over $|z| < 1$. Since $h(\bar{z}) = \bar{h}(z)$, it follows that $h(\mathbb{D})$ is symmetrical about the real axis and so we need only to discuss the case $0 < \theta \leq \pi$. If we set $z = e^{i\theta}$ and $t = \cot\theta/2$, then obviously $t \geq 0$. Also, for $z = (it-1)/(it+1)$, after a simple computation, we have

$$\begin{aligned} h(e^{i\theta}) &= (it)^{\eta-1}\left[(\mu-\nu)it - \frac{\eta(1+t^2)}{2}\right] \\ &= (it)^{\eta-1}H(t) \end{aligned}$$

where

$$H(t) = (\mu-\nu)it - \frac{\eta(1+t^2)}{2}.$$

Writing $H(t) = U(t) + iV(t)$, where $U(t) = -\eta(1+t^2)/2$ and $V(t) = (\mu-\nu)t$. If $\mu > \nu$, then a calculation shows that $\min_{t\geq 0}\arg H(t)$ occurs at $t = 1$ and

$$\min_{t\geq 0}\arg H(t) = \pi - \arctan\frac{\mu-\nu}{\eta}.$$

Thus, we have

$$\min_{|z|<1} \arg h(z) = \frac{(\eta+1)\pi}{2} - \arctan \frac{\mu-\nu}{\eta}.$$

If $\mu = \nu$, then $\arg H(t) = \pi$ and $\min_{|z|<1} \arg h(z) = (\eta+1)\pi/2$. Thus for $\mu \geq \nu$, we have

$$\min_{|z|<1} \arg h(z) = \frac{(\eta+1)\pi}{2} - \arctan \frac{\mu-\nu}{\eta}.$$

Thus the proof is complete now. □

Letting $\mu = 1 = \nu$ in Corollary 3.3.26, we get the following result which provides a sufficient condition for a function to be strongly starlike.

Example 3.3.27. Let $0 < \eta \leq 1$. If $f \in \mathcal{A}$ and satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \frac{(\eta+1)\pi}{2},$$

then $f \in \mathcal{SS}^*(\eta)$.

The Dominant: $q(z) = \sqrt{1+z}$.

Taking $q(z) = \sqrt{1+z}$ as dominant in the subordination part of Theorem 3.2.1 for multiplier transform, we obtain the following corollary:

Corollary 3.3.28. *Let $\lambda \neq -p$ be a complex number and* $\operatorname{Re}[(\mu-\nu)(\lambda+p)] \geq 0$. *If $f \in \mathcal{A}_p$, and satisfies the subordination*

$$\Omega^r_{I,\mu,\nu}(f(z))\left(\mu\Omega^{r+1}_{I,1,1}(f(z)) - \nu\Omega^r_{I,1,1}(f(z))\right) \prec (\mu-\nu)\sqrt{1+z} + \frac{z}{2(\lambda+p)\sqrt{1+z}},$$

then $\Omega^r_{I,\mu,\nu}(f(z)) \prec \sqrt{1+z}$ and $\sqrt{1+z}$ is the best dominant.

Putting $p = 1, \lambda = 0$ and $r = 0$ in Corollary 3.3.28, we have the following corollary.

Corollary 3.3.29. *Let $\mu \geq \nu$. If $f \in \mathcal{A}$ and satisfies the subordination*

$$(f'(z))^\mu \left(\frac{z}{f(z)}\right)^\nu \left(\mu\left(1 + \frac{zf''(z)}{f'(z)}\right) - \nu\frac{zf'(z)}{f(z)}\right) \prec (\mu-\nu)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}} := h(z),$$

then

$$(f'(z))^{\mu}\left(\frac{z}{f(z)}\right)^{\nu} \prec \sqrt{1+z}$$

and $\sqrt{1+z}$ *is the best dominant.*

Example 3.3.30. If $f \in \mathcal{A}$ and satisfies

$$\left|\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{1}{2\sqrt{2}} \approx 0.35,$$

then $f \in \mathcal{S}_L^*$.

Proof. Putting $\mu = \nu = 1$ in Corollary 3.3.29 the dominant $h(z)$ reduces to

$$h(z) = \frac{z}{2\sqrt{1+z}}.$$

Writing $h(e^{i\theta}) = u(\theta) + iv(\theta), -\pi < \theta < \pi$, we have

$$u(\theta) = \frac{\cos 3(\theta/4)}{2\sqrt{2\cos(\theta/2)}} \quad \text{and} \quad v(\theta) = \frac{\sin 3(\theta/4)}{2\sqrt{2\cos(\theta/2)}}.$$

A simple calculation gives

$$u^2(\theta) + v^2(\theta) = \frac{1}{8\cos(\theta/2)} \geq \frac{1}{8}.$$

Here $h(0) = 0$ and $h(-1) = -\infty$ and image domain of $h(\mathbb{D})$ is the interior of a domain bounded by parabola opening towards left on the real axis and contains the interior of the circle $u^2 + v^2 = 1/8$. □

Chapter 4

Sandwich Theorems for Analytic Functions Involving a Generalized Linear Operator

4.1 Introduction

There are several linear operators defined in univalent function theory and studied in past few decades. Recently, Lupaş [**2–5**] in his series of papers considered the linear combination of two linear operators and discussed subordination results related to them. Let us have a look on results carried out by Lupaş first. For function f given by (1.1.1) the generalized Sălăgean operator [**24**] is defined by

$$D_\lambda^m f(z) = z + \sum_{k=2}^{\infty}(1+(k-1)\lambda)^m a_k z^k \ \ (\lambda \geq 0, m \in \mathbb{N}).$$

For $\lambda = 1$, this operator reduces to the Sălăgean operator [**162**]. The generalized Sălăgeăn operator is further generalized by Ramdan and Darus [**143**] and discussed univalent criteria for this operator. Using the Ruscheweyh operator [**160**], for function $f \in \mathcal{A}$, defined by $R^m f(z) = \sum_{k=2}^{\infty} C_{m+k-1}^{m} a_k z^k$ and the generalized multiplier

The contents of this chapter appeared in [**175**].

transform [**3**] defined by

$$I(m,\lambda,l)f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{l+1}\right)^m a_k z^k \ \ (l,\lambda \geq 0, m \in \mathbb{N}\cup\{0\}),$$

which is another generalization of the generalized Sălăgean operator [**24**], Lupaş [**5**], in 2010, considered a new operator defined as follows:

$$RI^{\alpha}(m,\lambda,l)f(z) = (1-\alpha)R^m f(z) + \alpha I(m,\lambda,l)f(z).$$

This operator was further generalized by Lupaş [**4**]. Recently in 2011, Lupaş [**2**] introduced another operator using the generalized Sălăgean operator [**24**] and the Ruscheweyh operator [**160**] as follows:

$$RD^m_{\lambda,\alpha} f(z) = (1-\alpha)R^m f(z) + \alpha D^m_{\lambda} f(z) \ \ (\alpha \geq 0, m \in \mathbb{N}).$$

In all these papers of Lupaş the main focus was to establish certain differential subordination results. It should be noted that all these operators, mentioned above, can be written in terms of the Hadamard product. Motivated by this fact, in this chapter, we have unified all those operator introduced by Lupaş [**2–5**] and Ramdan and Darus [**143**] using the Hadamard product. We have derived some differential sandwich results associated with the newly defined operator for normalized analytic functions. The results proved in this chapter generalize results proved by Lupaş in his series of papers. Lupaş in his papers proved results for some particular operators but our results hold good for any linear operator which can be written in terms of Hadamard product. Our results also generalize several results including that of Owa et al. [**99**], Hallenbeck [**69**], Miller and Mocanu [**108**], Obradović [**125**] and Kumar et al. [**180**] and others.

Definition 4.1.1. For $f \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, the operator $\mathcal{O}_{g,h}(\alpha)$ is defined by

$$\mathcal{O}_{g,h}(\alpha)f(z) = (1-\alpha)(f*g)(z) + \alpha(f*h)(z),$$

where $g, h \in \mathcal{A}$ are given by $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$ and $h(z) = z + \sum_{k=2}^{\infty} h_k z^k$.

Remark 4.1.1. By taking suitable values for α and appropriate functions g, h, the operator $\mathcal{O}_{g,h}(\alpha)$ reduces to the several known operators introduced in [**2–5, 24, 142, 143, 162, 181**]. For instance, if we set $\alpha \geq 0$,

$$g(z) = z + \sum_{k=2}^{\infty} C_{m+k-1}^{m} z^k \text{ and } h(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda(k-1) + l}{l+1} \right)^m z^k \ (m \in \mathbb{N}),$$

then the operator $\mathcal{O}_{g,h}(\alpha)$ reduces to the operator $RI^{\alpha}(m, \lambda, l)$ introduced by Lupaş [**5**]. Further if we take

$$g(z) = z + \sum_{k=2}^{\infty} C_{m+k-1}^{m} z^k \text{ and } h(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^m z^k,$$

then $\mathcal{O}_{g,h}(\alpha) = RD_{\lambda,\alpha}^{m}$ $(m \in \mathbb{N})$ is the operator introduced by Lupaş [**2**].

Preliminaries

The following lemmas are required to prove our main results:

Lemma 4.1.1. [**69, 108**] *Let ϕ be a convex function in $\mathbb{D}$, with $\phi(0) = a, \gamma \neq 0$ and* $\mathrm{Re}(\gamma) \geq 0$. *If $p \in \mathcal{H}[a, n]$ satisfies the subordination*

$$p(z) + \frac{zp'(z)}{\gamma} \prec \phi(z),$$

then $p(z) \prec q(z) \prec \phi(z)$, where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z \phi(t) t^{\gamma/n-1} dt.$$

The function q is convex function and is the best (a, n)-dominant.

Lemma 4.1.2. [**105**] *Let ϕ be a convex function in $\mathbb{D}$, with $\phi(0) = a, \gamma \neq 0$ and* $\mathrm{Re}\,\gamma \geq 0$. *Let $p \in \mathcal{H}[a, n] \cap \mathcal{Q}$ be such that $p(z) + (zp'(z))/\gamma$ is univalent in $\mathbb{D}$ and satisfies the subordination*

$$\phi(z) \prec p(z) + \frac{zp'(z)}{\gamma}.$$

Then

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z \phi(t) t^{\gamma/n-1} dt \prec p(z).$$

The function q is convex and is the best (a,n)-subordinant.

Lemma 4.1.3. [**109**] *Let w be a convex function in $\mathbb{D}$ and let the function ϕ be given by $\phi(z) = w(z) + n\delta z w'(z)$, where $\delta > 0$ and n is a positive integer. If the function $p(z) = w(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots$, is analytic in $\mathbb{D}$ and $p(z) + \delta z p'(z) \prec \phi(z)$, then $p \prec w$ and this result is sharp.*

Lemma 4.1.4. [**207**] *Let a, b and c $(c \neq 0, -1, -2, -3, \cdots)$ be real or complex parameters. Then*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z) \quad (\operatorname{Re} c > \operatorname{Re} b > 0);$$

$$(b+1)\ {}_2F_1(1,b;b+1;z) = (b+1) + bz\ {}_2F_1(1,b+1;b+2;z).$$

First we shall give a generalization to the result [**5**, Theorem 2.1] of Lupaş. For convenience, let us denote

$$B_\alpha^\delta(g,h) := \{f \in \mathcal{A} : \operatorname{Re}(\mathcal{O}_{g,h}(\alpha)f(z))' > \delta, 0 \leq \delta < 1\}.$$

The following result shows the convexity of the set $B_\alpha^\delta(g,h)$.

Theorem 4.1.5. *The set $B_\alpha^\delta(g,h)$ is convex.*

Proof. Let us assume that

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{kj} z^k \in B_\alpha^\delta(g,h) \quad (j = 1, 2).$$

Define the function H by

$$H(z) = \gamma_1 f_1(z) + \gamma_2 f_2(z),$$

where γ_1 and γ_2 are non-negative real number such that $\gamma_1 + \gamma_2 = 1$. To prove the convexity of the set $B_\alpha^\delta(g,h)$ it suffices to show that the function $H \in B_\alpha^\delta(g,h)$ or $\operatorname{Re}(\mathcal{O}_{g,h}(\alpha)H(z))' > \delta$. From the definition of H, we have

$$H(z) = z + \sum_{k=2}^{\infty}(\gamma_1 a_{k1} + \gamma_2 a_{k2})z^k.$$

A simple computation gives

$$\mathcal{O}_{g,h}(\alpha)H(z) = z + \sum_{k=2}^{\infty}(\alpha h_k + (1-\alpha)g_k)(\gamma_1 a_{k1} + \gamma_2 a_{k2})z^k$$

and

$$\operatorname{Re}(\mathcal{O}_{g,h}(\alpha)H(z))' = 1 + \operatorname{Re}\sum_{k=2}^{\infty}(\alpha h_k + (1-\alpha)g_k)[\gamma_1 a_{k1} + \gamma_2 a_{k2}]kz^{k-1}. \qquad (4.1.1)$$

Since $f_j \in B_\alpha^\delta(g,h)$, it follows that

$$\operatorname{Re}\left\{\sum_{k=2}^{\infty}(\alpha h_k + (1-\alpha)g_k)ka_{kj}z^{k-1}\right\} > \delta - 1 \quad (j = 1,2). \qquad (4.1.2)$$

It is clear from (4.1.1) and (4.1.2) that $\operatorname{Re}(\mathcal{O}_{g,h}(\alpha)H(z))' > \delta$, and hence $B_\alpha^\delta(g,h)$ is convex. □

Remark 4.1.2. Theorem 4.1.5 can be extended as follows:

If $f_j \in B_\alpha^\delta(g,h)$ $(j = 1,2,3...n)$, then $\sum_{j=1}^{n}\gamma_j f_j \in B_\alpha^\delta(g,h)$, where γ_i are non-negative real numbers such that $\sum_{j=1}^{n}\gamma_j = 1$. Note that Theorem 4.1.5 generalizes the result [**5**, Theorem 2.1] of Lupaş.

4.2 Sandwich Theorems

Theorem 4.2.1. *Let ϕ be a convex function in $\mathbb{D}$ with $\phi(0) = 1$. Assume that $\operatorname{Re}(c) > -1$, and $f \in \mathcal{A}$. Let the function F be defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt. \qquad (4.2.1)$$

1. *If $(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi(z)$, then $((\mathcal{O}_{g,h}(\alpha))F(z))' \prec q(z)$, where q is a convex function and is the best dominant given by*

$$q(z) = \frac{c+1}{z^{c+1}}\int_0^z \phi(t)t^c dt. \qquad (4.2.2)$$

2. *Let $(\mathcal{O}_{g,h}(\alpha)f(z))'$ be analytic univalent in $\mathbb{D}$. If $\phi(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))'$ and $(\mathcal{O}_{g,h}(\alpha)F(z))' \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then $q(z) \prec ((\mathcal{O}_{g,h}(\alpha))F(z))'$, where q given by (4.2.8), is a convex function and is the best subordinant.*

Proof. It can be easily seen from (4.2.1) that

$$cF(z) + zF'(z) = (c+1)f(z). \tag{4.2.3}$$

Now convoluting both sides of (4.2.3) with $h(z) = z + \sum_{k=2}^{\infty}(\alpha h_k + (1-\alpha)g_k)z^k$ and differentiating, we get

$$(\mathcal{O}_{g,h}(\alpha)F(z))' + \frac{z(\mathcal{O}_{g,h}(\alpha)F(z))''}{c+1} = (\mathcal{O}_{g,h}(\alpha)f(z))'. \tag{4.2.4}$$

In view of the first assumption of Theorem 4.2.1 namely $(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi(z)$, and (4.2.4), we have

$$(\mathcal{O}_{g,h}(\alpha)F(z))' + \frac{z(\mathcal{O}_{g,h}(\alpha)F(z))''}{c+1} \prec \phi(z). \tag{4.2.5}$$

If we assume $p(z) = (\mathcal{O}_{g,h}(\alpha)F(z))'$, then (4.2.5) becomes

$$p(z) + \frac{zp'(z)}{c+1} \prec \phi(z). \tag{4.2.6}$$

Now an application of Lemma 4.1.1 with $n = 1,\ \gamma = c + 1$ implies

$$p \prec q \quad \text{or equivalently} \quad (\mathcal{O}_{g,h}(\alpha)F(z))' \prec q(z),$$

where q given by (4.2.2), is convex and is the best dominant. The second half of the proof follows in a similar way by using Lemma 4.1.2. □

Theorem 4.2.1 leads to the following sandwich result:

Corollary 4.2.2. *Let ϕ_i $(i = 1, 2)$ be convex function in $\mathbb{D}$ with $\phi_i(0) = 1$. Let $\operatorname{Re} c > -1$, $f \in \mathcal{A}$ and F is given by (4.2.1). Assume that $(\mathcal{O}_{g,h}(\alpha)f(z))'$ is analytic univalent in $\mathbb{D}$ and $(\mathcal{O}_{g,h}(\alpha)F(z))' \in \mathcal{H}[1,1] \cap \mathcal{Q}$.*

If

$$\phi_1(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi_2(z),$$

then

$$q_1(z) \prec (\mathcal{O}_{g,h}(\alpha)F(z))' \prec q_2(z),$$

where the function q_i is given by

$$q_i(z) = \frac{c+1}{z^{c+1}} \int_0^z \phi_i(t)t^c dt \quad (i = 1, 2). \tag{4.2.7}$$

Theorem 4.2.3. *Let ϕ be a convex function in $\mathbb{D}$, with $\phi(0) = 1$ and $f \in \mathcal{A}$.*

1. *If $(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi(z)$, then $((\mathcal{O}_{g,h}(\alpha))f(z))/z \prec q(z)$, where q is a convex function and is the best dominant given by*
$$q(z) = \frac{1}{z}\int_0^z \phi(t)dt. \tag{4.2.8}$$
2. *Let $(\mathcal{O}_{g,h}(\alpha)f(z))'$ be analytic univalent in $\mathbb{D}$. If $\phi(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))'$ holds and $((\mathcal{O}_{g,h}(\alpha))f(z))/z \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then $q(z) \prec ((\mathcal{O}_{g,h}(\alpha))f(z))/z$, where q given by (4.2.2), is a convex function and is the best subordinant.*

Proof. Let us prove the first part of theorem. For this we assume
$$p(z) = \frac{\mathcal{O}_{g,h}(\alpha)f(z)}{z}. \tag{4.2.9}$$
It is clear that $p(0) = 1$ and $p \in \mathcal{H}[1,1]$. From (4.2.9), we have
$$p(z) + zp'(z) = (\mathcal{O}_{g,h}(\alpha)f(z))'. \tag{4.2.10}$$
Since $(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi(z)$, it follows from (4.2.10) that $p(z) + zp'(z) \prec \phi(z)$. Now an application of Lemma 4.1.1 with $n = 1$ and $\gamma = 1$ leads to
$$p \prec q \;\text{ or }\; (\mathcal{O}_{g,h}(\alpha)f(z))/z \prec q(z),$$
where q is given by (4.2.8) is a convex function and is the best dominant. The second half of the proof follows in a similar way by using Lemma 4.1.2. □

The following corollary is the sandwich result obtained from Theorem 4.2.3:

Corollary 4.2.4. *Let ϕ_i $(i = 1, 2)$ be convex in $\mathbb{D}$ with $\phi_i(0) = 1$. Assume that $f \in \mathcal{A}$, $((\mathcal{O}_{g,h}(\alpha))f(z))/z \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and $(\mathcal{O}_{g,h}(\alpha)f(z))'$ is analytic univalent in $\mathbb{D}$. If*
$$\phi_1(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi_2(z),$$
then
$$q_1(z) \prec ((\mathcal{O}_{g,h}(\alpha))f(z))/z \prec q_2(z),$$
where q_i be given by
$$q_i(z) = \frac{1}{z}\int_0^z \phi_i(t)dt. \tag{4.2.11}$$

Theorem 4.2.5. *Let ϕ be a convex function in $\mathbb{D}$, with $\phi(0) = 0$ and $f \in \mathcal{A}$.*

1. *If*
$$\mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi(z),$$
then $\mathcal{O}_{g,h}(\alpha)f(z) \prec q(z)$, where the function q, given by (4.2.8), is convex and is the best dominant.

2. *Let $\mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))'$ be analytic univalent in $\mathbb{D}$. If f satisfies*
$$\phi(z) \prec \mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))'$$
and $\mathcal{O}_{g,h}(\alpha)f(z) \in \mathcal{H}[0,1] \cap \mathcal{Q}$, then $q(z) \prec \mathcal{O}_{g,h}(\alpha)f(z)$, where the function q given by (4.2.8), is convex and is the best subordinant.

Proof. Let us assume that
$$p(z) = \mathcal{O}_{g,h}(\alpha)f(z). \tag{4.2.12}$$
Then clearly $p(0) = 0$ and $p \in \mathcal{H}[0,1]$. A computation using (4.2.12) yields
$$p(z) + zp'(z) = \mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))'. \tag{4.2.13}$$
Since $\mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi(z)$, (4.2.13) becomes $p(z) + zp'(z) \prec \phi(z)$. Now an application of Lemma 4.1.1 with $\gamma = n = 1$ completes the proof of first part of the theorem. The second half of the proof follows similarly using Lemma 4.1.2. □

From Theorem 4.2.5, we have the following sandwich result:

Corollary 4.2.6. *Let ϕ_i $(i = 1, 2)$ be convex function in $\mathbb{D}$, with $\phi_i(0) = 0$ and $f \in \mathcal{A}$. Assume that $\mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))'$ is analytic univalent in $\mathbb{D}$ and $\mathcal{O}_{g,h}(\alpha)f(z) \in \mathcal{H}[0,1] \cap \mathcal{Q}$. If*
$$\phi_1(z) \prec \mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi_2(z),$$
then
$$q_1(z) \prec \mathcal{O}_{g,h}(\alpha)f(z) \prec q_2(z),$$
where the function q_i $(i = 1, 2)$ is given by (4.2.11).

Theorem 4.2.7. *Let ϕ be convex function in $\mathbb{D}$, with $\phi(0) = 1$ and $f \in \mathcal{A}$.*

1. *If $(\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))'' \prec \phi(z)$, then $(\mathcal{O}_{g,h}(\alpha)f(z))' \prec q(z)$, where the function q, given by (4.2.8) is convex and is the best dominant.*

2. *Let $(\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))''$ is analytic univalent in $\mathbb{D}$. If f satisfies*

$$\phi(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))'',$$

   and $\mathcal{O}_{g,h}(\alpha)f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then $q(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))'$, where the function q, given by (4.2.8), is convex and is the best subordinant.

Proof. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = (\mathcal{O}_{g,h}(\alpha)f(z))'. \tag{4.2.14}$$

Then clearly $p(0) = 0$ and $p \in \mathcal{H}[0,1]$. A computation using (4.2.14) yields

$$p(z) + zp'(z) = (\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))''. \tag{4.2.15}$$

Since $(\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))'' \prec \phi(z)$, the expression given in (4.2.15) becomes $p(z) + zp'(z) \prec \phi(z)$. Now an application of Lemma 4.1.1 with $\gamma = n = 1$ completes the proof of first part of the theorem. The second half of the proof follows by a similar application of Lemma 4.1.2. □

We have the following sandwich result from Theorem 4.2.7:

Corollary 4.2.8. *Let ϕ_i $(i = 1,2)$ be convex function in $\mathbb{D}$, with $\phi_i(0) = 1$. Assume that $f \in \mathcal{A}$, $(\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))''$ is analytic univalent in $\mathbb{D}$ and $\mathcal{O}_{g,h}(\alpha)f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$. If the following holds*

$$\phi_1(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))'' \prec \phi_2(z),$$

then $q_1(z) \prec (\mathcal{O}_{g,h}(\alpha)f(z))' \prec q_2(z)$, where the function q_i $(i = 1,2)$ is given by (4.2.11).

