1. Objective

The aim of this study is to develop a quick and accurate method to investigate the elastic-plastic large deflection behaviour of aluminium/steel plates subjected to a combination of biaxial compression/tension, biaxial in-plane bending, edge shear and lateral pressure loads, until the ultimate limit state is reached. The welding induced residual stresses are included in the method as initial parameters. It is assumed that the plating is simply supported at all four edges which are kept straight.

2. Introduction

- Ultimate strength of plates and stiffened plates is the most fundamental strength for marine structures, and a great deal of progress has been achieved in this area in the past decades. There are a variety of methods and computer codes available for the ultimate strength analysis of plates and stiffened plates, ranging from simple analytical formulas to complicated numerical methods. The analysis costs typically increase with the level of detail modeling and the fidelity of the analysis procedure used. Therefore, the studies on ultimate strength of plates and stiffened plates have been and shall continue to be a large area of active researches in marine structures.

- The geometry of plating found in Ship and offshore structures is normally rectangular and the material used is usually mild or high tensile steel. The use of aluminum alloys is now increasing in the design and fabrication of high-speed vessel structures.

- The initial imperfections in the forms of initial distortion and welding residual stress are inevitable in marine structures due to the limits of fabrication technology. They have very significant effects on the ultimate strength of plates and stiffened plates and should be accounted in the ultimate strength evaluation of marine structures.

- The ship plating is generally subjected to combined in-plane and lateral pressure loads. In-plane loads include biaxial compression/tension, biaxial in-plane bending and edge shear, which are mainly induced by overall hull girder bending and/or torsion of the vessel. Lateral pressure loads are due to water pressure and/or cargo.

- These load components are not always applied simultaneously, but more than one load component will normally exist and interact. Hence, for more advanced ultimate strength design of ship structures, it is of crucial importance to better understand the characteristics of the ultimate strength for ship plating under combined loads.

- The FEM have been increasingly applied to predict ultimate strength of structural components, such as plates and stiffened plates. However, there has been little development in improving the computational efficiency of FEM analysis to evaluate ultimate strength in the recent past. It has thus been recognized that semi-analytical methods can in specific cases compute the nonlinear behaviour of structural elements more efficiently and with the required accuracy.

- A unique feature of the developed method is that geometric nonlinearity is handled by analytically solving the nonlinear governing differential equations of the elastic large deflection plate theory, while material nonlinearity associated with plasticity is dealt with by an implicit numerical technique. Also, the ultimate strength characteristics of ship plating are investigated and discussed by varying the plate dimensions, load application etc.

3. The Incremental Galerkin Technique : An Insight

The basic purpose behind the introduction of this method was to accommodate the geometric nonlinearity associated with buckling by analytically formulating the incremental forms of the governing differential equations for the elastic large deflection plate theory. Upon solving these newly formed set of incremental governing differential equations using the Galerkin method, a set of linear first order simultaneous equations for the unknowns will be obtained, which can be easily solved. Such a method will not only reduce the computational effort drastically but also the solution thus determined will be unique, unlike the traditional potential energy based approach. In this paper, the incremental Galerkin method mentioned above is improvised to better accommodate material nonlinearity associated with plasticity as well as geometric nonlinearity due to large lateral deflection.

Assumptions in formulating the Incremental Galerkin Technique:

- The plating is rectangular and simply supported on all four sides. The material of the plating is isotropic homogeneous steel or aluminum alloy. In-plane movements of the boundary are freely allowed, while keeping their edges straight.

- The plate is normally subjected to combined loads. The number of potential load Components acting on the plate are six, namely biaxial compression/tension, edge shear, biaxial in-plane bending moment and uniform lateral pressure loads, as shown in Fig. 1.

- The shape of initial deflection existing in the plate is normally complex, but it can be expressed by a Fourier series function.

- The welding induced residual stresses which may exist in the plating will affect the large deflection behavior as well as the plasticity.

- Residual stresses can develop in the plate in both *x* and *y* directions, since welding is normally carried out in the two directions. As shown in Fig. 2, the distribution of welding induced residual stresses for the plate is idealized to be composed of two stress blocks, namely a compressive residual stress block and a tensile residual stress block.