Theorem 4.2.9. *Let q be a convex function in $\mathbb{D}$ with $\phi(0) = 1$. Further assume that $f \in \mathcal{A}$ and F is defined by (4.2.1). If the subordination*

$$(\mathcal{O}_{g,h}(\alpha)f(z))' \prec q(z) + \frac{zq'(z)}{c+1} \quad (c > -1),$$

then $(\mathcal{O}_{g,h}(\alpha)F(z))' \prec q(z)$.

Proof. Let $p(z) = (\mathcal{O}_{g,h}(\alpha)F(z))'$. Proceeding as described in the proof of Theorem 4.2.1, we arrive at

$$p(z) + \frac{zp'(z)}{c+1} \prec q(z) + \frac{zq'(z)}{c+1}.$$

Now the result follows at once by an application of Lemma 4.1.3. □

Theorem 4.2.10. *Let q be a convex function in $\mathbb{D}$ with $q(0) = 1$. If $f \in \mathcal{A}$ satisfies $(\mathcal{O}_{g,h}(\alpha)f(z))' \prec q(z) + zq'(z)$, then $((\mathcal{O}_{g,h}(\alpha))f(z))/z \prec q(z)$. The function q is convex and is the best dominant.*

Proof. Proceeding as in the proof of Theorem 4.2.3, we have (4.2.10). Now (4.2.10) together with $\phi(z) = q(z) + zq'(z)$ lead to $p(z) + zp'(z) \prec q(z) + zq'(z)$. Now an application of Lemma 4.1.3 completes the proof. □

The following theorems can be proved in a similar way to that of Theorem 4.2.10:

Theorem 4.2.11. *Let q be a convex function in $\mathbb{D}$ and $\phi(z) = q(z) + zq'(z)$. Assume that $f \in \mathcal{A}$. If the subordination $\mathcal{O}_{g,h}(\alpha)f(z) + z(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \phi(z)$ holds, then $\mathcal{O}_{g,h}(\alpha)f(z) \prec q(z)$.*

Theorem 4.2.12. *Let q be a convex function in $\mathbb{D}$ and $\phi(z) := q(z) + zq'(z)$. Assume that $f \in \mathcal{A}$. If the subordination $(\mathcal{O}_{g,h}(\alpha)f(z))' + z(\mathcal{O}_{g,h}(\alpha)f(z))'' \prec \phi(z)$ holds, then $(\mathcal{O}_{g,h}(\alpha)f(z))' \prec q(z)$.*

4.3 Applications

In this section, we shall choose distinct functions as dominant satisfying the conditions of general subordination theorems proved in Section 4.2. As a consequence, we shall get some new and interesting results.

Corollary 4.3.1. *Let $f \in \mathcal{A}$ and F be defined by (4.2.1). If* $\operatorname{Re} c > -1$ *and*

$$f'(z) + (1-\alpha)zf''(z) \prec \frac{1+(2\beta-1)z}{1+z} \quad (\beta < 1), \tag{4.3.1}$$

then the following holds:

$$F'(z) + (1-\alpha)zF''(z) \prec 2(1-\beta)\ {}_2F_1(1, c+1; c+2; -z) + 2\beta - 1. \tag{4.3.2}$$

The function on the right of (4.3.2) is convex and is the best dominant.

Proof. If we take $g(z) = z/(1-z)$ and $h(z) = z/(1-z)^2$, then it is easy to see that $(f * g)(z) = f(z)$, $(f * h)(z) = zf'(z)$ and $\mathcal{O}_{g,h}(\alpha)f(z) = (1-\alpha)f(z) + \alpha zf'(z)$ and thus, $(\mathcal{O}_{g,h}(\alpha)f(z))' = f'(z) + \alpha zf''(z)$, and $(\mathcal{O}_{g,h}(\alpha)F(z))' = F'(z) + \alpha zF''(z)$.

Now by setting $\phi(z) = (1+(1-2\beta)z)/(1+z)$ $(\beta < 1)$ in the first part of Theorem 4.2.1 yields (4.3.1). Further we have

$$q(z) = \frac{c+1}{z^{c+1}} \int_0^z \frac{1-(1-2\beta)t}{1+t} t^c dt.$$

Now a computation using Lemma 4.1.4, yields

$$q(z) = 2(1-\beta)\ {}_2F_1(1, c+1; c+2; -z) + 2\beta - 1.$$

This completes the proof. □

Setting $c = 0$ and $\alpha = 1$, Corollary 4.3.1 reduces to the following result.

Corollary 4.3.2. [**108**, Lemma 5.5k] *Let $f \in \mathcal{A}$ and F be defined by (4.2.1). If the following subordination holds*

$$f'(z) \prec \frac{1-(1-2\beta)z}{1+z} \quad (\beta < 1),$$

then we have $F'(z) \prec 2(1-\beta)\ {}_2F_1(1, 1; 2; -z) + 2\beta - 1$. The function on the right of (4.3.2) is convex and is the best dominant.

By taking

$$g(z) = \frac{z}{1-z}, \quad h(z) = \frac{z}{(1-z)^2} \quad \text{and} \quad \phi(z) = \frac{1+(1-2\beta)z}{1-z} \quad (0 \le \beta < 1)$$

in Theorem 4.2.3, we obtain the following result:

Corollary 4.3.3. *Let* $f \in \mathcal{A}$ *satisfies* $\operatorname{Re}(f'(z) + (1-\alpha)zf''(z)) > \beta$. *Then*

$$\operatorname{Re}\left(\frac{\alpha f(z) + (1-\alpha)zf'(z)}{z}\right) > 2(\beta - 1)\ln 2 + 2\beta - 1.$$

Remark 4.3.1. The above result generalizes the result of Owa et al. [**128**, Corollary 1] and when $\alpha = 0$, the result reduces to the result [**69**, Theorem 6] of Hallenbeck.

Corollary 4.3.4. [**69**, Theorem 6] *If* $f \in \mathcal{A}$ *satisfies* $\operatorname{Re}(f'(z) + zf''(z)) > \beta$, *then*

$$\operatorname{Re} f'(z) > 2(\beta - 1)\ln 2 + 2\beta - 1.$$

Corollary 4.3.5. [**128**, Corollary 1] *If* $f \in \mathcal{A}$ *satisfies* $\operatorname{Re} f'(z) > \beta$, *then*

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > 2(\beta - 1)\ln 2 + 2\beta - 1.$$

Remark 4.3.2. Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by

$$q(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \le \beta < 1).$$

Then q is convex in $\mathbb{D}$ with $q(0) = 1$. Further by setting $g(z) = z/(1-z)$ and $h(z) = z/(1-z)^2$ in Theorem 4.2.10 we see that

$$\frac{\alpha f(z) + (1-\alpha)zf'(z)}{z} \prec \frac{1 + (1-2\beta)z}{1-z}$$

whenever the following subordination holds:

$$f'(z) + (1-\alpha)zf''(z) \prec \frac{1 + (1-2\beta)z}{1-z} + \frac{2(1-\beta)z}{(1-z)^2}.$$

Thus, we have the following result:

Corollary 4.3.6. *If* $f \in \mathcal{A}$ *satisfies*

$$\operatorname{Re}(f'(z) + (1-\alpha)zf''(z)) > \frac{3\beta - 1}{2} \quad (0 \le \beta < 1),$$

then

$$\operatorname{Re}\left(\frac{\alpha f(z) + (1-\alpha)zf'(z)}{z}\right) > \beta.$$

Corollary 4.3.6 reduces to the following results for the choices of $\alpha = 0$, and 1 respectively.

Corollary 4.3.7. [**180**, Example 3.5] *If* $f \in \mathcal{A}$ *satisfies*

$$\operatorname{Re}(f'(z) + zf''(z)) > \frac{3\beta - 1}{2} \quad (0 \leq \beta < 1),$$

then $\operatorname{Re} f'(z) > \beta$.

Corollary 4.3.8. [**125**, Theorem 2] *If* $f \in \mathcal{A}$ *satisfies*

$$\operatorname{Re} f'(z) > \frac{3\beta - 1}{2} \quad (0 \leq \beta < 1),$$

then $\operatorname{Re}(f(z)/z) > \beta$.

Setting $g(z) = h(z) = \phi(z) = z/(1-z)^2$ in the first part of Theorem 4.2.5, we obtain the following result:

Example 4.3.9. Let $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}(\alpha f(z) + (2-\alpha) z f'(z) + (1-\alpha) z^2 f''(z)) > -\frac{1}{2}.$$

Then $\operatorname{Re}(\alpha f(z) + (1-\alpha) z f'(z)) > \ln 2 - 1$.

By taking

$$g(z) = h(z) = \frac{z}{1-z} \text{ and } \phi(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \leq \beta < 1)$$

in the first part of Theorems 4.2.7 and 4.2.12, we deduce the following results respectively:

Example 4.3.10. Let $f \in \mathcal{A}$ satisfies $\operatorname{Re}(f(z) + zf'(z)) > \beta$ $(0 \leq \beta < 1)$, then

$$\operatorname{Re} f(z) > 2(1-\beta)\ln 2 + 2\beta - 1.$$

Example 4.3.11. If $f \in \mathcal{A}$ satisfies the inequality

$$\operatorname{Re}(f'(z) + zf''(z)) > 2(1-\beta)\ln 2 + 2\beta - 1 \quad (0 \leq \beta < 1),$$

then $\operatorname{Re} f'(z) > \beta$.

Corollary 4.3.12. *Let $f \in \mathcal{A}$ and the function F be as defined in (4.2.1). If the following inequality holds*

$$\mathrm{Re}(\mathcal{O}_{g,h}(\alpha)f(z))' > \beta + \frac{\beta - 1}{2(c+1)} \quad (0 \leq \beta < 1, c > -1), \tag{4.3.3}$$

then $\mathrm{Re}(\mathcal{O}_{g,h}(\alpha)F(z))' > \beta$*. The result is best possible.*

Proof. Let us define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1).$$

Clearly q is convex in $\mathbb{D}$ with $q(0) = 1$. A calculation shows that

$$q(z) + \frac{zq'(z)}{c+1} = \frac{1 + (1 - 2\beta)z}{1 - z} - \frac{2(1 - \beta)z}{(c+1)(1 - z)^2}.$$

Now an application of Theorem 4.2.9 yields

$$(\mathcal{O}_{g,h}(\alpha)f(z))' \prec \frac{1 + (1 - 2\beta)z}{1 - z} - \frac{2(1 - \beta)z}{(c+1)(1 - z)^2}.$$

This implies (4.3.3) and hence the result follows at once. □

Remark 4.3.3. By setting $\beta = 0$ in Corollary 4.3.12, we have the inequality

$$\mathrm{Re}(\mathcal{O}_{g,h}(\alpha)f(z))' > \frac{-1}{2(c+1)} \quad (c > -1)$$

which implies that $\mathrm{Re}(\mathcal{O}_{g,h}(\alpha)F(z))' > 0$. This shows that $\mathcal{O}_{g,h}(\alpha)F(z)$ is univalent.

Chapter 5

Fekete-Szegö Coefficient Inequality for Certain Classes of Analytic Functions

5.1 Introduction

Finding the necessary conditions for functions satisfying certain geometric properties have been a core area of research. The Bierberbach conjecture and its proof by de Branges was the main attraction for many researchers working in GFT till 1985. Since many of them were attempting to prove or disprove the Bierberbach conjecture, the Fekete-Szegö coefficient problem could not get much attention during that time, but at the same time the Fekete–Szegö coefficient problems for the classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{CC}$ were settled by Keogh and Merkes [**83**]. Further Ma–Minda [**101**] and Ravichandran [**18**] explored the Fekete–Szegö coefficient problem for more general classes. In this direction, Obradović [**123**], in 2010, introduced a class of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1}\right\} > 0 \quad (0 < \lambda < 1).$$

Most of the results of this chapter appeared in [**176–178**].

He discussed the starlikeness criteria of the functions satisfying the above condition. Tuneksi and Darus [**202**] generalized this class by introducing the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left\{ f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1} \right\} > \alpha \quad (0 \leq \alpha < 1, 0 < \lambda < 1) \tag{5.1.1}$$

and they obtained the estimate on Fekete-Szegö functional for functions in this class. The above expressions are some what similar to the analytic representation of starlike functions. It should be noted that the analytic representations of convex and starlike functions can be written in terms of Hadamard product. The expression $zf'(z)/f(z)$ can be written as $(f*K_1)(z)/(f*K_2)(z)$ where $K_1(z) = z/(1-z)^2$ and $K_2(z) = z/(1-z)$. Similarly $1+zf''(z)/f'(z)$ can also be written by taking $K_1(z) = (z+z^2)/(1-z)^3$ and $K_2(z) = z/(1-z)$. Using this fact Murugusundaramoorthy et al. [**112**], in 2007, for $\varphi \in \mathcal{P}$, with the image domain $\varphi(\mathbb{D})$ symmetrical with respect to the real axis, starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 1$, introduced a class $\mathcal{M}_{g,h}(\varphi)$ such that $f \in \mathcal{A}$ satisfying

$$\frac{(f*g)(z)}{(f*h)(z)} \prec \varphi(z) \quad (g_n > h_n > 0),$$

where $g, h \in \mathcal{A}$, and are given by

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \quad h(z) = z + \sum_{n=2}^{\infty} h_n z^n \tag{5.1.2}$$

and obtained the estimate on the Fekete-Szegö functional for the class $\mathcal{M}_{g,h}(\varphi)$. All the linear operators investigated in GFT can be written as $L(f,g) = f*g$ for some suitable $g \in \mathcal{A}$. Motivated by the above defined classes and using the generalized Sălăgean operator [**24**], Răducanu [**151**] introduced the class $\mathcal{M}_{\alpha,\beta}^{n,\lambda}(\varphi)$ defined by

$$\left(\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}\right)^{\alpha} \left(\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)}\right)^{\beta} \prec \varphi(z) \quad (\alpha, \beta, \lambda \geq 0; f \in \mathcal{A})$$

and discussed the Fekete-Szegö problem for functions in this class.

It has been observed that the univalence or starlikeness of φ is required only in proving the growth, distortion and covering theorems. For estimate on coefficients we do not require any such assumptions, we can take the superordinate function with suitable normalization with positive real part. Since our main motive in this chapter is to discuss the Fekete-Szegö coefficient estimate, here after throughout this chapter, we assume that $\varphi \in \mathcal{P}$ has the form $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, B_1 > 0$, and $B_2 \in \mathbb{R}$. Further, in Sections 5.2, 5.3 and 5.4, we shall consider three different classes of analytic functions defined in terms of convolution and subordination as follows:

Definition 5.1.1. Let α and β be real numbers. Assume that g and h are given by

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad \text{and} \quad h(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{5.1.3}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$, if it satisfies

$$\left(\frac{(f*g)(z)}{z}\right)^{\alpha} \left(\frac{(f*h)(z)}{z}\right)^{\beta} \prec \varphi(z),$$

where the powers are principal one.

Note that the definition explicitly assumes that $(f*g)(z) \neq 0, (f*h)(z) \neq 0$ for $z \neq 0$. For appropriate choice of the functions g, h, φ and constants α and β, the class $\mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$ reduces to the following classes:

1. $\mathcal{M}_{g,h}^{1,-1}(\varphi) =: \mathcal{M}_{g,h}(\varphi)$, the class introduced by Murugusundaramoorthy et al. [**112**].
2. $\mathcal{M}^{1,-1}_{\frac{z}{(1-z)^2}, \frac{z}{1-z}}(\varphi) =: \mathcal{S}^*(\varphi)$, the class of Ma-Minda Starlike functions.
3. $\mathcal{M}^{1,-1}_{\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}}(\varphi) =: \mathcal{K}(\varphi)$, the class of Ma-Minda convex functions.
4. If $g(z) = z/(1-z)^2$, $h(z) = z/(1-z)$ and $\varphi(z) = (1+z)/(1-z)$, then class $\mathcal{M}_{g,h}^{1,-(\lambda+1)}(\varphi)$ $(0 < \lambda < 1)$ reduces to the class introduced by Obradović [**123**].

Definition 5.1.2. Let g and h be given by (5.1.3) with $g_n > h_n > 0$. A function $f \in \mathcal{A}$ given by (1.1.1) is said to be in the class $\mathcal{N}_{g,h}(\alpha, \varphi)$, if it satisfies

$$(1-\alpha)\frac{(f*g)(z)}{(f*h)(z)} + \alpha\frac{(f*g)'(z)}{(f*h)'(z)} \prec \varphi(z) \quad (\alpha \geq 0). \tag{5.1.4}$$

Remark 5.1.1. For various choices of the functions g, h, φ and the real number α, the class $\mathcal{N}_{g,h}(\alpha,\varphi)$ reduces to several known classes, we enlist a few of them below:

1. The class $\mathcal{N}_{g,h}(0,\varphi) =: M_{g,h}(\varphi)$, introduced and studied by Murugusundaramoorthy et al. [**112**].

2. If we set
$$g(z) = \frac{z}{(1-z)^2}, \quad h(z) = \frac{z}{(1-z)} \tag{5.1.5}$$
and $\varphi(z) = (1+z)/(1-z)$, then the class $\mathcal{N}_{g,h}(\alpha,\varphi)$ reduces to the class $\mathcal{M}(\alpha)$ of α–convex functions.

3. $\mathcal{N}_{\frac{z}{(1-z)^2},\frac{z}{(1-z)}}(\alpha,\varphi) =: \mathcal{M}(\alpha,\varphi)$, the class introduced by Ali et al. [**21**].

4. $\mathcal{N}_{\frac{z}{(1-z)^2},\frac{z}{(1-z)}}(0,\varphi) =: \mathcal{S}^*(\varphi)$ and $\mathcal{N}_{\frac{z}{(1-z)^2},\frac{z}{(1-z)}}(1,\varphi) =: \mathcal{C}(\varphi)$, the classes introduced by Ma and Minda [**101**].

Definition 5.1.3. Let $\alpha \geq 0$. For a fixed function $g \in \mathcal{A}$ given by (5.1.3), the class $\mathcal{S}_g^\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ of the form (1.1.1) satisfying
$$1 + \frac{z(f*g)'(z)}{(f*g)(z)} + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{(1-\alpha)z^2(f*g)''(z) + z(f*g)'(z)}{(1-\alpha)z(f*g)'(z) + \alpha(f*g)(z)} \prec \varphi(z). \tag{5.1.6}$$

Note that the above class $\mathcal{S}_g^\alpha(\varphi)$, in fact generalizes several known classes, a few are enlisted below:

1. For $g(z) = z/(1-z)$, we have $\mathcal{S}_g^0(\varphi) =: \mathcal{S}^*(\varphi)$ and $\mathcal{S}_g^1(\varphi) =: \mathcal{K}(\varphi)$, the classes introduced by Ma and Minda [**101**].

2. If we take $g(z) = z + \sum_{n=2}^\infty n^m z^n$, then $(f*g)(z)$ becomes the Sălăgean [**162**] operator $\mathcal{D}^m$ defined by
$$\mathcal{D}^m f(z) = z + \sum_{n=2}^\infty n^m a_n z^n \quad (m \in \{0,1,2,3,\ldots\}).$$

3. Further, if we set $\varphi(z) = (1+z)/(1-z)$ and $g = z + \sum_{n=2}^\infty n^m z^n$ in the above Definition 5.1.3, then the class $\mathcal{S}_g^\alpha(\varphi)$ reduces to the class $\mathcal{HS}_m^*(\alpha)$ introduced by Răducanu [**151**]. He investigated the relationship property between the classes $\mathcal{HS}_m^*(\alpha)$ and $\mathcal{S}^*$ and obtained the Fekete-Szegö inequality for the class $\mathcal{HS}_m^*(\alpha)$.

Preliminaries

Recently Ali et al. [**18**] reformulated the results of Ma and Minda (see Lemma 1.1.1) and Keogh and Merkes (see Lemma 1.1.2) as mentioned below:

Lemma 5.1.1. [**18**] *If* $w \in \mathbf{B}$ *and* $w(z) = w_1 z + w_2 z^2 + \cdots (z \in \mathbb{D})$, *then*

$$|w_2 - tw_1^2| \leq \begin{cases} -t & (t \leq -1), \\ 1 & (-1 \leq t \leq 1), \\ t & (t \geq 1). \end{cases}$$

For $t < -1$ *or* $t > 1$, *equality holds if and only if* $w(z) = z$ *or one of its rotations. For* $-1 < t < 1$, *equality holds if and only if* $w(z) = z^2$ *or one of its rotations. Equality holds for* $t = -1$ *if and only if* $w(z) = z(\lambda + z)/(1 + \lambda z)$ $(0 \leq \lambda \leq 1)$ *or one of its rotations, while for* $t = 1$, *equality holds if and only if* $w(z) = -z(\lambda + z)/(1 + \lambda z)$ $(0 \leq \lambda \leq 1)$ *or one of its rotations. Also the sharp upper bound can be improved for* $-1 < t < 1$,

$$|w_2 - tw_1^2| + (1 + t)|w_1|^2 \leq 1 \quad (-1 < t \leq 0) \tag{5.1.7}$$

and

$$|w_2 - tw_1^2| + (1 - t)|w_1|^2 \leq 1 \quad (0 \leq t < 1). \tag{5.1.8}$$

Lemma 5.1.2. [**18, 83**] *If* $w \in \mathbf{B}$ *and* $w(z) = w_1 z + w_2 z^2 + \cdots (z \in \mathbb{D})$, *then for any complex number* t,

$$|w_2 - tw_1^2| \leq \max\{1; |t|\}$$

and the result is sharp for the functions given by $w(z) = z^2$ *or* $w(z) = z$.

Further Ali et al. [**18**] utilized the above results to discuss the estimate on the Fekete-Szegö functional for $p-$valent analytic functions. This motivates us to discuss the Fekete-Szegö problem for classes $\mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$, $\mathcal{N}_{g,h}(\alpha, \varphi)$ and $\mathcal{S}_g^{\alpha}(\varphi)$ of analytic functions by considering $\varphi \in \mathcal{P}$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$.

5.2 Fekete-Szegö Inequality for $\mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$

Throughout this section, it is assumed that $\alpha g_n + \beta h_n > 0$, and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Theorem 5.2.1. *If $f \in \mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$, then for any real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1 A}{(\alpha g_3 + \beta h_3)} & (\mu \leq \sigma_1), \\ \frac{B_1}{(\alpha g_3 + \beta h_3)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ -\frac{B_1 A}{(\alpha g_3 + \beta h_3)} & (\mu \geq \sigma_2), \end{cases}$$

where

$$A := \frac{B_2}{B_1} - \frac{[\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]B_1}{2(\alpha g_2 + \beta h_2)^2},$$

$$\sigma_1 := \frac{2(B_2 - B_1)(\alpha g_2 + \beta h_2)^2 - [\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2]B_1^2}{2(\alpha g_3 + \beta h_3)B_1^2}$$

and

$$\sigma_2 := \frac{2(B_2 + B_1)(\alpha g_2 + \beta h_2)^2 - [\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2]B_1^2}{2(\alpha g_3 + \beta h_3)B_1^2}.$$

The result is sharp.