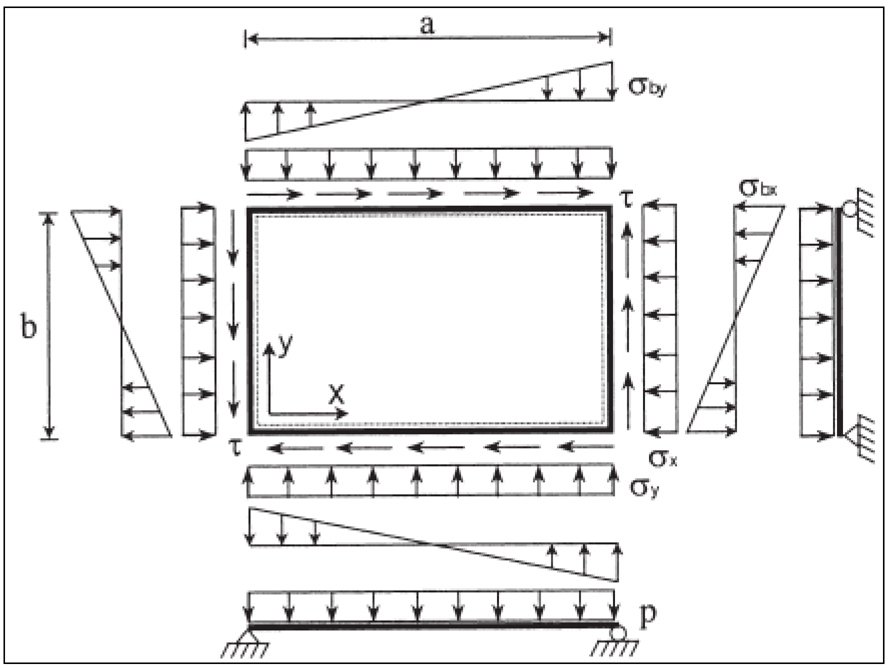


Fig.1. A ship plating under combined biaxial compression/tension, biaxial in-plane bending, edge shear and lateral pressure loads

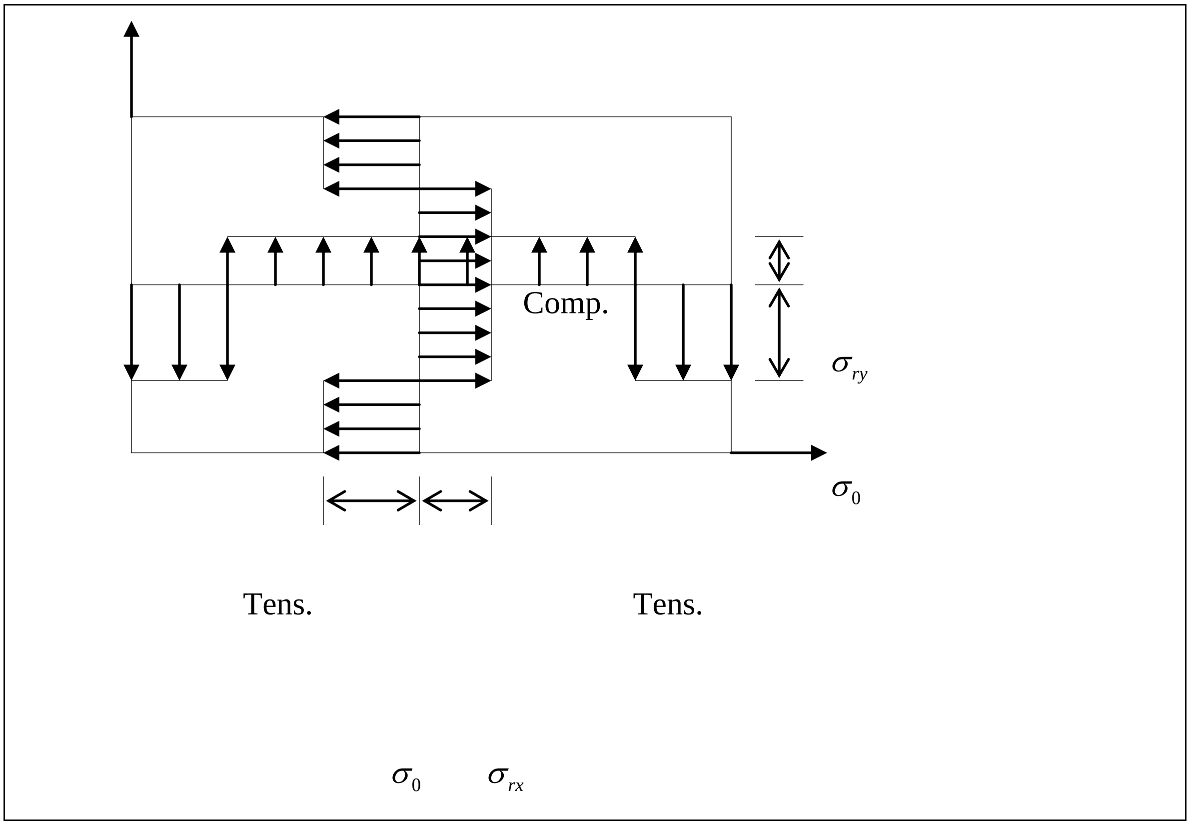


Fig.2. Idealized welding induced residual stresses distribution inside the plate *x* and *y* directions

- For the approximate evaluation of the plasticity, it is assumed that the plate is composed of a number of membrane fibers in the *x* and *y* directions. Each membrane fiber is considered to have a number of layers in the *z* direction, as shown in Fig. 3.