Proof. Since $f \in \mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$, there exists $w(z) = w_1 z + w_2 z^2 + \cdots \in \mathbf{B}$ such that

$$\left(\frac{(f*g)(z)}{z}\right)^{\alpha} \left(\frac{(f*h)(z)}{z}\right)^{\beta} = \varphi(w(z)). \tag{5.2.1}$$

By a computation, we get

$$\left(\frac{(f*g)(z)}{z}\right)^{\alpha} = 1 + \alpha a_2 g_2 z + \left(\alpha a_3 g_3 + \frac{\alpha(\alpha-1)}{2} a_2^2 g_2^2\right) z^2 + \cdots$$

and

$$\left(\frac{(f*h)(z)}{z}\right)^{\beta} = 1 + \beta a_2 h_2 z + \left(\beta a_3 h_3 + \frac{\beta(\beta-1)}{2} a_2^2 h_2^2\right) z^2 + \cdots .$$

Substituting these in (5.2.1) and comparing the coefficients, we have

$$(\alpha g_2 + \beta h_2) a_2 = B_1 w_1 \tag{5.2.2}$$

and

$$(\alpha g_3 + \beta h_3) a_3 + (\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2)\frac{a_2^2}{2} = B_1 w_2 + B_2 w_1^2. \tag{5.2.3}$$

From (5.2.2) and (5.2.3), we obtain

$$a_3 - \mu a_2^2 = \frac{B_1}{2(\alpha g_3 + \beta h_3)}(w_2 - t w_1^2),$$

where

$$t := -\frac{B_2}{B_1} + \frac{[\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]B_1}{2(\alpha g_2 + \beta h_2)^2}. \tag{5.2.4}$$

The result now follows by an application of Lemma 5.1.1. Further if

$$-\frac{B_2}{B_1} + \frac{[\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]B_1}{2(\alpha g_2 + \beta h_2)^2} \leq -1,$$

then, for

$$\mu \leq \frac{2(B_2 - B_1)(\alpha g_2 + \beta h_2)^2 - [\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2]B_1^2}{2(\alpha g_3 + \beta h_3)B_1^2} := \sigma_1$$

and A be as given in the statement of Theorem 5.2.1, we have

$$|a_3 - \mu a_2^2| \leq \frac{B_1 A}{(\alpha g_3 + \beta h_3)}.$$

Further if $-1 \leq t \leq 1$, then for $\sigma_1 \leq \mu \leq \sigma_2$, we have

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{(\alpha g_3 + \beta h_3)}.$$

Similarly if $t \geq 1$, for $\mu \geq \sigma_2$, we have

$$|a_3 - \mu a_2^2| \leq -\frac{B_1 A}{(\alpha g_3 + \beta h_3)}.$$

This completes the proof. To show sharpness of bounds, we now define the functions K_{ϕ_n} $(n = 2, 3, 4, \cdots)$ by

$$\left(\frac{(K_{\phi_n} * g)(z)}{z}\right)^{\alpha} \left(\frac{z}{(K_{\phi_n} * h)(z)}\right)^{\beta} = \phi(z^{n-1}), \quad K_{\phi_n}(0) = 0 = (K_{\phi_n})'(0) - 1$$

and the functions G_γ and H_γ $(0 \leq \gamma \leq 1)$ are defined by

$$\left(\frac{(G_\gamma * g)(z)}{z}\right)^{\alpha} \left(\frac{z}{(G_\gamma * h)(z)}\right)^{\beta} = \phi\left(\frac{z(z+\gamma)}{1+\gamma z}\right), \text{ with } G_\gamma(0) = 0 = (G_\gamma)'(0) - 1$$

and

$$\left(\frac{(H_\gamma * g)(z)}{z}\right)^{\alpha} \left(\frac{z}{(H_\gamma * h)(z)}\right)^{\beta} = \phi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right), \text{ with } H_\gamma(0) = 0 = (H_\gamma)'(0) - 1.$$

It is clear that the functions K_{ϕ_n} $(n = 2, 3, 4, \cdots), G_\gamma$ and H_γ $(0 \leq \gamma \leq 1)$ are in the class $M_{g,h}^{\alpha,\beta}(\phi)$. In either cases $\mu < \sigma_1$ or $\mu > \sigma_2$, the equality holds if and only if f is K_{ϕ_2} or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality occurs if and only if f is K_{ϕ_3} or one of its rotations. If $\mu = \sigma_1$ then equality holds if and only if f is G_λ or one of its rotations. If $\mu = \sigma_2$ then the quantity holds if and only if f is H_λ or one of its rotations. □

Remark 5.2.1. If $\sigma_1 \leq \mu \leq \sigma_2$, then the estimate given in Theorem 5.2.1 can be improved by bifurcating the interval as follows: Let

$$\sigma_3 := \frac{2B_2(\alpha g_2 + \beta h_2)^2 - [\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2]B_1^2}{2(\alpha g_3 + \beta h_3)B_1^2}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + R_1 \leq \frac{B_1}{\alpha g_3 + \beta h_3},$$

where

$$R_1 := \frac{2(B_1 - B_2)(\alpha g_2 + \beta h_2)^2 + [\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]B_1^2}{2(\alpha g_3 + \beta h_3)B_1^2}|a_2|^2.$$

Similarly if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + R_2 \leq \frac{B_1}{\alpha g_3 + \beta h_3},$$

where

$$R_2 := \frac{2(B_2 + B_1)(\alpha g_2 + \beta h_2)^2 + [\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]B_1^2}{2(\alpha g_3 + \beta h_3)B_1^2}|a_2|^2.$$

Remark 5.2.2. When $\alpha = 1$ and $\beta = -1$, Theorem 5.2.1 reduces to the result [**112**, Theorem 2.1] of Murugusundaramoorthy et al.

Theorem 5.2.2. *If $f \in \mathcal{M}_{g,h}^{\alpha,\beta}(\varphi)$, then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(\alpha g_2 + \beta h_2)} \max\{1; |R|\},$$

where

$$R := \frac{[\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]B_1}{2(\alpha g_2 + \beta h_2)^2} - \frac{B_2}{B_1}.$$

Proof. Proceeding in the similar way as in the proof of Theorem 5.2.1, using (5.2.4) and Lemma 1.1.2, the result immediately follows. □

Here below, we discuss some special cases of our main results of this section.

Theorem 5.2.3. *Let the functions g and h are given by*

$$g(z) = z + \sum_{n=2}^{\infty} \frac{n\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n-\delta+1)} z^n \quad \text{and} \quad h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n-\delta+1)} z^n.$$

If $f \in M^{\alpha,\beta}_{g,h}(\varphi)$*, then for any real number* μ*, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\delta)(3-\delta)AB_1}{6(3\alpha+\beta)} & (\mu \leq \sigma_1), \\ \frac{(2-\delta)(3-\delta)B_1}{6(3\alpha+\beta)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ -\frac{(2-\delta)(3-\delta)AB_1}{6(3\alpha+\beta)} & (\mu \geq \sigma_2), \end{cases}$$

where

$$A := \frac{B_2}{B_1} - \frac{[(4\alpha(\alpha-1)+\beta(\beta-1)+4\alpha\beta)(3-\delta)+3\mu(2-\delta)(3\alpha+\beta)]B_1}{(2\alpha+\beta)^2(3-\delta)},$$

$$\sigma_1 := \frac{(3-\delta)[2(B_1-B_2)(2\alpha+\beta)^2-(4\alpha(\alpha-1)+\beta(\beta-1)+4\alpha\beta)B_1^2]}{3(2-\delta)(3\alpha+\beta)B_1^2}$$

and

$$\sigma_2 := \frac{(3-\delta)[2(B_1+B_2)(2\alpha+\beta)^2-(4\alpha(\alpha-1)+\beta(\beta-1)+4\alpha\beta)B_1^2]}{3(2-\delta)(3\alpha+\beta)B_1^2}.$$

Remark 5.2.3. If we set $\alpha = 1$ and $\beta = -1$ in the Theorem 5.2.3, it reduces to the result [**112**, Corollary 3.2] of Murugusundaramoorthy et al. For $\alpha = 1$, $\beta = -1$, $B_1 = 8/\pi^2$, $B_2 = 16/3\pi^2$ and $\delta = 1$, Theorem 5.2.3 reduces to the result [**102**, Theorem 2] of Ma and Minda.

If we set $g(z) = z/(1-z)^2, h(z) = z/(1-z)$ and $\varphi(z) = (1+z)/(1-z)$ in Theorem 5.2.1, we have the following result.

Corollary 5.2.4. [**83, 188**] *If* $f \in \mathcal{S}^*$*, then for any real number* μ*, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu & \text{if } \mu \leq \frac{1}{2}; \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases}$$

If we take

$$\varphi(z) = \frac{1+Cz}{1+Dz} \quad (-1 \leq D < C \leq 1)$$

in Theorem 5.2.2, it reduces to the following:

Corollary 5.2.5. *Let* $f \in \mathcal{M}^{\alpha,\beta}_{g,h}((1+Cz)/(1+Dz))$*. Then for any complex number* μ*, we have*

$$|a_3 - \mu a_2^2| \leq \frac{C-D}{2(\alpha g_2 + \beta h_2)} \max\{1; |R_1|\},$$

where

$$R_1 = D + \frac{[\alpha(\alpha-1)g_2^2 + \beta(\beta+1)h_2^2 - 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 - \beta h_3)](C-D)}{2(\alpha g_2 + \beta h_2)^2}.$$

Setting $g(z) = z/(1-z)^2, h(z) = z/(1-z), \alpha = 1$ and $\beta = -\lambda - 1,\ \lambda < 1$ in Theorem 5.2.2, we deduce the following result:

Corollary 5.2.6. *Let $f \in \mathcal{A}$ satisfies*

$$f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1} \prec \frac{1+Cz}{1+Dz},$$

then for any complex number μ, we have

$$|a_3 - \mu a_2^2| \leq \frac{C-D}{2-\lambda} \max\left\{1; \left|D + \frac{(1+\lambda-2\mu)(\lambda-2)(C-D)}{(1-\lambda)^2}\right|\right\}.$$

Remark 5.2.4. For $C = 1-2a, 0 \leq a < 1,\ 0 < \lambda < 1$ and $D = -1$, the Corollary 5.2.6 reduces to [**202**, Theorem 1] of Tuneski and Darus. Note that our proof is quite different from the one given by Tuneski and Darus [**202**]. There was a typographical error in the assertion of [**202**, Theorem 1], and it is rectified in the following result.

Corollary 5.2.7. [**202**, Theorem 1] *Let $0 \leq a < 1$ and $0 < \lambda < 1$. If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1}\right\} > a,$$

then for any complex number μ, we have

$$|a_3 - \mu a_2^2| \leq \frac{2(1-a)}{2-\lambda} \max\left\{1; \left|1 + \frac{(1+\lambda-2\mu)(2-\lambda)(1-a)}{(1-\lambda)^2}\right|\right\}.$$

Remark 5.2.5. For $a = 0$, the Corollary 5.2.7 reduces to the result [**202**, Corollary 1] of Tuneski and Darus. Setting $C = k\ (0 < k \leq 1)$ and $D = 0$ in Corollary 5.2.6, we obtain the following result of Tuneski and Darus [**202**, Theorem 2].

Corollary 5.2.8. [**202**, Theorem 2] *Let $0 < \lambda < 1$ and $0 < k \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1} - 1\right| < k,$$

then for all complex number μ, we have

$$|a_3 - \mu a_2^2| \leq \frac{k}{1-\lambda} \max\left\{1; \frac{k}{(1-\lambda)^2}\left|\frac{(1+\lambda-2\mu)(\lambda-2)}{2}\right|\right\}.$$

5.3 Fekete-Szegö Inequality for $\mathcal{N}_{g,h}(\alpha, \varphi)$

Throughout this section we shall assume that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = z + \sum_{n=2}^{\infty} g_n z^n \text{ and } h(z) = z + \sum_{n=2}^{\infty} h_n z^n.$$

Theorem 5.3.1. *If $f \in \mathcal{N}_{g,h}(\alpha, \varphi)$, then for any real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1 A}{(1+2\alpha)(g_3-h_3)} & (\mu \leq \sigma_1), \\ \frac{B_1}{(1+2\alpha)(g_3-h_3)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{B_1 A}{(1+2\alpha)(h_3-g_3)} & (\mu \geq \sigma_2), \end{cases} \tag{5.3.1}$$

where

$$A = \frac{B_2}{B_1} - \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2},$$

$$\sigma_1 := \frac{(B_2 - B_1)(1+\alpha)^2(g_2 - h_2)^2 - (1+3\alpha)(h_2^2 - h_2 g_2)B_1^2}{(1+2\alpha)(g_3 - h_3)B_1^2}$$

and

$$\sigma_2 := \frac{(B_2 + B_1)(1+\alpha)^2(g_2 - h_2)^2 - (1+3\alpha)(h_2^2 - h_2 g_2)B_1^2}{(1+2\alpha)(g_3 - h_3)B_1^2},$$

and for any complex number μ

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(1+2\alpha)(g_3 - h_3)} \max\{1; |t|\}, \tag{5.3.2}$$

where

$$t := -\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2}. \tag{5.3.3}$$

The inequality is sharp.

Proof. If $f \in \mathcal{N}_{g,h}(\alpha, \varphi)$, then there exists $w(z) = w_1 z + w_2 z^2 + \cdots \in \mathbf{B}$ such that

$$(1-\alpha)\frac{(f*g)(z)}{(f*h)(z)} + \alpha\frac{(f*g)'(z)}{(f*h)'(z)} = \varphi(w(z)). \tag{5.3.4}$$

A computation shows that

$$\frac{(f*g)(z)}{(f*h)(z)} = 1 + a_2(g_2 - h_2)z + [a_3(g_3 - h_3) + a_2^2(h_2^2 - h_2 g_2)]z^2 + \cdots, \tag{5.3.5}$$

$$\frac{(f*g)'(z)}{(f*h)'(z)} = 1 + 2a_2(g_2 - h_2)z + [3a_3(g_3 - h_3) + 4a_2^2(h_2^2 - h_2 g_2)]z^2 + \cdots \tag{5.3.6}$$

and

$$\varphi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \cdots . \tag{5.3.7}$$

From (5.3.4), (5.3.5), (5.3.6) and (5.3.7), we have

$$(1+\alpha)(g_2 - h_2)a_2 = B_1 w_1 \tag{5.3.8}$$

and

$$(1+2\alpha)(g_3 - h_3)a_3 + (1+3\alpha)(h_2^2 - h_2 g_2)a_2^2 = B_1 w_2 + B_2 w_1^2. \tag{5.3.9}$$

A computation using (5.3.8) and (5.3.9) gives

$$a_3 - \mu a_2^2 = \frac{B_1}{(1+2\alpha)(g_2 - h_2)}(w_2 - t w_1^2), \tag{5.3.10}$$

where t is given by (5.3.3). Now the first inequality (5.3.1) is established by an application of Lemma 5.1.1 as follows:

If

$$-\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2} \leq -1,$$

then

$$\mu \leq \frac{(B_2 - B_1)(1+\alpha)^2(g_2 - h_2)^2 - (1+3\alpha)(h_2^2 - h_2 g_2)B_1^2}{(1+2\alpha)(g_3 - h_3)B_1^2} := \sigma_1$$

and hence Lemma 5.1.1 implies

$$|a_3 - \mu a_2^2| \leq \frac{B_1 A}{(1+2\alpha)(g_3 - h_3)},$$

where

$$A = \frac{B_2}{B_1} - \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2}.$$

For

$$-1 \leq -\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2} \leq 1,$$

we have $\sigma_1 \leq \mu \leq \sigma_2$, where σ_1 and σ_2 are as stated Theorem 5.3.1. Now an application of Lemma 5.1.1 yields

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{(1+2\alpha)(g_3 - h_3)}.$$

For

$$-\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2} \geq 1,$$

we have $\mu \geq \sigma_2$ and it follows from Lemma 5.1.1 that

$$|a_3 - \mu a_2^2| \leq \frac{B_1 A}{(1+2\alpha)(h_3 - g_3)}.$$

Now the second inequality (5.3.2) follows by an application of Lemma 1.1.2 as follows:

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{B_1}{(1+2\alpha)(g_2 - h_2)}[w_2 - t w_1^2] \\ &\leq \frac{B_1}{(1+2\alpha)(g_3 - h_3)} \max\{1; |t|\}, \end{aligned}$$

where t is given by (5.3.3). Sharpness can be verified in a similar way as in the proof of Theorem 5.2.1. □

Remark 5.3.1. If we set $\alpha = 1$, g and h are as given by (5.1.5), then Theorem 5.3.1 reduces to [**101**, Theorem 3] of Ma and Minda. When $\alpha = 0$, Theorem 5.3.1 reduces to the result [**112**, Theorem 2.1], proved by Murugusundaramoorthy et al. There were few typographical errors in the assertion of the result [**112**, Theorem 2.1] and it is rectified in the following corollary:

Corollary 5.3.2. [**112**, Theorem 2.1] *If $f \in M_{g,h}(\varphi)$, then for any real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{g_3 - h_3}\left(\frac{B_2}{B_1} - \frac{[(h_2^2 - h_2 g_2) + \mu(g_3 - h_3)]B_1}{(g_2 - h_2)^2}\right) & (\mu \leq \sigma_1), \\ \frac{B_1}{g_3 - h_3} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{B_1}{g_3 - h_3}\left(\frac{[(h_2^2 - h_2 g_2) + \mu(g_3 - h_3)]B_1}{(g_2 - h_2)^2} - \frac{B_2}{B_1}\right) & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{(B_2 - B_1)(g_2 - h_2)^2 - (h_2^2 - h_2 g_2)B_1^2}{(g_3 - h_3)B_1^2}$$

and

$$\sigma_2 := \frac{(B_2 + B_1)(g_2 - h_2)^2 - (h_2^2 - h_2 g_2)B_1^2}{(g_3 - h_3)B_1^2}.$$

We now discuss some applications of Theorem 5.3.1.

Corollary 5.3.3. *Assume that*

$$g(z) = z + \sum_{n=2}^{\infty} \frac{n\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n-\delta+1)} z^n \ \text{ and } \ h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n-\delta+1)} z^n.$$

If $f \in \mathcal{N}_{g,h}(\alpha,\varphi)$, then for any real number μ, we have

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{(2-\delta)(3-\delta)B_1}{12(1+2\alpha)}\left(\frac{B_2}{B_1}-\frac{[12\mu(1+2\alpha)(2-\delta)-4(3-\delta)(1+3\alpha)]B_1}{4(3-\delta)(1+\alpha)^2}\right) & (\mu \leq \sigma_1), \\ \frac{(2-\delta)(3-\delta)B_1}{12(1+2\alpha)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{(2-\delta)(3-\delta)B_1}{12(1+2\alpha)}\left(\frac{[12\mu(1+2\alpha)(2-\delta)-4(3-\delta)(1+3\alpha)]B_1}{4(3-\delta)(1+\alpha)^2}-\frac{B_2}{B_1}\right) & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{(3-\delta)[(B_1-B_2)(1+\alpha)^2+(1+3\alpha)B_1^2]}{3(2-\delta)(1+2\alpha)B_1^2}$$

and

$$\sigma_2 := \frac{(3-\delta)[(B_1+B_2)(1+\alpha)^2+(1+3\alpha)B_1^2]}{3(2-\delta)(1+2\alpha)B_1^2}.$$

Remark 5.3.2. Taking $\alpha = 8/\pi^2, B_2 = 16/3\pi^2$ and $\delta = 1$ in Corollary 5.3.3, we have the result of Ma and Minda [**102**, Theorem 2]. When $\alpha = 0$, Corollary 5.3.3 reduces to the result [**112**, Corollary 3.2] of Murugusundaramoorthy et al. There were few typographical errors in the assertion of [**112**, Corollary 3.2] and we corrected it as follows:

Corollary 5.3.4. [**112**, Corollary 3.2] *If $f \in M_{g,h}(\varphi)$, then for any real number μ, we have*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{(2-\delta)(3-\delta)B_1}{12}\left(\frac{B_2}{B_1}-\frac{[12\mu(2-\delta)-4(3-\delta)]B_1}{4(3-\delta)}\right) & (\mu \leq \sigma_1), \\ \frac{(2-\delta)(3-\delta)B_1}{12} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{(2-\delta)(3-\delta)B_1}{12}\left(\frac{[12\mu(2-\delta)-4(3-\delta)]B_1}{4(3-\delta)}-\frac{B_2}{B_1}\right) & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{(3-\delta)[B_1-B_2+B_1^2]}{3(2-\delta)B_1^2}$$

and

$$\sigma_2 := \frac{(3-\delta)[B_1+B_2+B_1^2]}{3(2-\delta)B_1^2}.$$

Putting $\varphi(z) = (1+z)/(1-z)$, g and h are as given by (5.1.5) in Theorem 5.3.1, we deduce the following result:

Corollary 5.3.5. *Let $f \in \mathcal{M}(\alpha)$, then for any real number μ, we have*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{(\alpha^2+8\alpha+3)-4\mu(1+2\alpha)}{(1+\alpha)^2(1+2\alpha)} & (\mu \leq \sigma_1), \\ \frac{1}{1+2\alpha} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{4\mu(1+2\alpha)-(\alpha^2+8\alpha+3)}{(1+\alpha)^2(1+2\alpha)} & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{1+3\alpha}{2(1+2\alpha)} \quad \textit{and} \quad \sigma_2 := \frac{\alpha^2+5\alpha+2}{2(1+2\alpha)}.$$

Note that for $\alpha = 0$, Corollary 5.3.5 reduces to a result in [**83**] (see also [**188**]). By taking $\varphi(z) = (1+z)/(1-z)$, g and h are given by (5.1.5), in second result of Theorem 5.1.1, we have the following result:

Corollary 5.3.6. *Let $f \in \mathcal{M}(\alpha)$, then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha} \max\left\{1; \left|\frac{4\mu(1+2\alpha) - (\alpha^2+8\alpha+3)}{(1+\alpha)^2}\right|\right\}.$$

Remark 5.3.3. For $\alpha = 1$, Corollary 5.3.6 reduces to the result [**83**, Corollary 1] of Keogh and Merkes.

5.4 Fekete-Szegö Inequality for $\mathcal{S}_g^\alpha(\varphi)$

Throughout this section, it is assumed that $\alpha \geq 0$,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

with b_2 and b_3 are non-zero real numbers unless otherwise stated specifically.

Theorem 5.4.1. *If $f \in \mathcal{S}_g^\alpha(\varphi)$, then for any real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right) & \textit{if} \quad \mu \leq \sigma_1; \\ \frac{B_1}{2(2\alpha+1)|b_3|} & \textit{if} \ \ \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2} - \frac{B_2}{B_1}\right) & \textit{if} \ \ \mu \geq \sigma_2, \end{cases} \tag{5.4.1}$$

where

$$\sigma_1 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1 b_3}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - 1\right)$$

and

$$\sigma_2 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1 b_3}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2}\right).$$

The inequality (5.4.1) is sharp.

Further when $\sigma_1 < \mu < \sigma_2$, the above result can be improved as follows:

Let

$$\sigma_3 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1 b_3}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2}\right).$$

If $\sigma_1 < \mu \leq \sigma_3$, *then*

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 - \frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{B_1}{2(2\alpha+1)|b_3|}$$

and if $\sigma_3 \leq \mu < \sigma_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{B_1}{2(2\alpha+1)|b_3|}.$$

Proof. Since $f \in \mathcal{S}_g^\alpha(\varphi)$, there exists an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \mathbf{B}$ such that

$$1 + \frac{z(f*g)'(z)}{(f*g)(z)} + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{(1-\alpha)z^2(f*g)''(z) + z(f*g)'(z)}{(1-\alpha)z(f*g)'(z) + \alpha(f*g)(z)} = \phi(w(z)). \tag{5.4.2}$$

A calculation shows that

$$\frac{z((f*g)'(z))}{(f*g)(z)} = 1 + a_2 b_2 z + (2a_3 b_3 - a_2^2 b_2^2)z^2 + \ldots,$$

$$1 + \frac{z(f*g)''(z)}{(f*g)'(z)} = 1 + 2a_2 b_2 z + (6a_3 b_3 - 4a_2^2 b_2^2)z^2 + \cdots$$

and

$$\frac{(1-\alpha)z^2(f*g)''(z) + z(f*g)'(z)}{(1-\alpha)z(f*g)'(z) + \alpha(f*g)(z)} = 1 + (2-\alpha)a_2 b_2 z + [(6-4\alpha)a_3 b_3 - (\alpha-2)^2 a_2^2 b_2^2]z^2 + \cdots.$$

Substituting these expressions in (5.4.2), we obtain

$$(1+\alpha)a_2 b_2 = B_1 w_1 \tag{5.4.3}$$

and

$$2(2\alpha+1)a_3 b_3 + (\alpha^2-4\alpha-1)a_2^2 b_2^2 = B_1 w_2 + B_2 w_1^2. \tag{5.4.4}$$

By using (5.4.3) and (5.4.4), we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(2\alpha+1)b_3}(w_2 - tw_1^2), \tag{5.4.5}$$

where

$$t := -\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2}. \tag{5.4.6}$$

If $t \leq -1$, then

$$-\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} \leq -1,$$

which implies

$$\mu \leq \frac{(1+\alpha)^2b_2^2}{2(2\alpha+1)B_1b_3}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - 1\right) := \sigma_1.$$

Now an application of Lemma 5.1.1 gives

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - \frac{\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2}\right) \quad (\mu \leq \sigma_1),$$

which is nothing but the first part of assertion (5.4.1).