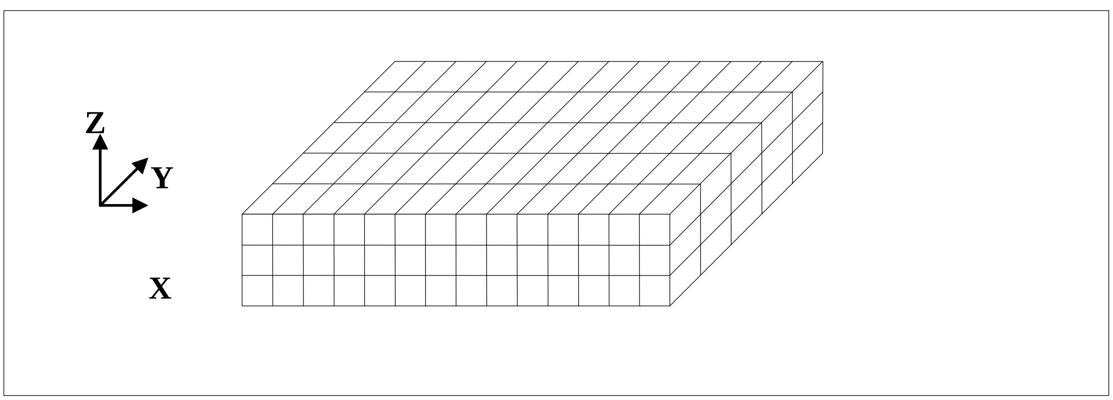


Fig.3. Example subdivision of plate mesh regions applied for treatment of plasticity

4. Analysis of elastic large deflection response

Basically, there are two different approaches to solve the nonlinear differential equations governing the large deflection of a simply supported plating *viz,*

- The traditional approach
- The incremental approach

In this study we analyze both these methods and eventually proceed with the incremental technique because of its simplicity and higher efficiency *vis-à-vis* the traditional method.

5. The traditional approach

The elastic large deflection response of steel or aluminum plates with initial imperfection is governed by two differential equations :

- The equilibrium equation.
- The compatibility equation.

These equations are as follows:

Equilibrium equation:

$$\Phi = D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - t\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^4 (w+w_0)}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 (w+w_0)}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 (w+w_0)}{\partial x \partial y}\right] - p = 0$$

(1)

Compatibility equation:

$$\frac{\partial^4 F}{\partial x^4}+2\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{\partial^4 F}{\partial y^4}-E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right]=0 \quad (2)$$

Where F = Airy's Stress function. When Airy's stress function, F, and the added deflection, *w*, are known, the stresses inside the plate can be calculates as follows:

$$\sigma_x=\frac{\partial^2 F}{\partial y^2}-\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)=0$$

$$\sigma_y=\frac{\partial^2 F}{\partial x^2}-\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)=0$$

$$\tau=-\frac{\partial^2 F}{\partial x\partial y}-\frac{Ez}{2(1+\nu)}\frac{\partial^2 w}{\partial x\partial y}=0$$

Also the corresponding strain components at a certain location inside the plate are given by:

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x}-z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial w}{\partial y}\frac{\partial w_0}{\partial y}-z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial y}-2z\frac{\partial^2 w}{\partial x\partial y}$$

where u , v = axial displacements in x and y directions .

Each strain component noted above is expressed as a function of stress components as follows:

$$\varepsilon_x=\frac{1}{\mathrm{E}}\left(\sigma_x-\nu\sigma_y\right)$$

$$\varepsilon_y=\frac{1}{\mathrm{E}}\left(\sigma_y-\nu\sigma_x\right)$$

$$\gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{\mathrm{E}}$$

Where z is the coordinate in the plate thickness direction with z = 0 at mid thickness. Above equations are often called the Marguerre equations. By solving the governing differential equations subject to the given boundary conditions, load application and initial imperfections , the membrane stress distribution inside the plate can be calculated and thus it is possible to examine the large- deflection behaviour of the plate.

In solving Eqs. (1) and (2) by the Galerkin method, the added deflection w and initial deflection w_0 can be assumed to be as follows:

$$w_0 = \sum_{m=1}^{\mathrm{M}} \sum_{n=1}^{\mathrm{N}} \mathrm{A}_{0mn}\, f_m(x) g_n(y) \tag{3}$$

$$w = \sum_{m=1}^{\mathrm{M}} \sum_{n=1}^{\mathrm{N}} \mathrm{A}_{mn}\, f_m(x) g_n(y) \tag{4}$$

where

$f_m(x)$ and $g_n(y)$ = basis functions which satisfy the boundary conditions for the plate.