Next, if $t \geq 1$, then

$$-\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} \geq 1$$

which implies

$$\mu \geq \frac{(1+\alpha)^2b_2^2}{2(2\alpha+1)B_1b_3}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2}\right) =: \sigma_2,$$

applying Lemma 5.1.1, we have

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} - \frac{B_2}{B_1}\right) \quad (\mu \geq \sigma_2),$$

which is essentially the third part of assertion (5.4.1).

Finally if $-1 \leq t \leq 1$, then

$$-1 \leq -\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} \leq 1$$

which shows that $\sigma_1 \leq \mu \leq \sigma_2$. Thus by an application of Lemma 5.1.1, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|} \quad (\sigma_1 \leq \mu \leq \sigma_2)$$

which is the second part of assertion (5.4.1). Sharpness can be verified in a similar way as in the proof of Theorem 5.2.1.

Further when $\sigma_1 < \mu < \sigma_2$ the above result can be improved as follows:
If $-1 < t \leq 0$, then

$$-1 < -\frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2} \leq 0,$$

which implies that $\sigma_1 < \mu \leq \sigma_3$. Now using (5.1.7), (5.4.5) and (5.4.6), we have

$$\frac{2(2\alpha+1)b_3}{B_1}|a_3 - \mu a_2^2| + \left(1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|w_1|^2 \leq 1. \tag{5.4.7}$$

Substituting the value of w_1 obtained from (5.4.3) in (5.4.7) and simplifying, we have

$$\begin{aligned}|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \\ \leq \frac{B_1}{2(2\alpha+1)|b_3|} \quad (\sigma_1 < \mu \leq \sigma_3).\end{aligned}$$

Further if $0 \leq t < 1$, then $\sigma_3 \leq \mu < \sigma_2$. Now a similar computation using (5.1.8), (5.4.3), (5.4.5) and (5.4.6) gives

$$\begin{aligned}|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \\ \leq \frac{B_1}{2(2\alpha+1)|b_3|}.\end{aligned}$$

This completes the proof. □

Remark 5.4.1. If we set $\alpha = 1$ and $g(z) = z/(1-z)$ in Theorem 5.4.1, then we have a result of Ma and Minda [**101**, Theorem 3]. By setting $\alpha = 0$ and $g(z) = z/(1-z)$ in Theorem 5.4.1, we obtain a result of Murugusundaramoorthy et al. [**112**, Corollary 2.2].

Using Lemma 1.1.2 and Equation (5.4.5), we deduce the following:

Theorem 5.4.2. *If $f \in \mathcal{S}_g^\alpha(\phi)$, then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|}\max\left\{1; \left|\frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} - \frac{B_2}{B_1}\right|\right\}.$$

From Theorem 5.4.1, we deduce the following result:

Corollary 5.4.3. *If $f \in \mathcal{S}_g^{\alpha}((1+Cz)/(1+Dz))$ $(-1 \leq D < C \leq 1)$, then for any real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{D-C}{2(2\alpha+1)|b_3|}\left(D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right) & \text{if} \quad \mu \leq \sigma_1; \\ \frac{C-D}{2(2\alpha+1)|b_3|} & \text{if} \ \ \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{C-D}{2(2\alpha+1)|b_3|}\left(D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right) & \text{if} \ \ \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(D-C)b_3}\left(1 + D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2}\right)$$

and

$$\sigma_2 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(C-D)b_3}\left(1 - D - \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2}\right).$$

The result is sharp.

The above result can be improved when $\sigma_1 < \mu < \sigma_2$ as follows:

Let

$$\sigma_3 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(D-C)b_3}\left(D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2}\right).$$

If $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(C-D)|b_3|}\tau_1 |a_2|^2 \leq \frac{C-D}{2(2\alpha+1)|b_3|},$$

$$\tau_1 := \left(1 + D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right)$$

and if $\sigma_3 \leq \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(C-D)|b_3|}\tau_2 |a_2|^2 \leq \frac{C-D}{2(2\alpha+1)|b_3|},$$

where

$$\tau_2 := \left(1 - D - \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right).$$

By taking $D = -1$ and $C = 1$ in Corollary 5.4.3, we obtain the following result:

Example 5.4.4. If $f \in \mathcal{S}_g^\alpha((1+z)/(1-z))$, then for any real number μ, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(1+\alpha)^2|b_3|}\left(\frac{3+10\alpha-\alpha^2}{2\alpha+1} - \frac{4\mu b_3}{b_2^2}\right) & \text{if } \mu \leq \sigma_1; \\ \frac{1}{(2\alpha+1)|b_3|} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{1}{(1+\alpha)^2|b_3|}\left(\frac{\alpha^2-10\alpha-3}{2\alpha+1} + \frac{4\mu b_3}{b_2^2}\right) & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(1+4\alpha-\alpha^2)b_2^2}{2(2\alpha+1)b_3} \text{ and } \sigma_2 := \frac{(3\alpha+1)b_2^2}{(2\alpha+1)b_3}.$$

The result can be improved when $\sigma_1 \leq \mu \leq \sigma_2$ as follows:

Let

$$\sigma_3 := \frac{(3+10\alpha-\alpha^2)b_2^2}{4(2\alpha+1)b_3}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{b_2^2}{2|b_3|}\left(\frac{\alpha^2-4\alpha-1}{2\alpha+1} + \frac{2\mu b_3}{b_2^2}\right)|a_2|^2 \leq \frac{1}{(2\alpha+1)|b_3|}$$

and if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{b_2^2}{|b_3|}\left(\frac{3\alpha+1}{2\alpha+1} - \frac{\mu b_3}{b_2^2}\right)|a_2|^2 \leq \frac{1}{(2\alpha+1)|b_3|}.$$

The result is sharp.

Remark 5.4.2. If we take $g(z) = z + \sum_{n=2}^\infty n^m z^n$ $(m \in \{0,1,2,3,\ldots\})$, in Example 5.4.4, it reduces to [**151**, Theorem 2] of Răducanu.

Taking $\varphi(z) = (1+Cz)/(1+Dz)$ $(-1 \leq D < C \leq 1)$ in Theorem 5.4.2, we deduce the following result:

Corollary 5.4.5. *If $f \in \mathcal{S}_g^\alpha((1+Cz)/(1+Dz))$, then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{C-D}{2(2\alpha+1)|b_3|}\max\{1; R\},$$

where

$$R = \left|\frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2} + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + D\right|.$$

If we take $g(z) = z + \sum_{n=2}^\infty n^m z^n$, $D = -1$ and $C = 1$ in Corollary 5.4.5, we have the following result:

Corollary 5.4.6. [**151**, Theorem 3, Răducanu] *If $f \in \mathcal{HS}^*_m(\alpha)$, then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{3^m(1+2\alpha)} \max\left\{1; \frac{|2^{2m-1}(\alpha^2 - 10\alpha - 3) + 2.3^m(1+2\alpha)\mu|}{2^{2m-1}(1+\alpha)^2}\right\}.$$

If we set $D = -1, C = 1$ and $g(z) = z/(1-z)$ in Corollary 5.4.5, then for $\alpha = 0$, we have the following result:

Corollary 5.4.7. [**83**, Theorem 1] *If $f \in \mathcal{S}^*$, then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \max\{1; |4\mu - 3|\}.$$

Setting $\alpha = 1, D = -1, C = 1$ and $g(z) = z/(1-z)$ in Corollary 5.4.5, we obtain the following result:

Corollary 5.4.8. [**83**, Corollary 1] *If $f \in \mathcal{K}$, then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{3}; |\mu - 1|\right\}.$$

Chapter 6

Initial Coefficients Estimate of Certain Bi-univalent Functions

6.1 Introduction

Recall that a function $f \in \mathcal{A}$ is called *bi-univalent* in $\mathbb{D}$ if both f and f^{-1} are univalent on $\mathbb{D}$. Lewin [**93**] introduced the class σ of bi-univalent analytic functions and showed that the second coefficient of every $f \in \sigma$ satisfy the inequality $|a_2| \leq 1.51$. Let σ_1 be the class of all functions $f = \phi \circ \psi^{-1}$ where ϕ, ψ map $\mathbb{D}$ onto a domain containing $\mathbb{D}$ and $\phi'(0) = \psi'(0)$. In 1969, Suffridge [**193**] constructed a function in $\sigma_1 \subset \sigma$, with the second coefficient $a_2 = 4/3$ and conjectured that $|a_2| \leq 4/3$ for all functions in σ. In 1969, Netanyahu [**116**] proved this conjecture for the subclass σ_1 of σ. Later in 1981, Styer and Wright [**192**] disproved the conjecture of Suffridge [**193**] by showing $a_2 > 4/3$ for some function in σ. For the counter example showing $\sigma \neq \sigma_1$, see [**35**]. For results related to bi-univalent polynomial, see [**82, 182**]. In 1979, Brannan [**27**] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. In 1985, Kedzierawski [**81**, Theorem 2] proved this conjecture for a special case when the functions f and f^{-1} both are starlike functions. In 1985, Tan [**195**] obtained the bound $|a_2| \leq 1.485$ for the class σ. At

Most of the results of this chapter will appear in [**179**].

that time the researchers were interested in finding estimate on the initial coefficients of the functions which are in some specific subclass of $\mathcal{S}$ along with their inverses also belong to the same subclass. In 1985, Kedzierawski [**81**], in an attempt to prove the Brannan's conjecture [**27**], considered a case when f and its inverse are in different subclasses of the class $\mathcal{S}$. He provided the following estimate for second coefficient of such functions:

$$|a_2| \leq \begin{cases} 1.5894, & f \in \mathcal{S},\ f^{-1} \in \mathcal{S}; \\ \sqrt{2}, & f \in \mathcal{S}^*,\ f^{-1} \in \mathcal{S}^*; \\ 1.507, & f \in \mathcal{S}^*,\ f^{-1} \in \mathcal{S}; \\ 1.224, & f \in \mathcal{K},\ f^{-1} \in \mathcal{S}. \end{cases}$$

In 1986, Brannan and Taha [**30**] obtained estimate on the initial coefficients for functions in the classes $\mathcal{K}_\sigma(\beta)$, $\mathcal{K}_\sigma(\beta)$ and $\mathcal{SS}^*_\sigma(\alpha)$. The work on bi-univalent functions gained momentum with the advent of a paper by Srivastava et al. [**189**], in 2010, in which they obtained the bounds of the initial coefficients for functions belonging to the classes

$$\mathcal{H}_\sigma(\beta) = \left\{ f \in \sigma : \operatorname{Re}(f'(z)) > \beta \text{ and } \operatorname{Re}(f^{-1})'(z) > \beta, 0 \leq \beta < 1 \right\}$$

and

$$\mathcal{H}_{\sigma,\alpha} = \left\{ f \in \sigma : |\arg f'(z)| \leq \frac{\alpha\pi}{2} \text{ and } |\arg(f^{-1})'(z)| \leq \frac{\alpha\pi}{2}, 0 < \alpha \leq 1 \right\}.$$

Recently, Ali et al. [**13**] extended the results of Brannan and Taha [**30**] by proving their results for more general classes. For similar coefficients estimate related problems of certain bi-univalent functions, see [**56, 189, 208**]. For a comprehensive survey and some open problems and survey related to bi-univalent functions, one may refer to [**63, 183**].

The results in this chapter are essentially motivated by the paper of Ali et al. [**13**], and Srivastava et al. [**189**]. This chapter mainly concerned with the estimation of coefficients of certain bi-univalent functions especially when both f and its inverse function f^{-1} belongs to the same class. Further the work of Kedzierawski [**81**] actuates us to consider cases when f is in some subclass of univalent functions and f^{-1} belongs to some other subclass of univalent functions. In Section 6.2, we find the estimate on the initial coefficients when f and f^{-1} belong to the same class, whereas in Section 6.3 estimates on the initial coefficients are derived when f and its inverse f^{-1} belong to different classes. Our results generalize several results derived in [**13, 56, 81, 189**] which are pointed out here. The classes we have considered are defined as follows: Throughout this chapter, unless stated clearly, we shall assume that $\varphi \in \mathcal{P}$ is an analytic function in $\mathbb{D}$ of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \tag{6.1.1}$$

with $B_1 > 0$ and B_2 is any real number. For the sake of convenience, we shall write $F := f^{-1}$.

Definition 6.1.1. Let $\lambda \geq 0$. A function $f \in \sigma$ is in the class $\mathcal{R}_\sigma(\lambda, \varphi)$, if it satisfies

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \varphi(z) \quad \text{and} \quad (1-\lambda)\frac{F(w)}{w} + \lambda F'(w) \prec \varphi(w).$$

On specializing the function φ and λ, the class $\mathcal{R}_\sigma(\lambda, \varphi)$ reduces to the known classes, a few are enlisted below:

1. $\mathcal{R}_\sigma(\lambda, (1 + (1-2\beta)z)/(1-z)) = \mathcal{R}_\sigma(\lambda, \beta)$ $(\lambda \geq 1;\ 0 \leq \beta < 1)$, introduced by Frasin and Aouf [**56**, Definition 3.1].
2. $\mathcal{R}_\sigma(\lambda, ((1+z)/(1-z))^\alpha) = \mathcal{R}_{\sigma,\alpha}(\lambda)$ $(\lambda \geq 1;\ 0 < \alpha \leq 1)$, introduced by Frasin and Aouf [**56**, Definition 2.1].
3. $\mathcal{R}_\sigma(1, \varphi) = \mathcal{R}_\sigma(\varphi)$, introduced by Ravichandran et al. [**13**, p. 345].
4. $\mathcal{R}_\sigma(1, (1 + (1-2\beta)z)/(1-z)). = \mathcal{R}_\sigma(\beta)$ $(0 \leq \beta < 1)$, introduced by Srivastava et al. [**189**, Definition 2]
5. $\mathcal{R}_\sigma(1, ((1+z)/(1-z))^\alpha). = \mathcal{R}_{\sigma,\alpha}$ $(0 < \alpha \leq 1)$, introduced by Srivastava et al. [**189**, Definition 1].

Definition 6.1.2. A function $f \in \sigma$ is in the class $\mathcal{S}^*_\sigma(\varphi)$, if it satisfies

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad \text{and} \quad \frac{wF'(w)}{F(w)} \prec \varphi(w).$$

Note that for a suitable choice of φ, the class $\mathcal{S}^*_\sigma(\varphi)$, reduces to the following well-known classes:

1. $\mathcal{S}^*_\sigma((1+(1-2\beta)z)/(1-z)) = \mathcal{S}^*_\sigma(\beta) \quad (0 \leq \beta < 1)$.
2. $\mathcal{S}^*_\sigma(((1+z)/(1-z))^\alpha) = \mathcal{SS}^*_\sigma(\alpha) \quad (0 < \alpha \leq 1)$.

Definition 6.1.3. A function $f \in \sigma$ is in the class $\mathcal{K}_\sigma(\varphi)$, if f and F satisfy the subordinations

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \text{ and } 1 + \frac{wF''(w)}{F'(w)} \prec \varphi(w).$$

Note that $\mathcal{K}_\sigma((1+(1-2\beta)z)/(1-z)) =: \mathcal{K}_\sigma(\beta) \;\; (0 \leq \beta < 1)$.

6.2 Functions f and f^{-1} belong to the same class

In this section, we shall consider the results when both f and f^{-1} are in the same class. Our first result provides estimate for the coefficient a_2 of functions f in $\mathcal{R}_\sigma(\lambda, \varphi)$.

Theorem 6.2.1. *If $f \in \mathcal{R}_\sigma(\lambda, \varphi)$, then*

$$|a_2| \leq \sqrt{\frac{B_1 + |B_1 - B_2|}{1 + 2\lambda}}. \tag{6.2.1}$$

Proof. Since $f \in \mathcal{R}_\sigma(\lambda, \varphi)$, there exist two analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi(r(z)) \text{ and } (1-\lambda)\frac{F(w)}{w} + \lambda F'(w) = \varphi(s(z)). \tag{6.2.2}$$

Define the functions p and q by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{6.2.3}$$

$$q(z) = \frac{1+s(z)}{1-s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots , \tag{6.2.4}$$

or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots\right) \tag{6.2.5}$$

and

$$s(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left(q_1 z + \left(q_2 - \frac{q_1^2}{2}\right)z^2 + \cdots\right). \tag{6.2.6}$$

It is clear that p and q are analytic in $\mathbb{D}$ and $p(0) = 1 = q(0)$. Also since p and q have positive real part in $\mathbb{D}$, it follows that $|p_i| \leq 2$ and $|q_i| \leq 2$. In view of (6.2.2), (6.2.5) and (6.2.6), clearly

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \text{ and } (1-\lambda)\frac{F(w)}{w} + \lambda F'(w) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{6.2.7}$$

On expanding (6.1.1) using (6.2.5) and (6.2.6), it is evident that

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2\right) z^2 + \cdots \tag{6.2.8}$$

and

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2\right) w^2 + \cdots . \tag{6.2.9}$$

Since $f \in \sigma$ has the Maclaurin series expansion given by (1.1.1), a computation shows that its inverse $F = f^{-1}$ has the expansion given by (1.2.1). It follows from (6.2.7), (6.2.8) and (6.2.9) that

$$(1+\lambda)a_2 = \frac{1}{2}B_1 p_1$$

$$(1+2\lambda)a_3 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2 \tag{6.2.10}$$

$$-(1+\lambda)a_2 = \frac{1}{2}B_1 q_1$$

$$(1+2\lambda)(2a_2^2 - a_3) = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2. \tag{6.2.11}$$

Now (6.2.10) and (6.2.11) yield

$$8(1+2\lambda)a_2^2 = 2(p_2+q_2)B_1 + (B_2 - B_1)(p_1^2 + q_1^2). \tag{6.2.12}$$

Finally an application of the known results, $|p_i| \leq 2$ and $|q_i| \leq 2$ in (6.2.12) yields the desired estimate for a_2 provided in (6.2.1). $\square$

Remark 6.2.1. Let $\varphi(z) = (1 + (1 - 2\beta)z)/(1 - z)$ with $0 \le \beta < 1$, which implies that $B_1 = B_2 = 2(1 - \beta)$. When $\lambda = 1$, Theorem 6.2.1 gives $|a_2| \le \sqrt{2(1-\beta)/3}$ for functions in the class $\mathcal{R}_\sigma(\beta)$, which in fact coincides with the result [**208**, Corollary 2] of Xu et al. In particular if $\beta = 0$, then $|a_2| \le \sqrt{2/3} \approx 0.816$ for functions $f \in \mathcal{R}_\sigma(0)$. Since the estimate on $|a_2|$ for $f \in \mathcal{R}_\sigma(0)$ is improved over the conjectured estimate $|a_2| \le \sqrt{2} \approx 1.414$ for $f \in \sigma$, functions in $\mathcal{R}_\sigma(0)$ are not the candidates for the sharpness of the estimate in the class σ.

Theorem 6.2.2. *If $f \in \mathcal{S}^*_\sigma(\varphi)$, then*

$$|a_2| \le \min\left\{\sqrt{B_1 + |B_2 - B_1|}, \sqrt{\frac{B_1^2 + B_1 + |B_2 - B_1|}{2}}, \frac{B_1\sqrt{B_1}}{\sqrt{B_1^2 + |B_1 - B_2|}}\right\}$$

and

$$|a_3| \le \min\left\{B_1 + |B_2 - B_1|, \frac{B_1^2 + B_1 + |B_2 - B_1|}{2}, R\right\},$$

where

$$R := \frac{1}{4}\left(B_1 + 3B_1 \max\left\{1; \left|\frac{B_1 - 4B_2}{3B_1}\right|\right\}\right).$$

Proof. Since $f \in \mathcal{S}^*_\sigma(\varphi)$, there exist analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with property $r(0) = 0 = s(0)$, such that

$$\frac{zf'(z)}{f(z)} = \varphi(r(z)) \quad \text{and} \quad \frac{wF'(w)}{F(w)} = \varphi(s(w)). \tag{6.2.13}$$

Let p and q be defined as in (6.2.3) and (6.2.4), then it is clear from (6.2.13), (6.2.5) and (6.2.6) that

$$\frac{zf'(z)}{f(z)} = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) \quad \text{and} \quad \frac{wF'(w)}{F(w)} = \varphi\left(\frac{q(w) - 1}{q(w) + 1}\right). \tag{6.2.14}$$

It follows from (6.2.14), (6.2.8) and (6.2.9) that

$$a_2 = \frac{1}{2}B_1 p_1 \tag{6.2.15}$$

$$2a_3 = \frac{B_1 p_1}{2}a_2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2 \tag{6.2.16}$$

$$-a_2 = \frac{1}{2}B_1 q_1 \tag{6.2.17}$$

and

$$4a_2^2 - 2a_3 = -\frac{B_1 q_1}{2} a_2 + \frac{1}{2} B_1 \left(q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2. \tag{6.2.18}$$

The equations (6.2.15) and (6.2.17) yield

$$p_1 = -q_1 \tag{6.2.19}$$

$$8a_2^2 = (p_1^2 + q_1^2) B_1^2 \tag{6.2.20}$$

and

$$2a_2 = \frac{B_1(p_1 - q_1)}{2}. \tag{6.2.21}$$

From (6.2.16), (6.2.18) and (6.2.21), it follows that

$$8a_2^2 = 2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2). \tag{6.2.22}$$

Further a computation using (6.2.16), (6.2.18), (6.2.15) and (6.2.19) gives

$$16a_2^2 = 2B_1^2 q_1^2 + 2B_1(p_2 + q_2) + (B_2 - B_1)(p_1^2 + q_1^2). \tag{6.2.23}$$

Similarly a computation using (6.2.16), (6.2.18), (6.2.21) and (6.2.20) yields

$$4(B_1^2 - B_2 + B_1) a_2^2 = B_1^3 (p_2 + q_2). \tag{6.2.24}$$

Now (6.2.22), (6.2.23) and (6.2.24) yield the desired estimate on $|a_2|$ as asserted in the theorem. To find estimate for $|a_3|$ subtract (6.2.16) from (6.2.18), to get

$$-4a_3 = -4a_2^2 + \frac{B_1(q_2 - p_2)}{2}. \tag{6.2.25}$$

Now a computation using (6.2.23) and (6.2.25) leads to

$$16a_3 = 2B_1^2 q_1^2 + 4B_2 p_2 + (B_1 - B_2)(p_1^2 + q_1^2). \tag{6.2.26}$$

From (6.2.15), (6.2.16), (6.2.17) and (6.2.18), it follows that

$$4a_3 = \frac{B_1}{2}(3p_2 + q_2) + (B_2 - B_1) p_1^2 \tag{6.2.27}$$

$$= \frac{B_1 q_2}{2} + \frac{3B_1}{2} \left(p_2 - \frac{2(B_1 - B_2)}{3B_1} p_1^2 \right). \tag{6.2.28}$$

An application of the result Lemma 1.1.2 in (6.2.28), yields

$$4|a_3| \leq B_1 + 3B_1 \max \left\{ 1; \left| \frac{B_1 - 4B_2}{3B_1} \right| \right\}. \tag{6.2.29}$$

Now the desired estimate on a_3 follows from (6.2.26), (6.2.27) and (6.2.29) at once. □

Remark 6.2.2. If $f \in \mathcal{S}^*_\sigma(\beta)$ $(0 \leq \beta < 1)$, then from Theorem 6.2.2, it is evident that

$$\begin{aligned} |a_2| &\leq \min\left\{\sqrt{2(1-\beta)}, \sqrt{(1-\beta)(3-2\beta)}\right\} \\ &= \begin{cases} \sqrt{2(1-\beta)}, & 0 \leq \beta \leq 1/2; \\ \sqrt{(1-\beta)(3-2\beta)}, & 1/2 \leq \beta < 1. \end{cases} \end{aligned} \tag{6.2.30}$$

Brannan and Taha [**29**, Theorem 3.1] proved that $|a_2| \leq \sqrt{2(1-\beta)}$ for $f \in \mathcal{S}^*_\sigma(\beta)$. We may expect a better estimate for functions in the class $\mathcal{S}^*_\sigma(\beta)$ in comparison to those in the class $\mathcal{S}^*(\beta)$. If we compare Brannan and Taha's estimate with the one namely $|a_2| \leq 2(1-\beta)$ for function $f \in \mathcal{S}^*(\beta)$, given by Robertson [**152**], we see that Brannan and Taha's estimate is better than the Robertson's only when $0 \leq \beta \leq 1/2$. However it may be noted that our estimate given in (6.2.30) improved the estimate of Brannan and Taha [**29**, Theorem 3.1].

Next if we take $\varphi(z) = ((1+z)/(1-z))^\alpha$ $(0 < \alpha \leq 1)$ in Theorem 6.2.2, we have $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Then we obtain the estimate on a_2 for functions f in $\mathcal{SS}^*_\sigma(\alpha)$ as:

$$|a_2| \leq \min\left\{\sqrt{4\alpha - 2\alpha^2}, \sqrt{\alpha^2 + 2\alpha}, \frac{2\alpha}{\sqrt{1+\alpha}}\right\} = \frac{2\alpha}{\sqrt{1+\alpha}}.$$

Brannan and Taha [**29**, Theorem 2.1] gave the same estimate for functions in the class $\mathcal{SS}^*_\sigma(\alpha)$.

Theorem 6.2.3. *If $f \in \mathcal{K}_\sigma(\varphi)$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{B_1^2 + B_1 + |B_2 - B_1|}{6}}, \frac{B_1}{2}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{B_1^2 + B_1 + |B_2 - B_1|}{6}, \frac{B_1(3B_1+2)}{12}\right\}.$$

Proof. Since $f \in \mathcal{K}_\sigma(\varphi)$, there exist analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with the property $r(0) = 0 = s(0)$, satisfying

$$1 + \frac{zf''(z)}{f'(z)} = \varphi(r(z)) \quad \text{and} \quad 1 + \frac{wF''(w)}{F'(w)} = \varphi(s(w)). \tag{6.2.31}$$

Let p and q be defined as in (6.2.3) and (6.2.4), then it is clear from (6.2.31), (6.2.5) and (6.2.6) that

$$1+\frac{zf''(z)}{f'(z)}=\varphi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad 1+\frac{wF''(w)}{F'(w)}=\varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{6.2.32}$$

It follows from (6.2.32), (6.2.8) and (6.2.9) that

$$2a_2=\frac{1}{2}B_1p_1 \tag{6.2.33}$$

$$6a_3=B_1p_1a_2+\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2 \tag{6.2.34}$$

$$-2a_2=\frac{1}{2}B_1q_1 \tag{6.2.35}$$

and

$$6(2a_2^2-a_3)=-B_1q_1a_2+\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2. \tag{6.2.36}$$

Now (6.2.33) and (6.2.35) yield

$$p_1=-q_1 \tag{6.2.37}$$

and

$$4a_2=\frac{B_1(p_1-q_1)}{2}. \tag{6.2.38}$$

From (6.2.34), (6.2.36), (6.2.37) and (6.2.33), it follows that

$$48a_2^2=2B_1^2p_1^2+2B_1(p_2+q_2)+(B_2-B_1)(p_1^2+q_1^2). \tag{6.2.39}$$

In view of $|p_i|\leq 2$ and $|q_i|\leq 2$ together with (6.2.38) and (6.2.39), yield the desired estimate on a_2 as asserted in the theorem. In order to find estimate for $|a_3|$, we subtract (6.2.34) from (6.2.36) and use (6.2.37) to obtain

$$-12a_3=-12a_2^2+\frac{B_1(q_2-p_2)}{2}. \tag{6.2.40}$$

A computation using (6.2.39) and (6.2.40) leads to

$$-48a_3=2B_1^2p_1^2-4B_2p_2+(B_1-B_2)(p_1^2+q_1^2). \tag{6.2.41}$$

From (6.2.38) and (6.2.40), it follows that

$$-12a_3=\frac{B_1(q_2-p_2)}{2}-\frac{3(p_1-q_1)^2B_1^2}{16}. \tag{6.2.42}$$

Now (6.2.41) and (6.2.42) yield the desired estimate on a_3. □

Remark 6.2.3. If $f \in \mathcal{K}_\sigma(\beta)$, $0 \leq \beta < 1$, then Theorem 6.2.3 gives

$$|a_2| \leq \min \left\{ \sqrt{\frac{(1-\beta)(3-2\beta)}{3}}, 1-\beta \right\} = 1-\beta$$

and

$$|a_3| \leq \min \left\{ \frac{(1-\beta)(3-2\beta)}{3}, \frac{(1-\beta)(4-3\beta)}{3} \right\} = \frac{(1-\beta)(3-2\beta)}{3},$$

which improves the following estimate given by Brannan and Taha.

Corollary 6.2.4. [**29**, Theorem 4.1] *Let* $f \in \mathcal{K}_\sigma(\beta)$, *then*

$$|a_2| \leq \sqrt{1-\beta} \quad \textit{and} \quad |a_3| \leq 1-\beta.$$

6.3 Functions f and f^{-1} belong to the different classes

In this section, we shall deal with the results of those bi-univalent functions where f and its inverse f^{-1} are in different classes. The function φ is taken to be the same as given in (6.1.1). Let us first recall the definitions of some classes.

$$\mathcal{R}(\varphi) := \{f \in \mathcal{A} : f'(z) \prec \varphi(z)\}, \quad \mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

Theorem 6.3.1. *Let* $f \in \sigma$ *and if* $f \in \mathcal{K}(\varphi)$ *and* $F \in \mathcal{R}(\varphi)$, *then*

$$|a_2| \leq \sqrt{\frac{3[B_1 + |B_2 - B_1|]}{8}} \quad \textit{and} \quad |a_3| \leq \frac{5[B_1 + |B_2 - B_1|]}{12}.$$

Proof. Since $f \in \mathcal{K}(\varphi)$ and $F \in \mathcal{R}(\varphi)$, there exist analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{zf''(z)}{f'(z)} = \varphi(r(z)) \quad \text{and} \quad F'(w) = \varphi(s(w)). \tag{6.3.1}$$

Let the functions p and q are defined by (6.2.3) and (6.2.4). It is clear that p and q are analytic in $\mathbb{D}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{D}$, and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. Proceeding as in the proof of Theorem 6.2.1 it follow from (6.3.1), (6.2.8) and (6.2.9) that

$$\begin{aligned} 2a_2 &= \frac{1}{2}B_1p_1 \\ 6a_3 - 4a_2^2 &= \frac{1}{2}B_1\Big(p_2 - \frac{1}{2}p_1^2\Big) + \frac{1}{4}B_2p_1^2 \\ -2a_2 &= \frac{1}{2}B_1q_1 \end{aligned} \tag{6.3.2}$$

and

$$3(2a_2^2 - a_3) = \frac{1}{2}B_1\Big(q_2 - \frac{1}{2}q_1^2\Big) + \frac{1}{4}B_2q_1^2. \tag{6.3.3}$$

A computation using (6.3.2) and (6.3.3) leads to

$$a_2^2 = \frac{2(p_2 + 2q_2)B_1 + (p_1^2 + 2q_1^2)(B_2 - B_1)}{32}. \tag{6.3.4}$$

and

$$a_3 = \frac{2(3p_2 + 2q_2)B_1 + (3p_1^2 + 2q_1^2)(B_2 - B_1)}{48}. \tag{6.3.5}$$

Now the desired estimates for a_2 and a_3 follow from (6.3.4) and (6.3.5) respectively using the fact that $|p_i| \leq 2$ and $|q_i| \leq 2$. □

Remark 6.3.1. If $f \in \mathcal{K}(\beta)$ and $F \in \mathcal{R}(\beta)$, then from Theorem 6.3.1 we see that

$$|a_2| \leq \sqrt{3(1-\beta)}/2 \text{ and } |a_3| \leq 5(1-\beta)/6.$$

In particular if $f \in \mathcal{K}$ and $F \in \mathcal{R}$, then $|a_2| \leq \sqrt{3}/2 \approx 0.867$ and $|a_3| \leq 5/6 \approx 0.833$. Recall the known estimate namely $|a_2| \leq 1$ and $|a_3| \leq 1$ for functions in the class $\mathcal{K}$. Thus our estimates are better than corresponding estimates known for the class $\mathcal{K}$.

Theorem 6.3.2. *Let $f \in \sigma$ and if $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{R}(\varphi)$, then*

$$|a_2| \leq \frac{\sqrt{5(B_1 + |B_2 - B_1|)}}{3}, \text{ and } |a_3| \leq \frac{7(B_1 + |B_2 - B_1|)}{9}.$$

Proof. Since $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{R}(\varphi)$, there exist analytic functions $r, s : \mathbb{D} \to \mathbb{D}$, with $r(0) = 0 = s(0)$, such that

$$\frac{zf'(z)}{f(z)} = \varphi(r(z)) \quad \text{and} \quad F'(w) = \varphi(s(w)). \tag{6.3.6}$$

Let the functions p and q be defined as in (6.2.3) and (6.2.4). Then

$$\frac{zf'(z)}{f(z)} = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad F'(w) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{6.3.7}$$

It follow from (6.3.7), (6.2.8) and (6.2.9) that

$$a_2 = \frac{1}{2}B_1 p_1$$

$$2a_3 - a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2 \tag{6.3.8}$$

$$-2a_2 = \frac{1}{2}B_1 q_1$$

$$3(2a_2^2 - a_3) = \frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2. \tag{6.3.9}$$

A computation using (6.3.8) and (6.3.9) leads to

$$a_2^2 = \frac{2(3p_2 + 2q_2)B_1 + (3p_1^2 + 2q_1^2)(B_2 - B_1)}{36} \tag{6.3.10}$$

and

$$a_3 = \frac{2(6p_2 + q_2)B_1 + (6p_1^2 + q_1^2)(B_2 - B_1)}{36}. \tag{6.3.11}$$

Now the bounds for $|a_2|$ and $|a_3|$ are obtained from (6.3.10) and (6.3.11) respectively using the fact that $|p_i| \leq 2$ and $|q_i| \leq 2$. □

Remark 6.3.2. If $f \in \mathcal{S}^*(\beta)$ and $F \in \mathcal{R}(\beta)$, then from Theorem 6.3.2, it is easy to see that

$$|a_2| \leq \sqrt{10(1-\beta)}/3 \quad \text{and} \quad |a_3| \leq 14(1-\beta)/9.$$

Further if $f \in \mathcal{S}^*$ and $F \in \mathcal{R}$, then $|a_2| \leq \sqrt{10}/3 \approx 1.054$ and $|a_3| \leq 14/9 \approx 1.56$. Recall the known estimates namely $|a_2| \leq 2$ and $|a_3| \leq 3$ for functions in the class $\mathcal{S}^*$. Thus our estimates are better than the corresponding known estimates for the class $\mathcal{S}^*$.

Theorem 6.3.3. *Let $f \in \sigma$ and if $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{K}(\varphi)$, then*

$$|a_2| \leq \sqrt{\frac{B_1 + |B_2 - B_1|}{2}} \quad \text{and} \quad |a_3| \leq \frac{B_1 + |B_2 - B_1|}{2}.$$

Proof. Let $f \in \mathcal{S}^*(\varphi)$ and $F \in \mathcal{K}(\varphi)$. Proceeding in the similar way as in the proof of Theorem 6.3.1, it is easy to see that

$$\begin{aligned} a_2 &= \frac{1}{2}B_1p_1 \\ 3a_3 - a_2^2 &= \frac{1}{2}B_1\Big(p_2 - \frac{1}{2}p_1^2\Big) + \frac{1}{4}B_2p_1^2 \qquad (6.3.12) \\ -2a_2 &= \frac{1}{2}B_1q_1 \\ 8a_2^2 - 6a_3 &= \frac{1}{2}B_1\Big(q_2 - \frac{1}{2}q_1^2\Big) + \frac{1}{4}B_2q_1^2. \qquad (6.3.13) \end{aligned}$$

A computation using (6.3.12) and (6.3.13) leads to

$$a_2^2 = \frac{2(2p_2 + q_2)B_1 + (2p_1^2 + q_1^2)(B_2 - B_1)}{24} \qquad (6.3.14)$$

and

$$a_3 = \frac{2(8p_2 + q_2)B_1 + (8p_1^2 + q_1^2)(B_2 - B_1)}{72}. \qquad (6.3.15)$$

Now using the result $|p_i| \leq 2$ and $|q_i| \leq 2$, the estimates on a_2 and a_3 follow from (6.3.14) and (6.3.15) respectively. □

Remark 6.3.3. Let $f \in \mathcal{S}^*(\beta)$ and $F \in \mathcal{K}(\beta)$, $0 \leq \beta < 1$. Then from Theorem 6.3.3, it is easy to see that

$$|a_2| \leq \sqrt{1-\beta} \ \text{ and } \ |a_3| \leq 1 - \beta.$$

In particular, if $f \in \mathcal{S}^*$ and $F \in \mathcal{K}$, then $|a_2| \leq 1$ and $|a_3| \leq 1$. Recall the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$ for functions in the class $\mathcal{S}^*$. Thus, our estimates are better than the so far known estimates for the class $\mathcal{S}^*$.

Chapter 7

Radius of Starlikeness for Analytic Functions with Fixed Second Coefficient

It is well-known that the estimate on the second coefficient gives several geometric properties like growth estimate, distortion estimate and covering theorem for functions in the class $\mathcal{S}$. This chapter focuses on the study of radius problems for functions with fixed second coefficient. We shall now see, how the second coefficient affects the radius constants for various classes of functions.

7.1 Introduction

We know that the radius of convexity of starlike functions is $2-\sqrt{3}$, see [**63**]. Goel [**59**], in 1971, generalized this result and obtained the radius of convexity for starlike function with fixed second coefficient. He also obtained the radius of starlikeness for functions $f(z) = z + 2bz^2 + \cdots$ $(0 \leq b \leq 1)$ satisfying $\operatorname{Re}(f(z)/z) > 0$. For growth and distortion estimates, order of convexity and radius of convexity for functions

Most of the results of this chapter are from [**12**], under review.

in $\mathcal{S}_b^*$, see [**197**]. McCarty [**103**], in 1972, introduced the class $\mathcal{P}_b'(\alpha)$ of functions $F(z) = z + b(1-\alpha)z^2 + \cdots$ such that $F' \in \mathcal{P}_b$. Further he obtained the growth and distortion estimates. He also determined the radius of convexity for functions in the class $\mathcal{P}_b'(\alpha)$, see [**103**, Theorem 4]. This result was sharp for $\alpha = 0$. In 1974, McCarty [**104**] proved the sharp result [**103**, Theorem 4] which is true for all $b \in [0,1]$ and $\alpha \in [0,1)$. He further generalized and proved the sharpen form of [**197**, Lemma 4] for functions in the class $\mathcal{S}_b^*(\alpha)$. Juneja and Mogra [**78**], in 1978, extended the results proved by McCarty [**104**].

Tuan and Anh [**198**] obtained the radii of convexity for functions in the classes

$$\mathcal{R}_{\gamma a} = \left\{ f(z) = z - 2az^2 + \cdots : \left| \frac{f(z)}{z} - \gamma \right| < \gamma, \gamma > 1, 0 \leq a \leq 1 - (2\gamma)^{-1} \right\}$$

and

$$\mathcal{T}_\gamma(G) = \left\{ f(z) = z + a_2 z^2 + \cdots : \left| \frac{f(z)}{g(z)} - \gamma \right| < \gamma, \gamma > 1 \right\},$$

where G is the class of functions $g \in \mathcal{A}$ satisfying $|g(z)/z - 1| < 1$.

Ali et al. [**10**] considered radius problems for several classes of functions defined either in terms of the ratio of f and g or the ratio of their derivatives, where $f, g \in \mathcal{A}$. Their results include (i) $\operatorname{Re}(f(z)/g(z)) > 0$, where the function g satisfies either $\operatorname{Re}(g(z)/z) > 0$ or $\operatorname{Re}(g(z)/z) > 1/2$ (ii) $|f(z)/g(z) - 1| < 1$, where $\operatorname{Re}(g(z)/z) > 0$ or g is convex and several other problems namely radius of uniform convexity for the classes $|f'(z)/g'(z) - 1| < 1$, where g is univalent or starlike or convex. The works presented in [**7, 10, 131, 172, 198**] motivates us to consider the problems carried out by Ali et al. [**10**] for analytic functions with the second fixed coefficient. Most of the results proved in [**10**] are generalized in this chapter.

Preliminaries

The following results are required in the present investigation. In the following results it is assumed that $|b| \leq 1$ and $0 \leq \alpha < 1$.

Lemma 7.1.1. [**103**, Theorem 2] *Let* $|b|\leq 1$ *and* $0\leq\alpha<1$. *If* $p\in\mathcal{P}_b(\alpha)$, *then, for* $|z|=r<1$, *we have*

$$\left|\frac{zp'(z)}{p(z)}\right|\leq\frac{2(1-\alpha)r}{1-r^2}\frac{|b|r^2+2r+|b|}{(1-2\alpha)r^2+2(1-\alpha)|b|r+1}.$$

Lemma 7.1.2. [**104**, Lemma 1] *Let* $|b|\leq 1$ *and* $0\leq\alpha<1$. *If* $p\in\mathcal{P}_b(\alpha)$, *then, for* $|z|=r<1$, *we have* $|p(z)-A_b|\leq D_b$, *where*

$$A_b=\frac{(1+|b|r)^2+(1-2\alpha)(|b|+r)^2r^2}{(1+2|b|r+r^2)(1-r^2)},\quad D_b=\frac{2(1-\alpha)(|b|+r)(1+|b|r)r}{(1+2|b|r+r^2)(1-r^2)}.$$

Lemma 7.1.3. [**104**, Theorem 1] *Let* $|b|\leq 1$ *and* $0\leq\alpha<1$. *Suppose* $p\in\mathcal{P}_b(\alpha)$. *Then, for* $|z|=r<1$, *we have*

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right)\geq\begin{cases}\frac{-2(1-\alpha)(|b|+2r+|b|r^2)r}{(1+2\alpha|b|r+(2\alpha-1)r^2)(1+2|b|r+r^2)}, & R'\leq R_b;\\ (2\sqrt{\alpha A_1}-A_1-\alpha)/(1-\alpha), & R'\geq R_b,\end{cases}$$

where $R_b=A_b-D_b, R'=\sqrt{\alpha A_1}$, A_b *and* D_b *are as given in Lemma 7.1.2.*

Lemma 7.1.4. [**14**, Theorem 5.1] *If* $f(z)=z+a_2z^2+\cdots\in\mathcal{K}$, *then* $f\in\mathcal{S}^*(\alpha)$, *where* α *is the smallest positive root of the equation* $2\alpha^3-|a_2|\alpha^2-4\alpha+2=0$, *in the interval* $[1/2,2/3]$.

Lemma 7.1.5. [**11**, Lemma 2.2] *For* $0<a<\sqrt{2}$, *let* r_a *be given by*

$$r_a=\begin{cases}(\sqrt{1-a^2}-(1-a^2))^{1/2}, & 0<a\leq 2\sqrt{2}/3;\\ \sqrt{2}-a, & 2\sqrt{2}/3\leq a<\sqrt{2},\end{cases}$$

and for $a>0$, *let* R_a *be given by*

$$R_a=\begin{cases}\sqrt{2}-a, & 0<a\leq 1/\sqrt{2};\\ a, & 1/\sqrt{2}\leq a.\end{cases}$$

Then $\{w:|w-a|<r_a\}\subset\{w:|w^2-1|<1\}\subset\{w:|w-a|<R_a\}$.

Lemma 7.1.6. [**165**, Section 3] *Let* $a>1/2$. *If the number* R_a *is given by*

$$R_a=\begin{cases}a-1/2, & 1/2<a\leq 3/2;\\ \sqrt{2a-2}, & a\geq 3/2,\end{cases}$$

then $\{w\in\mathbb{C}:|w-a|<R_a\}\subset\{w\in\mathbb{C}:|w-a|<\operatorname{Re}w\}$.

7.2 Radii of Starlikeness

For notational convenience, let us denote by $\mathcal{A}(b)$, the class of normalized analytic functions of the form $f(z) = z + bz^2 + \cdots$. Since the coefficients of functions with positive real part are bounded by 2, for the function $f(z) = z + a_2z^2 + \cdots$ satisfying $\operatorname{Re}(f(z)/z) > 0$, it follows that $|a_2| \leq 2$. Hence, the functions with the property $\operatorname{Re}(f(z)/z) > 0$ have the Maclaurin series expansion of the form

$$f(z) = z + 2bz^2 + \cdots \quad (|b| \leq 1).$$

Definition 7.2.1. The class $\mathcal{F}_b^1$ is defined by

$$\mathcal{F}_b^1 := \left\{ f \in \mathcal{A}(2b) : \operatorname{Re}\left(\frac{f(z)}{z}\right) > 0, |b| \leq 1 \right\}.$$

Theorem 7.2.1. *For the class* $\mathcal{F}_b^1$*,*

(1) $\mathcal{S}_L^*-$*radius* r_0 *is the smallest positive root in* $(0,1]$ *of the equation*

$$(\sqrt{2}-1)r^4 + 2\sqrt{2}|b|r^3 + 4r^2 + 2|b|(2-\sqrt{2})r - \sqrt{2} + 1 = 0, \tag{7.2.1}$$

(2) $\mathcal{M}(\beta)-$*radius* r_1 *is the smallest positive root in* $(0,1]$ *of the equation*

$$(\beta-1)r^4 + 2|b|\beta r^3 + 4r^2 + 2|b|(2-\beta)r - \beta + 1 = 0. \tag{7.2.2}$$

(3) $\mathcal{S}^*(\alpha)-$*radius* r_2 *is the smallest positive root in* $(0,1]$ *of the equation*

$$(1-\alpha)r^4 + 2|b|(2-\alpha)r^3 + 4r^2 + 2|b|\alpha r + \alpha - 1 = 0. \tag{7.2.3}$$

(4) $\mathcal{S}_P^*-$ *radius* r_3 *is the smallest positive root in* $(0,1]$ *of the equation*

$$r^4 + 6|b|r^3 + 8r^2 + 2|b|r - 1 = 0. \tag{7.2.4}$$

Proof. (1) Given that the function $p(z) = f(z)/z = 1 + 2bz + \cdots \in \mathcal{P}_b$. Using the fact $zp'(z)/p(z) = zf'(z)/f(z) - 1$ and Lemma 7.1.1 with $\alpha = 0$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r(|b|r^2 + 2r + |b|)}{(1-r^2)(r^2 + 2|b|r + 1)}. \tag{7.2.5}$$

Now Lemma 7.1.5 reveals that the function f satisfies $|(zf'(z)/f(z))^2-1|<1$, in $|z|<r$, if the following inequality holds:

$$\frac{2r(|b|r^2+2r+|b|)}{(1-r^2)(r^2+2|b|r+1)}\leq\sqrt{2}-1,$$

or equivalently, if the inequality

$$1-\sqrt{2}+2|b|\left(2-\sqrt{2}\right)r+4r^2+2\sqrt{2}|b|r^3+\left(\sqrt{2}-1\right)r^4\leq 0$$

holds. Therefore, $\mathcal{S}_L^*$–radius of the class $\mathcal{F}_b^1$ is the smallest positive root $r_0\in(0,1]$ of the Equation (7.2.1).

To prove the sharpness, consider the function f_0 defined by

$$f_0(z)=\frac{z(1+2bz+z^2)}{1-z^2}. \tag{7.2.6}$$

If we set $w(z):=z(z+b)/(1+bz)$ ($|b|\leq 1$), the Schwarz function, then we see that

$$\frac{f_0(z)}{z}=\frac{1+w(z)}{1-w(z)}$$

and hence $\operatorname{Re}(f_0(z)/z)>0$ in $\mathbb{D}$ or $f_0\in\mathcal{F}_b^1$. Since, for $z=r_0$,

$$\frac{zf_0'(z)}{f_0(z)}=\frac{1+4br_0+4r_0^2-r_0^4}{(1-r_0^2)\left(1+2br_0+r_0^2\right)}=\sqrt{2},$$

it follows that, for $z=r_0$,

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2-1\right|=1.$$

This shows sharpness of the result. Figures given below illustrate sharpness of the result:

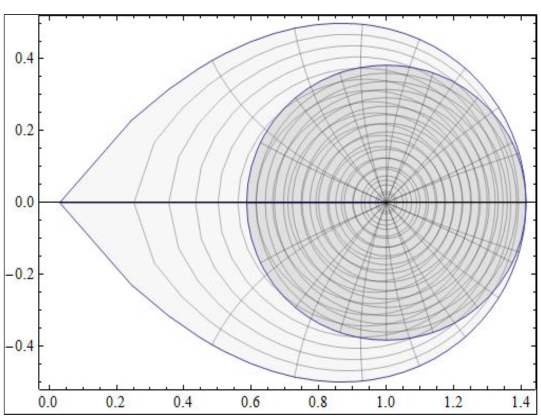

Figure 7.1. The $\mathcal{S}_L^*$-radius $r_0\approx 0.19891$ for $b=1$ is sharp.

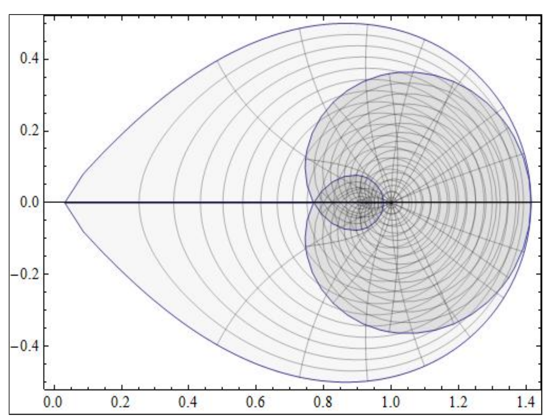

Figure 7.2. The $\mathcal{S}_L^*$-radius $r_0\approx 0.2479$ for $b=0.5$ is sharp.

(2) The inequality (7.2.5) shows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq 1+\frac{2r(|b|r^2+2r+|b|)}{(1-r^2)(r^2+2|b|r+1)} \leq \beta$$

if the following inequality holds:

$$(\beta-1)r^4+2|b|\beta r^3+4r^2+2(2-\beta)|b|r+1-\beta \leq 0.$$

Therefore, $\mathcal{M}(\beta)$-radius of the class $\mathcal{F}_b^1$ is the smallest positive root $r_1 \in (0,1]$ of the Equation (7.2.2). The result is sharp for the function given in (7.2.6). Since for $z = r_1$,

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1+4br_1+4r_1^2-r_1^4}{(1-r_1^2)(1+2br_1+r_1^2)} = \beta.$$

Figures shown below illustrate the sharpness of result:

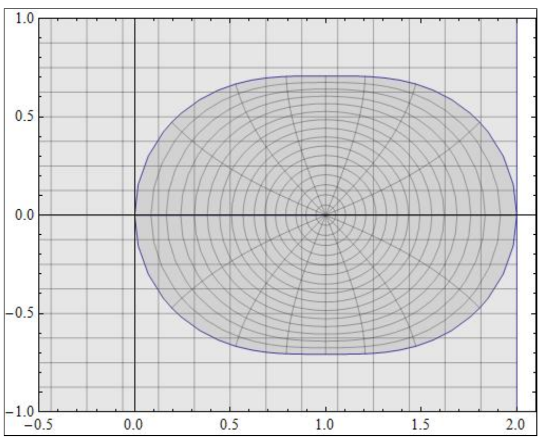

Figure 7.3. The $\mathcal{M}(2)$-radius $r_1 \approx 0.4142$ for $b = 1$ radius is sharp.

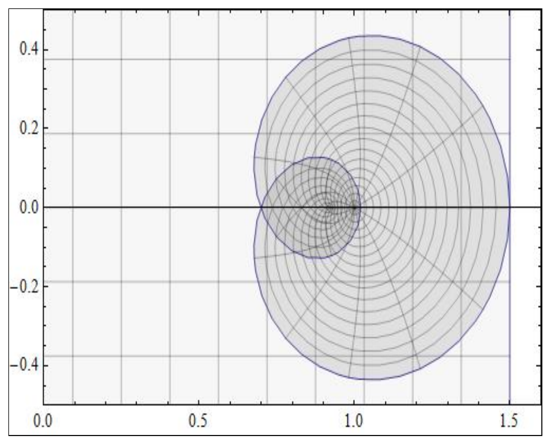

Figure 7.4. The $\mathcal{M}(3/2)$-radius $r_1 \approx 0.2833$ for $b = 0.5$ is sharp.

(3) In view of (7.2.5), it follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 1-\frac{2r(|b|r^2+2r+|b|)}{(1-r^2)(r^2+2|b|r+1)} \geq \alpha$$

if the following inequality holds:

$$(1-\alpha)r^4+2|b|(2-\alpha)r^3+4r^2+2|b|\alpha r+\alpha-1 \leq 0.$$

Thus, $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}_b^1$ is the smallest positive root $r_2 \in (0,1]$ of the Equation (7.2.3).

The function f_0 defined by

$$f_0(z) = \frac{z(1-z^2)}{1-2bz+z^2} \tag{7.2.7}$$

is in the class $\mathcal{F}_b^1$ because for the function f_0 defined in (7.2.7), we have

$$\frac{f_0(z)}{z} = \frac{1-w(z)}{1+w(z)},$$

where $w(z) = z(z-b)/(1-bz)$ is an analytic function satisfying the conditions of Schwarz's lemma in the unit disk $\mathbb{D}$, and hence $\operatorname{Re}(f_0(z)/z) > 0$ in $\mathbb{D}$. The result is sharp for the function given in (7.2.7) as, for

$$\operatorname{Re}\left(\frac{zf_0'(z)}{f_0(z)}\right) = \frac{zf_0'(z)}{f_0(z)} = \frac{1-r_2^2\,(4-4br_2+r_2^2)}{(1-r_2^2)\,(1-2br_2+r_2^2)} = \alpha \quad (z=-r_2),$$

which demonstrates sharpness.

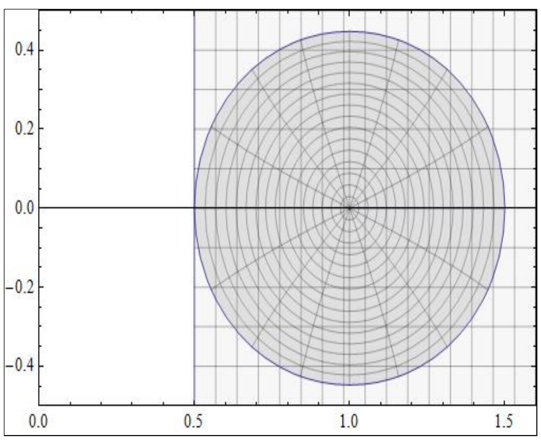

Figure 7.5. The $\mathcal{S}^*(1/2)$-radius $r_2 \approx 0.2360$ for $b=1$ is sharp.

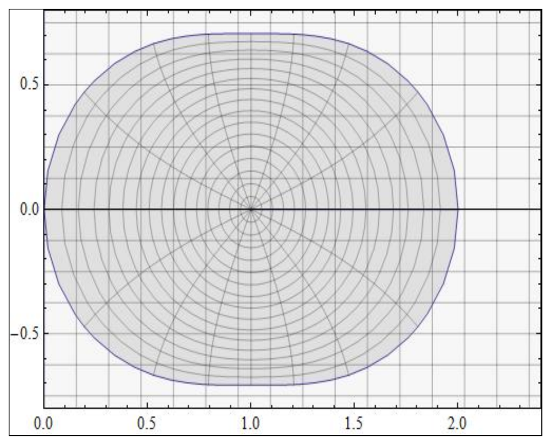

Figure 7.6. The $\mathcal{S}^*$-radius $r_2 \approx 0.41421$ for $b=1$ is sharp.

(4) Lemma 7.1.6 shows that the disk given in (7.2.5) lies inside the parabolic domain $\Omega = \{w : |w-1| < \operatorname{Re} w\}$ provided that

$$\frac{2r(|b|r^2+2r+|b|)}{(1-r^2)(r^2+2|b|r+1)} \leq \frac{1}{2},$$

or equivalently, if the inequality $r^4+6|b|r^3+8r^2+2|b|r-1 \leq 0$ holds. Thus, $\mathcal{S}_P^*-$radius of the class $\mathcal{F}_b^1$ is the smallest positive root $r_3 \in (0,1]$ of the Equation (7.2.4).

The function defined in (7.2.7) satisfies

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1 - r_3^2\,(4 - 4br_3 + r_3^2)}{(1 - r_3^2)\,(1 - 2br_3 + r_3^2)} = \frac{1}{2} \quad (z = -r_3)$$

which demonstrates sharpness. Sharpness of the result evident form the following figures. □

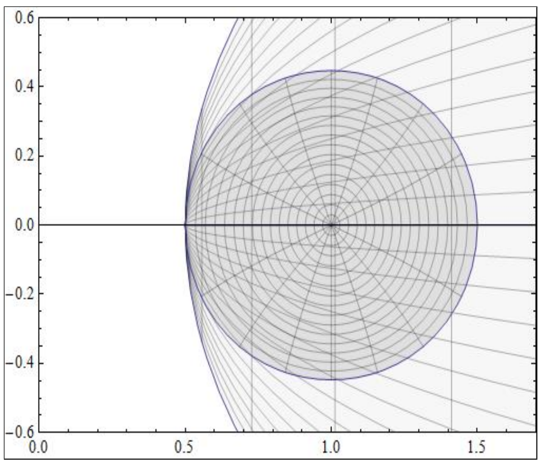

Figure 7.7. The $\mathcal{S}_P^*$−radius $r_3 \approx 0.2360$ for $b = 1$ is sharp.

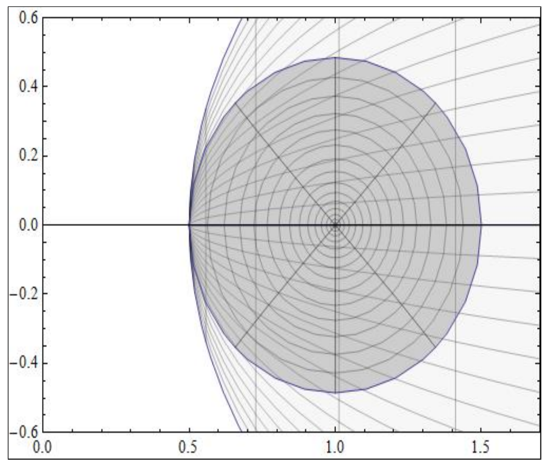

Figure 7.8. The $\mathcal{S}_P^*$−radius $r_3 \approx 0.3509$ for $b = 0$ is sharp.

Remark 7.2.1. For $\alpha = 0$, part (3) of Theorem 7.2.1 reduces to the following result of Goel [**59**, Theorem 2]:

If $f(z) = z + bz^2 + \cdots$ is analytic in $\mathbb{D}$ and satisfies the condition $\operatorname{Re}(f(z)/z) > 0$, then f is univalent and starlike for $|z| < r_2$, where $r_2 \in (0, 1]$ is the smallest positive root of the equation $1 - 4r^2 - 4br^3 - r^4 = 0$.

7.3 Radii Problems for Functions Defined by Ratio of Functions

Let $f(z) = z + a_2z^2 + \cdots$ satisfies $\operatorname{Re}(f(z)/g(z)) > 0$ for some normalized analytic function $g(z) = z + g_2z^2 + \cdots$ such that $\operatorname{Re}(g(z)/z) > 0$ in $\mathbb{D}$. Then the second coefficient g_2 of g is bounded by 2, and $a_2 = g_2 + c_1$, where c_1 is the coefficient of a function with positive real part and so $|c_1| \leq 2$, and hence $|a_2| \leq 4$. Our next theorem focuses on these functions with fixed second coefficient which are given by respectively

$$f(z) = z + 4bz^2 + \cdots \quad \text{and} \quad g(z) = z + 2cz^2 + \cdots .$$

Definition 7.3.1. For $|b| \leq 1$ and $|c| \leq 1$, we define the class of function $\mathcal{F}^2_{b,c}$ as follows:

$$\mathcal{F}^2_{b,c} := \left\{ f \in \mathcal{A}(4b) : \operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \text{ and } \operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, g \in \mathcal{A}(2c) \right\}.$$

Theorem 7.3.1. *Assume that $\gamma := |2b - c|$. For the class $\mathcal{F}^2_{b,c}$,*

(1) $\mathcal{S}^*_L-$radius r_0 is the smallest positive root in $(0,1]$ of the equation

$$(\sqrt{2}-1)r^6 + (|c|+\gamma)2\sqrt{2}r^5 + (7+\sqrt{2}+4(1+\sqrt{2})|c|\gamma)r^4 + 12(|c|+\gamma)r^3 \\ + (9-\sqrt{2}+4(3-\sqrt{2})|c|\gamma)r^2 + 2(2-\sqrt{2})(|c|+\gamma)r - \sqrt{2}+1 = 0. \quad (7.3.1)$$

(2) $\mathcal{M}(\beta)-$radius r_1 is the smallest positive root in $(0,1]$ of the equation

$$(\beta-1)r^6 + 2\beta(|c|+\gamma)r^5 + (7+\beta+4(1+\beta)|c|\gamma)r^4 + 12(|c|+\gamma)r^3 \\ + (9-\beta+4(3-\beta)|c|\gamma)r^2 + 2(2-\beta)(|c|+\gamma)r - \beta + 1 = 0. \quad (7.3.2)$$

(3) $\mathcal{S}^*(\alpha)-$radius r_2 is the smallest positive root in $(0,1]$ of the equation

$$(1-\alpha)r^6 + 2(2-\alpha)(|c|+\gamma)r^5 + (9-\alpha+4(3-\alpha)|c|\gamma)r^4 + 12(|c|+\gamma)r^3 \\ + (7+\alpha+4(1+\alpha)|c|\gamma)r^2 + 2(|c|+\gamma)\alpha r + \alpha - 1 = 0. \quad (7.3.3)$$

(4) $\mathcal{S}^*_P-$radius r_3 is the smallest positive root in $(0,1]$ of the equation

$$r^6 + 6(|c|+\gamma)r^5 + (17+20\gamma|c|)r^4 + 24(|c|+\gamma)r^3 + (15+12\gamma|c|)r^2 \\ + 2(|c|+\gamma)r - 1 = 0. \quad (7.3.4)$$

Proof. (1) Let the functions p and h be defined by

$$p(z) = \frac{g(z)}{z} \quad \text{and} \quad h(z) = \frac{f(z)}{g(z)}.$$

Then

$$p(z) = 1 + 2cz + \cdots \quad \text{and} \quad h(z) = 1 + 2(2b - c)z + \cdots$$

or $p \in \mathcal{P}_c$ and $h \in \mathcal{P}_{2b-c}$. Since $f(z) = zp(z)h(z)$, we obtain from Lemma 7.1.1 with $\alpha = 0$ the following

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zh'(z)}{h(z)}\right| \\ &\leq \frac{2r}{1-r^2}\left(\frac{|c|r^2 + 2r + |c|}{r^2 + 2|c|r + 1} + \frac{\gamma r^2 + 2r + \gamma}{r^2 + 2\gamma r + 1}\right). \end{aligned} \tag{7.3.5}$$

From Lemma 7.1.5 it follows that, the function f satisfies

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1 \quad (|z| < r)$$

if the inequality

$$\frac{2r}{1-r^2}\left(\frac{|c|r^2 + 2r + |c|}{r^2 + 2|c|r + 1} + \frac{\gamma r^2 + 2r + \gamma}{r^2 + 2\gamma r + 1}\right) \leq \sqrt{2} - 1$$

holds or equivalently

$$\begin{aligned} &(\sqrt{2} - 1)r^6 + (|c| + \gamma)2\sqrt{2}r^5 + (7 + \sqrt{2} + 4(1 + \sqrt{2})|c|\gamma)r^4 + 12(|c| + \gamma)r^3 \\ &\quad + (9 - \sqrt{2} + 4(3 - \sqrt{2})|c|\gamma)r^2 + 2(2 - \sqrt{2})(|c| + \gamma)r - \sqrt{2} + 1 \leq 0. \end{aligned}$$

Therefore, the $\mathcal{S}^*_L$-radius of the class $\mathcal{F}^2_{b,c}$ is the smallest positive root $r_0 \in (0, 1]$ of the Equation (7.3.1).

Consider the functions defined by

$$f_0(z) = \frac{z\left(1 + (4b - 2c)z + z^2\right)\left(1 + 2cz + z^2\right)}{(1 - z^2)^2} \quad \text{and} \quad g_0(z) = \frac{z\left(1 + 2cz + z^2\right)}{(1 - z^2)}. \tag{7.3.6}$$

The function f_0 with the choice g_0, defined above, is in the class $\mathcal{F}^2_{b,c}$ because

$$\frac{f_0(z)}{g_0(z)} = \frac{1 + w_1(z)}{1 - w_1(z)} \quad \text{and} \quad \frac{g_0(z)}{z} = \frac{1 + w_2(z)}{1 - w_2(z)},$$

where

$$w_1(z) = \frac{z(z + 2b - c)}{1 + (2b - c)z} \text{ and } w_2(z) = \frac{z(z + c)}{1 + cz}$$

are analytic functions satisfying the conditions of Schwarz's lemma on the unit disk $\mathbb{D}$, and hence

$$\operatorname{Re}\left(\frac{g_0(z)}{z}\right) > 0 \text{ and } \operatorname{Re}\left(\frac{f_0(z)}{g_0(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Since, for $z = r_0$

$$\frac{zf_0'(z)}{f_0(z)} = 1 + \frac{2}{1-r_0} + \frac{2}{1+r_0} - \frac{2(1+cr_0)}{1+2cr_0+r_0^2} - \frac{2+4br_0-2cr_0}{1+r_0(4b-2c+r_0)} = \sqrt{2}, \quad (7.3.7)$$

we have

$$\left| \left(\frac{zf_0(z)}{f_0(z)} \right)^2 - 1 \right| = 1.$$

Thus, the result is sharp. The following figures illustrate sharpness of the result:

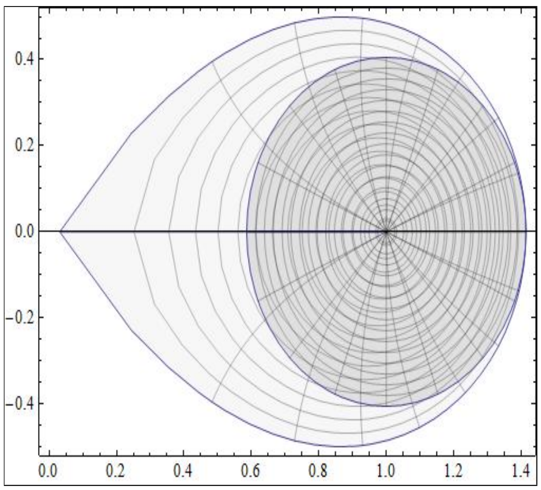

Figure 7.9. The $\mathcal{S}_L^*$-radius $r_0 \approx 0.1025$ for $b = 1 = c$ is sharp.

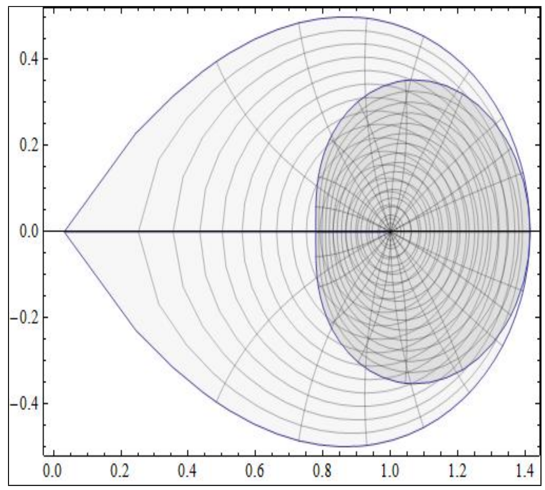

Figure 7.10. The $\mathcal{S}_L^*$-radius $r_0 \approx 0.3509$ for $b = 0.5$ and $c = 1$ is sharp.

(2) The inequality (7.3.5) shows that

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &\leq 1 + \frac{2r}{1-r^2}\left(\frac{|c|r^2+2r+|c|}{r^2+2|c|r+1} + \frac{\gamma r^2+2r+\gamma}{r^2+2\gamma r+1}\right) \\ &\leq \beta \end{aligned}$$

if the following inequality holds:

$$\begin{aligned} (\beta-1)r^6 + 2\beta(|c|+\gamma)r^5 + (7+\beta+4(1+\beta)|c|\gamma)r^4 \\ + 12(|c|+\gamma)r^3 + (9-\beta+4(3-\beta)|c|\gamma)r^2 \\ + 2(2-\beta)(|c|+\gamma)r - \beta + 1 \leq 0. \end{aligned}$$

Hence $\mathcal{M}(\beta)$-radius of the class $\mathcal{F}_{b,c}^2$ is the smallest positive root $r_1 \in (0,1]$ of the Equation (7.3.2).

The result is sharp for the functions given in (7.3.6). For $z = r_1$, we have

$$\begin{aligned}\frac{zf_0'(z)}{f_0(z)} &= 1 + \frac{2}{1-r_1} + \frac{2}{1+r_1} - \frac{2(1+cr_1)}{1+2cr_1+r_1^2} - \frac{2+4br_1-2cr_1}{1+r_1(4b-2c+r_1)} \\ &= \beta.\end{aligned}$$

This shows sharpness of the result. The following figures illustrate sharpness of the result:

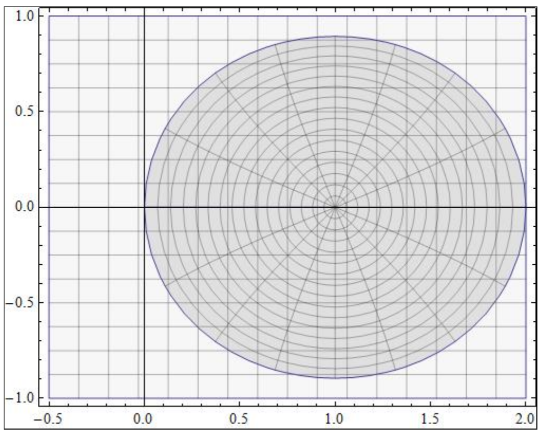

Figure 7.11. The $\mathcal{M}(2)$-radius $r_1 \approx 0.2360$ for $b = 1 = c$ is sharp.

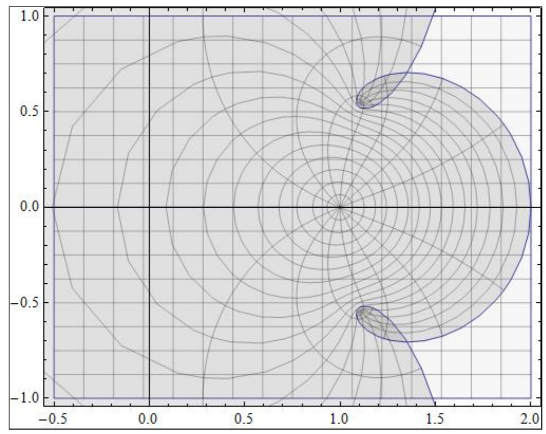

Figure 7.12. The $\mathcal{M}(2)$-radius $r_1 \approx 0.2659$ for $b = 1$ and $c = 0$ is sharp.

(3) In view of (7.3.5), it follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 1 - \frac{2r}{1-r^2}\left(\frac{|c|r^2+2r+|c|}{r^2+2|c|r+1} + \frac{\gamma r^2+2r+\gamma}{r^2+2\gamma r+1}\right) \geq \alpha$$

if the following inequality holds:

$$\begin{aligned}&(1-\alpha)r^6 + 2(2-\alpha)(|c|+\gamma)r^5 + (9-\alpha+4(3-\alpha)|c|\gamma)r^4 + 12(|c|+\gamma)r^3 \\ &\quad + (7+\alpha+4(1+\alpha)|c|\gamma)r^2 + 2(|c|+\gamma)\alpha r + \alpha - 1 \leq 0.\end{aligned}$$

Thus, $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}_{b,c}^2$ is the smallest positive root $r_2 \in (0,1]$ of the Equation (7.3.3).

Consider the functions defined by

$$f_0(z) = \frac{z\,(1-z^2)^2}{(1-(4b-2c)z+z^2)\,(1-2cz+z^2)} \text{ and } g_0(z) = \frac{z\,(1-z^2)}{(1-2cz+z^2)}. \quad (7.3.8)$$

The function f_0 with the choice g_0, defined in (7.3.8), is in the class $\mathcal{F}^2_{b,c}$ because

$$\frac{f_0(z)}{g_0(z)} = \frac{1-w_1(z)}{1+w_1(z)} \quad \text{and} \quad \frac{g_0(z)}{z} = \frac{1-w_2(z)}{1+w_2(z)},$$

where

$$w_1(z) = \frac{z(z-(2b-c))}{1-(2b-c)z} \quad \text{and} \quad w_2(z) = \frac{z(z-c)}{1-cz} \quad (|2b-c| \leq 1)$$

are analytic functions satisfying the conditions of Schwarz's lemma and hence

$$\operatorname{Re}\left(\frac{g_0(z)}{z}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{f_0(z)}{g_0(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

The functions defined in (1.4.2) satisfy

$$\frac{zf_0'(z)}{f_0(z)} = 1 - \frac{2}{1+r_2} - \frac{2}{1-r_2} + \frac{2+2cr_2}{1+2cr_2+r_2^2} + \frac{2(1+2br_2-cr_2)}{1+r_2(4b-2c+r_2)} = \alpha \quad (z=-r_2)$$

which demonstrates the sharpness. Figures given below illustrate sharpness of the result:

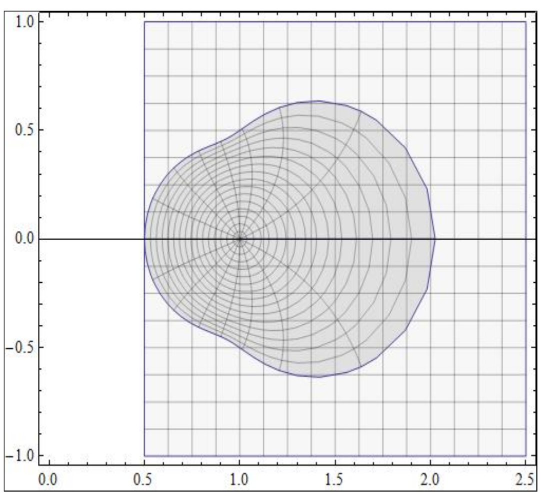

Figure 7.13. The $\mathcal{S}^*(1/2)$-radius $r_2 \approx 0.1406$ for $b=1$ and $c=0$ is sharp.

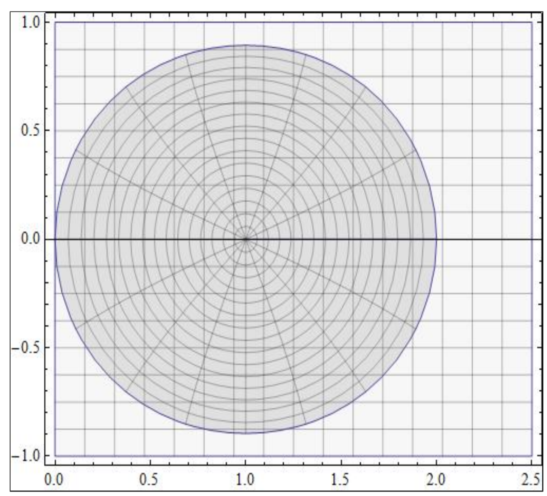

Figure 7.14. The $\mathcal{S}^*$-radius $r_2 \approx 0.2360$ for $b = 1 = c$ is sharp.

(4) In view of Lemma 7.1.6, the disk given in (7.3.5) lies inside the parabolic region $\Omega = \{w : |w-1| < \operatorname{Re} w\}$, if $M \leq 1/2$ which on simplification becomes

$$r^6 + (6q+6\gamma)r^5 + (17+20q\gamma)r^4 + (24q+24\gamma)r^3 + (15+12q\gamma)r^2 + (2q+2\gamma)r - 1 \leq 0.$$

The result is sharp for the functions defined in (7.3.8) as it can be seen from

$$\left(\frac{zf'(z)}{f(z)}\right)_{z=-r_0} = 1 - \frac{2}{1+r_0} - \frac{2}{1-r_0} + \frac{2+2q_1r_0}{1+2q_1r_0+r_0^2} + \frac{2(1+2p_1r_0-q_1r_0)}{1+r_0(4p_1-2q_1+r_0)} = \frac{1}{2}.$$

This confirms the sharpness of result. The following figures illustrate sharpness of the result. □

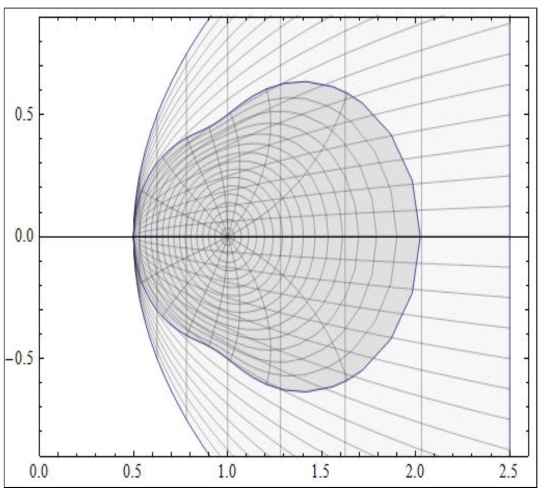

Figure 7.15. The $\mathcal{S}_P^*$-radius $r_3 \approx 0.1406$ for $b = 1$ and $c = 0$ is sharp.

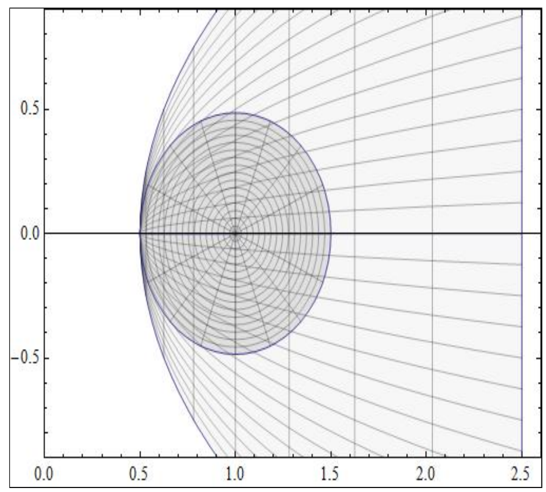

Figure 7.16. The $\mathcal{S}_P^*$-radius $r_3 \approx 0.1231$ for $b = 1 = c$ is sharp.

Remark 7.3.1. Setting $b = 1 = c$, in Theorem 7.3.1 obtain the following result of Ali et al. [**10**, Theorem 2.1].

For the class $\mathcal{F}_{1,1}^2$,

(i) the $\mathcal{S}_L^*-$radius, $r_0 = \frac{\sqrt{2}-1}{2+\sqrt{7-2\sqrt{2}}}$,

(ii) the $\mathcal{M}(\beta)-$radius, $r_1 = \frac{\beta-1}{2+\sqrt{4+(\beta-1)^2}}$,

(iii) the $\mathcal{S}^*(\alpha)-$radius, $r_2 = \frac{1-\beta}{2+\sqrt{4+(1-\alpha)^2}}$,

(iv) the $\mathcal{S}_P^*-$radius, $r_3 = \frac{1}{4+\sqrt{17}}$.

Let $f(z) = z + a_2z^2 + \cdots$ satisfies $\mathrm{Re}(f(z)/g(z)) > 0$ for some normalized analytic function $g(z) = z + g_2z^2 + \cdots$ such that $\mathrm{Re}(g(z)/z) > 1/2$ in $\mathbb{D}$. Then the second coefficient g_2 of g is bounded by 1, and $a_2 = g_2 + c_1$, where c_1 is the coefficient of a function with positive real part, and hence $|a_2| \leq 3$. In the next theorem, we shall focus on functions with fixed second coefficient which are given respectively by

$$f(z) = z + 3bz^2 + \cdots \quad \text{and} \quad g(z) = z + cz^2 + \cdots .$$

Definition 7.3.2. For $|b| \leq 1$ and $|c| \leq 1$, let

$$\mathcal{F}^3_{b,c} := \left\{ f \in \mathcal{A}_{3b} : \operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \text{ and } \operatorname{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2},\ g \in \mathcal{A}_c \right\}.$$

Theorem 7.3.2. *Assume that* $\gamma_1 = |3b - c|$. *For the class* $\mathcal{F}^3_{b,c}$,

(1) $\mathcal{S}^*_L-$*radius* r_0 *is the smallest positive root in* $(0, 1]$ *of the equation*

$$\begin{aligned} &\sqrt{2}|c|r^5 + (1+\sqrt{2})(1+|c|\gamma_1)r^4 + (6|c| + \sqrt{2}(1+\sqrt{2})\gamma_1)r^3 \\ &\quad + (6 + (3-\sqrt{2})|c|\gamma_1)r^2 + \sqrt{2}(\sqrt{2}-1)(|c|+\gamma_1)r - \sqrt{2} + 1 = 0. \end{aligned} \tag{7.3.9}$$

The result is sharp.

(2) $\mathcal{M}(\beta)-$*radius* r_1 *is the smallest positive root in* $(0, 1]$ *of the equation*

$$\begin{aligned} &|c|\beta r^5 + (1+\beta)(1+|c|\gamma_1)r^4 + (6|c| + (2+\beta)\gamma_1)r^3 \\ &\quad + (6 + (3-\beta)|c|\gamma_1)r^2 + (2-\beta)(|c|+\gamma_1)r - \beta + 1 = 0. \end{aligned} \tag{7.3.10}$$

The result is sharp.

(3) $\mathcal{S}^*(\alpha)-$*radius* r_2 *is the smallest positive root in* $(0, 1]$ *of the equation*

$$\begin{aligned} &-|c|\alpha r^7 + (|c|(1-\alpha)(\gamma_1 + 2|c|) - 1 - \alpha)r^6 + (|c|(2-\alpha)(3+2|c|\gamma_1) - \alpha\gamma_1)r^5 \\ &\quad + (5 + 8|c|^2 - \alpha + 2(3-\alpha)|c|\gamma_1)r^4 + ((12+\alpha)|c| + 2(2+|c|^2\alpha)\gamma_1)r^3 \\ &\quad + (5 - 2|c|^2 + \alpha + 2|c|^2\alpha + (1+3\alpha)|c|\gamma_1)r^2 + (2|c| + 3|c|\alpha + \alpha\gamma_1)r \\ &\quad + \alpha - 1 = 0. \end{aligned} \tag{7.3.11}$$

(4) $\mathcal{S}^*_P-$*radius* r_3 *is the smallest positive root in* $(0, 1]$ *of the equation*

$$\begin{aligned} &|c|r^7 + (1 + 4|c|^2 + 3|c|\gamma_1)r^6 + (17|c| + 3\gamma_1 + 8|c|^2\gamma_1)r^5 \\ &\quad + (13 + 20|c|^2 + 16|c|\gamma_1)r^4 + (31|c| + 8\gamma_1 + 4|c|^2\gamma_1)r^3 \\ &\quad + (11 + 5|c|\gamma_1)r^2 + (\gamma_1 - |c|)r - 1 = 0. \end{aligned} \tag{7.3.12}$$

Proof. (1) Define the functions p and h by $p(z)=g(z)/z$ and $h(z)=f(z)/g(z)$

$$p(z)=1+cz+\cdots \quad \text{and} \quad h(z)=\frac{f(z)}{g(z)}=1+(3b-c)z+\cdots \tag{7.3.13}$$

or $p\in\mathcal{P}_{c/2}(1/2)$ and $h\in\mathcal{P}_{(3b-c)/2}$. Lemma 7.1.1 with $\alpha=0$ and $\alpha=1/2$ respectively lead to

$$\left|\frac{zh'(z)}{h(z)}\right|\leq\frac{r}{1-r^2}\left(\frac{\gamma_1 r^2+4r+\gamma_1}{r^2+\gamma_1 r+1}\right) \tag{7.3.14}$$

and

$$\left|\frac{zp'(z)}{p(z)}\right|\leq\frac{r}{1-r^2}\left(\frac{|c|r^2+2r+|c|}{|c|r+1}\right). \tag{7.3.15}$$

From (7.3.13), $f(z)/z=p(z)h(z)$, and so the inequalities in (7.3.14) and (7.3.15) yield

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-1\right| &\leq \left|\frac{zh'(z)}{h(z)}\right|+\left|\frac{zp'(z)}{p(z)}\right| \\ &\leq \frac{r}{1-r^2}\left(\frac{\gamma_1 r^2+4r+\gamma_1}{r^2+\gamma_1 r+1}+\frac{|c|r^2+2r+|c|}{|c|r+1}\right).\end{aligned} \tag{7.3.16}$$

By Lemma 7.1.5, the function f satisfies $|(zf'(z)/f(z))^2-1|<1$, in $|z|<r$, if the following inequality holds

$$\frac{r}{1-r^2}\left(\frac{\gamma_1 r^2+4r+\gamma_1}{r^2+\gamma_1 r+1}+\frac{|c|r^2+2r+|c|}{|c|r+1}\right)\leq\sqrt{2}-1$$

or equivalently, if the following inequality holds:

$$\begin{aligned}&\sqrt{2}|c|r^5+(1+\sqrt{2})(1+|c|\gamma_1)r^4+(6|c|+\sqrt{2}(1+\sqrt{2})\gamma_1)r^3 \\ &\quad+(6+(3-\sqrt{2})|c|\gamma_1)r^2+\sqrt{2}(\sqrt{2}-1)(|c|+\gamma_1)r-\sqrt{2}+1\leq 0.\end{aligned}$$

Therefore, $\mathcal{S}^*_L$-radius of the class $\mathcal{F}^3_{b,c}$ is the smallest positive root $r_0\in(0,1]$ of the Equation (7.3.9). Consider the functions defined by

$$f_0(z)=\frac{z(1+(3b-c)z+z^2)(1+cz)}{(1-z^2)^2} \quad \text{and} \quad g_0(z)=\frac{z(1+cz)}{(1-z^2)}. \tag{7.3.17}$$

The function f_0 with the choice g_0, defined in (7.3.17), is in the class $\mathcal{F}^3_{b,c}$ because

$$\frac{f_0(z)}{g_0(z)}=\frac{1+w_1(z)}{1-w_1(z)} \quad \text{and} \quad \frac{g_0(z)}{z}=\frac{1+w_2(z)}{1-w_2(z)},$$

where

$$w_1(z)=\frac{z\left(z+\frac{3b-c}{2}\right)}{1+\frac{(3b-c)z}{2}} \quad \text{and} \quad w_2(z)=\frac{z\left(z+\frac{c}{2}\right)}{1+\frac{cz}{2}} \quad (|3b-c|\leq 2)$$

are analytic functions satisfying the conditions of Schwarz's lemma in the unit disk $\mathbb{D}$. Hence $\operatorname{Re}(g_0(z)/z) > 1/2$ and $\operatorname{Re}(f_0(z)/g_0(z)) > 0$ in $\mathbb{D}$. Since

$$\frac{zf'(z)}{f(z)} = \frac{2}{1-r_0} + \frac{2}{1+r_0} - \frac{1}{1+cr_0} - \frac{2+3br_0-cr_0}{1+r_0(3b-c+r_0)} = \sqrt{2} \quad (z = r_0),$$

we have

$$\left| \left(\frac{zf_0(z)}{f_0(z)} \right)^2 - 1 \right| = 1.$$

Thus the result is sharp. Following figures illustrate sharpness of the result:

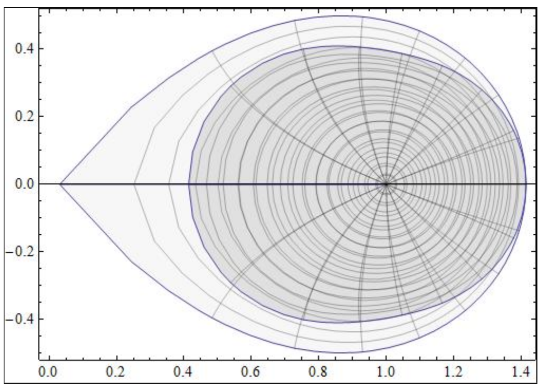

Figure 7.17. The $\mathcal{S}^*_L$-radius $r_0 \approx 0.1452$ for $b = 1$ and $c = 0$ is sharp.

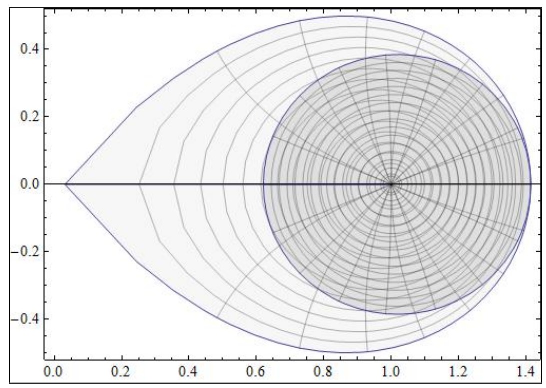

Figure 7.18. The $\mathcal{S}^*_L$-radius $r_0 \approx 0.1300$ for $b = 1$ and $c = 1$ is sharp.

(2) From inequality (7.3.16), we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq \frac{r}{1-r^2}\left(\frac{\gamma_1 r^2 + 4r + \gamma_1}{r^2 + \gamma_1 r + 1} + \frac{|c|r^2 + 2r + |c|}{|c|r + 1}\right) + 1 \leq \beta$$

if the following inequality holds:

$$\begin{aligned} &\beta|c|r^5 + (1+\beta)(1+|c|\gamma_1)r^4 + (6|c| + (2+\beta)\gamma_1)r^3 \\ &\quad + (6 + (3-\beta)|c|\gamma_1)r^2 + (2-\beta)(|c| + \gamma_1)r - \beta + 1 \leq 0. \end{aligned}$$

Therefore $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}^3_{b,c}$ is the smallest positive root $r_1 \in (0,1]$ of the Equation (7.3.10). The result is sharp for the functions given in (7.3.17) as it can be seen that, for $z = r_1$,

$$\frac{zf'(z)}{f(z)} = \frac{2}{1-r_1} + \frac{2}{1+r_1} - \frac{1}{1+cr_1} - \frac{2+3br_1-cr_1}{1+r_1(3b-c+r_1)} = \beta.$$

Following figures illustrate sharpness of the result:

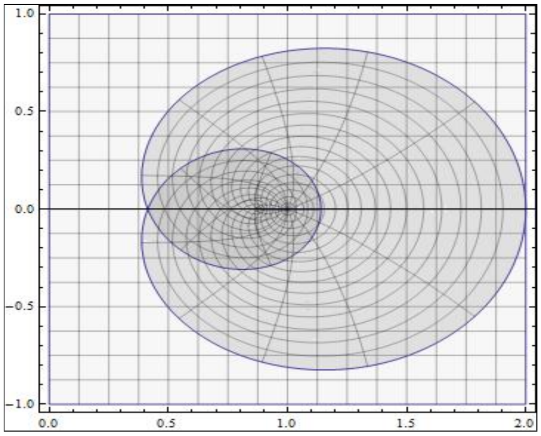

Figure 7.19. $\mathcal{M}(2)-$radius $r_1 \approx 0.3277$ for $b = 0.5$ and $c = 0.5$ is sharp.

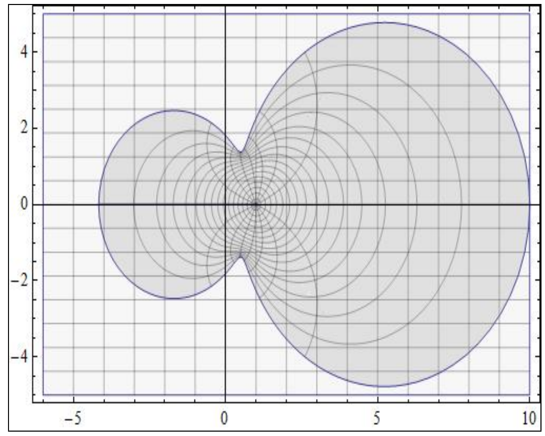

Figure 7.20. $\mathcal{M}(10)-$radius $r_1 \approx 0.8104$ for $b = 1$ and $c = 1$ is sharp.

(3) Since $f(z)/z = p(z)h(z)$, it follows from Lemma 7.1.1 and Lemma 7.1.3 that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 1 - \frac{(\gamma_1 r^2 + 4r + \gamma_1)r}{(r^2 + \gamma_1 r + 1)(1 - r^2)} + \frac{(|c| + 2r + |c|r^2)r}{(1 + 2|c|r + r^2)(1 + |c|r)} \geq \alpha, \quad (7.3.18)$$

if the following inequality holds:

$$\begin{aligned} &-|c|\alpha r^7 + (|c|(1-\alpha)(\gamma_1 + 2|c|) - 1 - \alpha)r^6 + (|c|(2-\alpha)(3 + 2|c|\gamma_1) - \alpha\gamma_1)r^5 \\ &\quad + (5 + 8|c|^2 - \alpha + 2(3-\alpha)|c|\gamma_1)r^4 + ((12+\alpha)|c| + 2(2 + |c|^2\alpha)\gamma_1)r^3 \\ &\quad + (5 - 2|c|^2 + \alpha + 2|c|^2\alpha + (1 + 3\alpha)|c|\gamma_1)r^2 + (2|c| + 3|c|\alpha + \alpha\gamma_1)r + \alpha - 1 \leq 0. \end{aligned}$$

Thus, $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}^3_{b,c}$ is the smallest positive root $r_2 \in (0, 1]$ of the Equation (7.3.11).

(4) From (7.3.16) and (7.3.18), it is clear that $|(zf'(z)/f(z)) - 1| < \operatorname{Re}(zf'(z)/f(z))$ provided

$$\begin{aligned} 1 - &\frac{(\gamma_1 r^2 + 4r + \gamma_1)r}{(r^2 + \gamma_1 r + 1)(1 - r^2)} + \frac{(|c| + 2r + |c|r^2)r}{(1 + 2|c|r + r^2)(1 + |c|r)} \\ &\geq \frac{r}{1 - r^2}\left(\frac{\gamma_1 r^2 + 4r + \gamma_1}{r^2 + \gamma_1 r + 1} + \frac{|c|r^2 + 2r + |c|}{|c|r + 1}\right) \end{aligned}$$

or equivalently, if the following inequality holds:

$$\begin{aligned} &|c|r^7 + (1 + 4|c|^2 + 3|c|\gamma_1)r^6 + (17|c| + 3\gamma_1 + 8|c|^2\gamma_1)r^5 + (13 + 20|c|^2 + 16|c|\gamma_1)r^4 \\ &\quad + (31|c| + 8\gamma_1 + 4|c|^2\gamma_1)r^3 + (11 + 5|c|\gamma_1)r^2 + (\gamma_1 - |c|)r - 1 \leq 0. \end{aligned}$$

The $\mathcal{S}_P^*$−radius of the class $\mathcal{F}_{b,c}^3$ is the smallest positive root r_3 in $(0,1]$ of the Equation (7.3.12). □

Remark 7.3.2. Putting $b = 1 = c$ in Theorem 7.3.2, we have the following result of Ali et al. [**10**, Theorem 2.2]:

For the class $\mathcal{F}_{1,1}^3$,

(i) the sharp $\mathcal{S}_L^*$−radius, $r_0 = \frac{4-2\sqrt{2}}{\sqrt{2}(\sqrt{17-4\sqrt{2}}+3)}$,

(ii) the sharp $\mathcal{M}(\beta)$−radius, $r_1 = \frac{2(\beta-1)}{3+\sqrt{9+4\beta(\beta-1)}}$,

(iii) the sharp $\mathcal{S}^*(\alpha)$−radius, $r_2 = \frac{2(1-\alpha)}{3+\sqrt{9-4\alpha+4\alpha^2}}$,

(iv) the $\mathcal{S}_P^*$−radius, $r_3 = \sqrt{10} - 3$.

Let $f(z) = z + a_2z^2 + \cdots$ satisfies $|f(z)/g(z) - 1| < 1$ for some normalized analytic function $g(z) = z + g_2z^2 + \cdots$ such that $\operatorname{Re}(g(z)/z) > 0$ in $\mathbb{D}$. Then the second coefficient g_2 of g is bounded by 2, and $a_2 = g_2 + c_1$, where c_1 is the coefficient of a function with positive real part and so $|c_1| \leq 2$, and hence $|a_2| \leq 3$. Our next theorem focuses on these functions with fixed second coefficients which are given respectively by $f(z) = z + 3bz^2 + \cdots$ and $g(z) = z + 2cz^2 + \cdots$.

Definition 7.3.3. For $|b| \leq 1$ and $|c| \leq 1$, let

$$\mathcal{F}_{b,c}^4 := \left\{ f \in \mathcal{A}_{3b} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1, \text{and } \operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \text{ where } g \in \mathcal{A}_{2c} \right\}.$$

Theorem 7.3.3. *Assume that $\delta := |2c - 3b|$. For the class $\mathcal{F}_{b,c}^4$,*

(1) $\mathcal{S}_L^*$−radius r_0 is the smallest positive root in $(0,1]$ of the equation

$$\sqrt{2}\delta r^5 + (1+\sqrt{2})(1+2\delta|c|))r^4 + 2(3\delta + \sqrt{2}(1+\sqrt{2})|c|)r^3 + 2(3 + (3-\sqrt{2})\delta|c|)r^2 + (2-\sqrt{2})(\delta + 2|c|)r - \sqrt{2} + 1 = 0. \tag{7.3.19}$$

(2) $\mathcal{M}(\beta)$−radius r_1 is the smallest positive root in $(0,1]$ of the equation

$$\beta\delta r^5 + (1+\beta)(1+2\delta|c|)r^4 + 2(3\delta + 2|c| + |c|\beta)r^3 + 2(3 + (3-\beta)\delta|c|)r^2 + (2-\beta)(\delta + 2|c|)r - \beta + 1 = 0. \tag{7.3.20}$$

(3) $f \in \mathcal{S}^*(\alpha)-$*radius* r_2 *is the smallest positive root in* $(0,1]$ *of the equation*

$$\begin{aligned}(2-\alpha)\delta r^5 + (1+2\delta|c|)(3-\alpha)r^4 + 2(3\delta + 4|c| - |c|\alpha)r^3 \\ + (\delta + 2|c|)\alpha r^2 + 2(3 + (1+\alpha)\delta|c|)r + \alpha - 1 = 0.\end{aligned} \tag{7.3.21}$$

(4) $\mathcal{S}^*_P-$*radius* r_3 *is the smallest positive root in* $(0,1]$ *of the equation*

$$3\delta r^5 + 5(1+2\delta|c|)r^4 + 2(6\delta + 7|c|)r^3 + 6(2+\delta|c|)r^2 + (\delta + 2|c|)r - 1 = 0. \tag{7.3.22}$$

Proof. (1) It is easy to see that $|f(z)/g(z) - 1| < 1$ if and only if $\operatorname{Re}(g(z)/f(z)) > 1/2$. Define the functions p and h by

$$p(z) = \frac{g(z)}{z} \quad \text{and} \quad h(z) = \frac{g(z)}{f(z)}.$$

Then

$$p(z) = 1 + 2cz + \cdots \quad \text{and} \quad h(z) = \frac{g(z)}{f(z)} = 1 + (2c - 3b)z + \cdots$$

or $p \in \mathcal{P}_c$ and $h \in \mathcal{P}_{(2c-3b)/2}(1/2)$. Lemma 7.1.1 with $\alpha = 0$ and $\alpha = 1/2$ respectively lead to

$$\left|\frac{zh'(z)}{h(z)}\right| \le \frac{2r(|c|r^2 + 2r + |c|)}{(1-r^2)(r^2 + 2|c|r + 1)} \quad \text{and} \quad \left|\frac{zp'(z)}{p(z)}\right| \le \frac{r(\delta r^2 + 2r + \delta)}{(1-r^2)(\delta r + 1)} \tag{7.3.23}$$

respectively, where $\delta := |2c - 3b|$. Since $zp(z) = f(z)h(z)$, from (7.3.23), we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\le \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zh'(z)}{h(z)}\right| \\ &\le \frac{r}{1-r^2}\left(\frac{2(|c|r^2 + 2r + |c|)}{r^2 + 2qr + 1} + \frac{(\delta r^2 + 2r + \delta)}{\delta r + 1}\right).\end{aligned} \tag{7.3.24}$$

By Lemma 7.1.5, the function f satisfies $|(zf'(z)/f(z))^2 - 1| < 1$, in $|z| < r$, if the following inequality holds:

$$\frac{r}{1-r^2}\left(\frac{2(|c|r^2 + 2r + |c|)}{r^2 + 2qr + 1} + \frac{(\delta r^2 + 2r + \delta)}{\delta r + 1}\right) \le \sqrt{2} - 1,$$

or equivalently, if

$$\begin{aligned}\sqrt{2}\delta r^5 + (1+\sqrt{2})(1+2\delta|c|))r^4 + 2(3\delta + \sqrt{2}(1+\sqrt{2})|c|)r^3 \\ + 2(3 + (3-\sqrt{2})\delta|c|)r^2 + (2-\sqrt{2})(\delta + 2|c|)r - \sqrt{2} + 1 \le 0.\end{aligned}$$

Therefore the $\mathcal{S}_L^*-$radius of the class $\mathcal{F}_{b,c}^4$ is the smallest positive root $r_0 \in (0,1]$ of the Equation (7.3.19).

(2) Using (7.3.24), we can get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq 1 + \frac{r}{1-r^2}\left(\frac{2(|c|r^2+2r+|c|)}{r^2+2qr+1} + \frac{(\delta r^2+2r+\delta)}{\delta r+1}\right) \leq \beta$$

if the following inequality holds:

$$\begin{aligned}\beta\delta r^5 &+ (1+\beta)(1+2\delta|c|)r^4 + 2(3\delta+2|c|+|c|\beta)r^3 \\ &+ 2(3+(3-\beta)\delta|c|)r^2 + (2-\beta)(\delta+2|c|)r - \beta + 1 \leq 0.\end{aligned}$$

Therefore, $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}_{b,c}^4$ is the smallest positive root $r_1 \in (0,1]$ of the Equation (7.3.20).

(3) Inequality in (7.3.24) implies that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 1 - \frac{r}{1-r^2}\left(\frac{2(|c|r^2+2r+|c|)}{r^2+2qr+1} + \frac{(\delta r^2+2r+\delta)}{\delta r+1}\right) \geq \alpha$$

if the following inequality holds:

$$\begin{aligned}(2-\alpha)\delta r^5 &+ (1+2\delta|c|)(3-\alpha)r^4 + 2(3\delta+4|c|-|c|\alpha)r^3 \\ &+ (\delta+2|c|)\alpha r^2 + 2(3+(1+\alpha)\delta|c|)r + \alpha - 1 \leq 0.\end{aligned}$$

Thus, $\mathcal{S}^*(\alpha)-$radius of the class $\mathcal{F}_{b,c}^4$ is the smallest positive root in $r_2 \in (0,1]$ of the Equation (7.3.21).

(4) Lemma 7.1.6 shows that the disk (7.3.24) lies inside the parabolic region $\Omega = \{w : |w-1| < \operatorname{Re} w\}$ provided

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{r}{1-r^2}\left(\frac{2(|c|r^2+2r+|c|)}{r^2+2qr+1} + \frac{(\delta r^2+2r+\delta)}{\delta r+1}\right) \leq \frac{1}{2}$$

if the following inequality holds:

$$3\delta r^5 + 5(1+2\delta|c|)r^4 + 2(6\delta+7|c|)r^3 + 6(2+\delta|c|)r^2 + (\delta+2|c|)r - 1 \leq 0.$$

Therefore, $\mathcal{S}_P^*-$radius of the class $\mathcal{F}_{b,c}^4$ is the smallest positive root $r_3 \in (0,1]$ of the Equation (7.3.12). □

Remark 7.3.3. In the special case when $b = 1 = c$ the parts (3) and (4) are sharp. Putting $b = 1 = c$ in Theorem 7.3.3, we obtain the following results of Ali et al. [**10**, Theorem 2.3]:

For the class $\mathcal{F}^4_{1,1}$,

(i) the $\mathcal{S}^*_L$−radius, $r_0 = \frac{2(2-\sqrt{2})}{\sqrt{2}(\sqrt{17-4\sqrt{2}}+3)}$,

(ii) the $\mathcal{M}(\beta)$−radius, $r_1 = \frac{2(\beta-1)}{3+\sqrt{9+4\beta(\beta-1)}}$,

(iii) the sharp $f \in \mathcal{S}^*(\alpha)$−radius, $r_2 = \frac{2(1-\alpha)}{3+\sqrt{9+4\beta(1-\alpha)(2-\alpha)}}$,

(iv) the sharp $\mathcal{S}^*_P$−radius, $r_3 = \frac{2\sqrt{3}-3}{3}$.

Let $f(z) = z + a_2 z^2 + \cdots$ satisfies $|f(z)/g(z) - 1| < 1$ for some convex function $g(z) = z + g_2 z^2 + \cdots$ in $\mathbb{D}$. Then the second coefficient g_2 of g is bounded by 1, and $a_2 = g_2 + c_1$, where c_1 is the coefficient of a function with positive real part and so $|c_1| \leq 2$, this implies $|a_2| \leq 2$. Our next theorem focuses on these functions with fixed second coefficient which are given respectively by

$$f(z) = z + 2bz^2 + \cdots \quad \text{and} \quad g(z) = z + cz^2 + \cdots .$$

Definition 7.3.4. For $|b| \leq 1$ and $|c| \leq 1$, let

$$\mathcal{F}^5_{b,c} := \left\{ f \in \mathcal{A}_{2b} : \left| \frac{f(z)}{g(z)} - 1 \right| < 1, \text{ and } \operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \text{ where } g \in \mathcal{A}_c \right\}.$$

Theorem 7.3.4. *Assume that* $\delta_1 := |c - 2b|$. *For the class* $\mathcal{F}^5_{b,c}$,

(1) $\mathcal{S}^*(\lambda)$*−radius* r_0 *is the smallest root in* $(0,1]$ *of the equation*

$$\begin{aligned}(\delta_1 + \beta_0\delta_1 - \delta_1\lambda)r^5 &+ (2 + \beta_0 + 3|c|\delta_1 + |c|\beta_0\delta_1 - \lambda - 2|c|\delta_1\lambda)r^4 \\ &+ (5|c| + |c|\beta_0 + 3\delta_1 - \beta_0\delta_1 - 2|c|\lambda)r^3 + (3 - \beta_0 + (1 - \beta_0 + 2\lambda)\delta_1|c|)r^2 \\ &+ (2|c|\lambda + \delta_1\lambda - |c| - |c|\beta_0)r + \lambda - 1 = 0, \qquad (7.3.25)\end{aligned}$$

where $\beta_0 = 2\alpha_0 - 1$ *and* $\alpha_0 \in (0,1]$ *is the smallest positive root of the equation* $2\alpha^3 - q\alpha^2 - 4\alpha + 2 = 0$ *in the interval* $[1/2, 2/3]$.

(2) $f \in \mathcal{S}_P^*-$*radius* r_1 *is the smallest root in* $(0,1]$ *of the equation*

$$\begin{aligned}(\delta_1+2\beta_0\delta_1)r^5&+(3+2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^4\\&+(8|c|-2|c|\beta_0+6\delta_1-2\beta_0\delta_1+2|c|\beta_0\delta_1-2|c|^2\beta_0\delta_1)r^3\\&+(6-2\beta_0+2|c|\beta_0-2|c|^2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^2\\&+(-2|c|\beta_0+\delta_1)r-1=0.\end{aligned}\tag{7.3.26}$$

(3) $f \in \mathcal{S}_L^*-$*radius* r_2 *is the smallest root in* $(0,1]$ *of the equation*

$$\begin{aligned}(\delta_1+\sqrt{2}\delta_1-\beta_0\delta_1)r^5&+(2+\sqrt{2}-\beta_0+3q\delta_1+2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^4\\&+(5|c|+2\sqrt{2}|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\&+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^2\\&+(3|c|-2\sqrt{2}|c|-|c|\beta_0+2\delta_1-\sqrt{2}\delta_1)r-\sqrt{2}+1=0.\end{aligned}\tag{7.3.27}$$

(4) $f \in \mathcal{M}(\beta)-$*radius* r_3 *is the smallest root in* $(0,1]$ *of the equation*

$$\begin{aligned}(\delta_1+\beta\delta_1-\beta_0\delta_1)r^5&+(2+\beta-\beta_0+3|c|\delta_1+2\beta|c|\delta_1-|c|\beta_0\delta_1)r^4\\&+(5|c|+2\beta|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\&+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2b|c|\delta_1-|c|\beta_0\delta_1)r^2\\&+(|c|-2\beta|c|-|c|\beta_0+2\delta_1-\beta\delta_1)r-\beta+1=0.\end{aligned}\tag{7.3.28}$$

Proof. (1) Define the functions h and p by

$$h(z)=\frac{g(z)}{f(z)} \text{ and } p(z)=\frac{zg'(z)}{g(z)}.$$

Then

$$h(z)=1+(c-2b)z+\cdots \text{ and } p(z)=1+cz+\cdots.$$

Since

$$\left|\frac{f(z)}{g(z)}-1\right|<1 \text{ if and only if } \operatorname{Re}\left(\frac{g(z)}{f(z)}\right)>\frac{1}{2},$$

we have $h \in \mathcal{P}_{(c-2b)/2}(1/2)$. Now an application of Lemma 7.1.1 to the function $h(z)$, gives

$$\left|\frac{zh'(z)}{h(z)}\right| \le \frac{(\delta_1 r^2+2r+\delta_1)r}{(\delta_1 r+1)(1-r^2)},\tag{7.3.29}$$

where $\delta_1 := |c-2b|$. Since $g(z) = z + cz^2 + \cdots \in \mathcal{K}_c$, it follows from Lemma 7.1.4 that

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > \alpha_0,$$

where α_0 is the smallest positive root of the equation $2\alpha^3 - |c|\alpha^2 - 4\alpha + 2 = 0$ in the interval $[1/2, 2/3]$. Thus, we have $\operatorname{Re}(p(z)) > \alpha_0$. An application of Lemma 7.1.2, for $\alpha = \alpha_0$, gives

$$|p(z) - A_c| \le D_c, \tag{7.3.30}$$

where

$$A_c = \frac{(1+|c|r)^2 - \beta_0(|c|+r)^2 r^2}{(1+2|c|r+r^2)(1-r^2)}, \quad D_c = \frac{(1-\beta_0)(|c|+r)(1+|c|r)r}{(1+2|c|r+r^2)(1-r^2)} \text{ and } \beta_0 = 2\alpha_0 - 1.$$

Since $h(z) = g(z)/f(z)$ and $p(z) = zg'(z)/g(z)$, we have

$$\left|\frac{zf'(z)}{f(z)} - A_c\right| \le |p(z) - A_c| + \left|\frac{zh'(z)}{h(z)}\right|. \tag{7.3.31}$$

From (7.3.30), (7.3.29) and (7.3.31), we have

$$\left|\frac{zf'(z)}{f(z)} - A_c\right| \le D_c + \frac{(\delta_1 r^2 + 2r + \delta_1)r}{(\delta_1 r + 1)(1-r^2)}. \tag{7.3.32}$$

Clearly $f \in \mathcal{S}^*(\lambda)$, provided that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge A_c - D_c - \frac{(\delta_1 r^2 + 2r + \delta_1)r}{(\delta_1 r + 1)(1-r^2)}) \ge \lambda$$

or equivalently, if the following inequality holds:

$$\begin{aligned}
&(\delta_1 + \beta_0\delta_1 - \delta_1\lambda)r^5 + (2 + \beta_0 + 3|c|\delta_1 + |c|\beta_0\delta_1 - \lambda - 2|c|\delta_1\lambda)r^4 \\
&\quad + (5|c| + |c|\beta_0 + 3\delta_1 - \beta_0\delta_1 - 2|c|\lambda)r^3 + (3 - \beta_0 + (\delta_1 - \beta_0\delta_1 + 2\delta_1\lambda)|c|)r^2 \\
&\quad + (-|c| - |c|\beta_0 + 2|c|\lambda + \delta_1\lambda)r - 1 + \lambda \le 0.
\end{aligned}$$

Thus, $\mathcal{S}^*(\lambda)-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_0 \in (0,1]$ of the Equation (7.3.25).

(2) In view of Lemma 7.1.6, the disk in (7.3.32) lies inside the parabolic region $|w-1| < \operatorname{Re} w$, provided

$$D_c + \frac{(\delta_1 r^2 + 2r + \delta_1)r}{(\delta_1 r + 1)(1-r^2)}) \le A_c - 1/2$$

or equivalently, if the following inequality holds:

$$\begin{aligned}&(\delta_1+2\beta_0\delta_1)r^5+(3+2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^4\\&\quad+(8|c|-2|c|\beta_0+6\delta_1-2\beta_0\delta_1+2|c|\beta_0\delta_1-2|c|^2\beta_0\delta_1)r^3\\&\quad+(6-2\beta_0+2|c|\beta_0-2|c|^2\beta_0+4|c|\delta_1-2|c|\beta_0\delta_1)r^2+(\delta_1-2|c|\beta_0)r-1\leq 0.\end{aligned}$$

Hence $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_1\in(0,1]$ of the Equation (7.3.26).

(3) From Lemma 7.1.5, the function f satisfies

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2-1\right|<1\quad(|z|<r),$$

if the following inequality holds:

$$D_c+\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)})\leq\sqrt{2}-A_c,$$

or equivalently, if the following inequality holds:

$$\begin{aligned}&(\delta_1+\sqrt{2}\delta_1-\beta_0\delta_1)r^5+(2+\sqrt{2}-\beta_0+3q\delta_1+2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^4\\&\quad+(5|c|+2\sqrt{2}|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\&\quad+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2\sqrt{2}|c|\delta_1-|c|\beta_0\delta_1)r^2\\&\quad+(3|c|-2\sqrt{2}|c|-|c|\beta_0+2\delta_1-\sqrt{2}\delta_1)r-\sqrt{2}+1\leq 0.\end{aligned}$$

Therefore the $\mathcal{S}^*_L-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_2\in(0,1]$ of the Equation (7.3.27).

(4) From (7.3.32), we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\leq A_c+D_c+\frac{(\delta_1r^2+2r+\delta_1)r}{(\delta_1r+1)(1-r^2)}\leq\beta.$$

if the following inequality holds:

$$\begin{aligned}&(\delta_1+\beta\delta_1-\beta_0\delta_1)r^5+(2+\beta-\beta_0+3|c|\delta_1+2\beta|c|\delta_1-|c|\beta_0\delta_1)r^4\\&\quad+(5|c|+2\beta|c|-|c|\beta_0+3\delta_1+2|c|^2\delta_1-\beta_0\delta_1-|c|\beta_0\delta_1-|c|^2\beta_0\delta_1)r^3\\&\quad+(3+2|c|^2-\beta_0-|c|\beta_0-|c|^2\beta_0+5|c|\delta_1-2b|c|\delta_1-|c|\beta_0\delta_1)r^2\\&\quad+(|c|-2\beta|c|-|c|\beta_0+2\delta_1-\beta\delta_1)r-\beta+1=0.\end{aligned}$$

Therefore, $\mathcal{M}(\beta)-$radius of the class $\mathcal{F}^5_{b,c}$ is the smallest positive root $r_3 \in (0,1]$ of the Equation (7.3.28). □

Remark 7.3.4. On setting $b = 1 = c$, Theorem 7.3.4 reduces to the following result of Ali et al. [**10**, Theorem 2.5]:

For the class $\mathcal{F}^5_{b,c}$,

(i) the $\mathcal{S}^*(\lambda)-$radius, $r_0 = \frac{1-\alpha}{1+\sqrt{2-2\alpha+\alpha^2}}$

(ii) the $\mathcal{S}^*_P-$radius, $r_1 = \frac{1}{2+\sqrt{5}}$.

(iii) the $\mathcal{S}^*_L-$radius, $r_2 = 3 - 2\sqrt{2}$,

(iv) the $\mathcal{M}(\beta)-$radius, $r_3 = \frac{\beta-1}{\beta}$.

Conclusion and Future Plans

- In the present work, we have established differential subordination implications associated with the analytic functions which maps the unit disk onto either a disk or right half-plane or right-half of the lemniscate of Bernoulli. We have also provided alternate proofs of some results of Ali et al. [**17**]. Using these results several sufficient conditions for normalized analytic functions to be Janowski and Sokół-Stankiewicz starlike are also derived. As a future task, similar type of results can be discussed for some other functions such as $1 + 4z/3 + 2z^2/3$, $\sin z$ and e^z.

- Several differential subordination, superordination and corresponding sandwich results for a class of linear operators have been established. Many interesting examples for different choices of dominant are investigated. Many existing results proved by Al-Kharsani and Al-Areefi [**23**], Kumar [**174**], Obradović [**121**] and Chichra [**43**] have been generalized. For further investigation, one may discuss some other properties of functions satisfying the given form of recurrence relation.

- The Fekete-Szegö inequality for certain subclasses of analytic functions has been obtained, which generalize the earlier results in this direction. Further, the estimate on initial coefficients of some subclasses of bi-univalent function are also derived in this thesis. The coefficient estimates proved for the bi-univalent functions are either generalizing earlier results or giving the best-known estimate on the coefficients. For further work, these kind of results can be discussed for more general classes and may try to find estimate on a_4 and a_5 and even on a_n.

- In the concluding chapter, we have generalized the results of Ali et al. [**10**] for functions with fixed second coefficient by obtaining sharp radius constants for specific classes. The work may be further extended by considering more general classes. For future task, these kind of problems may be considered for other classes of analytic functions with fixed second coefficient.

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List of Research Papers:

[1] S. Sivaprasad Kumar, Virendra Kumar, V. Ravichandran and N. E. Cho, Sufficient conditions for starlike functions associated with the lemniscate of Bernoulli, J. Inequal. Appl. **2013**(2013), Art. 176, 13pp.

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