A_{mn} = unknown deflection coefficient

A_{0mn} = known deflection coefficient

Substituting Eqs. (3) and (4) into Eq. (2), the stress function F can be obtained by:

$$F = F_{\mathrm{H}} + \sum_{r=1}^{\mathrm{M}} \sum_{s=1}^{\mathrm{N}} \mathrm{K}_{rs}\, p_r(x) q_s(y) \tag{5}$$

where it is evident from Eq. (2) that the coefficients K_{rs} will be second order functions with regard to the unknown deflection coefficients . F_H is a homogeneous solution of the stress function which satisfies the applied loading condition.

To compute the unknown coefficients A_{mn}, one may use the Galerkin method for the equilibrium Eq. (1), resulting in the following equation:

$$\iiint \Phi f_r(x) g_s(y) dvol = 0, r = 1, 2, 3, ..., s = 1, 2, 3, ... \qquad (6)$$

Substituting Eqs. (3)–(5) into Eq. (6), and performing the integration over the whole volume of the plate, a set of third order simultaneous equations with regard to the unknown coefficients A_{mn} will be obtained.

Anomalies in the traditional approach:

- Since a cubic equation is obtained for each of the unknown coefficients, solving the simultaneous equations to get the coefficients A_{mn} normally requires an iteration process.
- Also the solution of each coefficient should be unique; therefore we will have to correctly select one among the three solutions obtained for each coefficient.
- It is not always an easy task to solve a set of third order simultaneous equations especially when the number of unknown coefficients A_{mn} is very large.

6. Large deflection analysis of a simply supported plate subjected to Combined Longitudinal Axial Load and Lateral Pressure by the Traditional Approach

Since it is known that for plates under predominantly longitudinal axial compressive loads the deflection term associated with the lowest bifurcation mode plays a dominant role in the elastic large deflection response. For this reason the initial and added deflection functions are simplified by including only the buckling mode initial deflection as follows:

$$w_0 = A_{0m} \sin\frac{m\pi x}{a}\cos\frac{\pi y}{b}$$

$$w = A_m \sin\frac{m\pi x}{a}\cos\frac{\pi y}{b}$$

where

m = buckling mode half wave number in the *x* direction .

A_{0m} = amplitude of the initial deflection for axial compressive loading.

A_m = unknown amplitude of the added deflection function.

The stress distribution inside the plate can be analyzed by solving the governing differential equations. First we determine the unknown amplitude of the added deflection, under the applied loading. On substituting w & w_0 in the compatibility equation (i.e. eqn (2)), we get:

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\mathrm{A}_m^2 m^2 \pi^4}{a^2 b^2}\cos^2\frac{m\pi x}{a}\cos^2\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = \frac{\mathrm{A}_m^2 m^2 \pi^4}{a^2 b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}$$

$$2\frac{\partial^2 w_0}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} = 2\frac{\mathrm{A}_{0m}\mathrm{A}_m m^2 \pi^4}{a^2 b^2}\cos^2\frac{m\pi x}{a}\cos^2\frac{\pi y}{b}$$

$$\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w}{\partial y^2}=\frac{\mathrm{A}_{0m}\mathrm{A}_m m^2\pi^4}{a^2b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}=\frac{\mathrm{A}_{0m}\mathrm{A}_m m^2\pi^4}{a^2b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}$$

$$\therefore \frac{\partial^4 F}{\partial x^4}+2\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{\partial^4 F}{\partial y^4}=\mathrm{E}\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right]$$

$$=\mathrm{E}\left(\frac{\mathrm{A}_m^2 m^2\pi^4}{a^2b^2}\cos^2\frac{m\pi x}{a}\cos^2\frac{\pi y}{b}-\frac{\mathrm{A}_m^2 m^2\pi^4}{a^2b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}+2\frac{\mathrm{A}_{0m}\mathrm{A}_m m^2\pi^4}{a^2b^2}\cos^2\frac{m\pi x}{a}\cos^2\frac{\pi y}{b}\right.$$
$$\left.-\frac{\mathrm{A}_{0m}\mathrm{A}_m m^2\pi^4}{a^2b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}-\frac{\mathrm{A}_{0m}\mathrm{A}_m m^2\pi^4}{a^2b^2}\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}\right)$$

$$=\mathrm{E}\left[\frac{\mathrm{A}_m^2 m^2\pi^4}{a^2b^2}\left(\cos^2\frac{m\pi x}{a}\cos^2\frac{\pi y}{b}-\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}\right)+2\frac{\mathrm{A}_{0m}\mathrm{A}_m m^2\pi^4}{a^2b^2}\left(\cos^2\frac{m\pi x}{a}\cos^2\frac{\pi y}{b}\right.\right.$$
$$\left.\left.-\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}\right)\right]$$

$$=\frac{\mathrm{E}m^2\pi^4\mathrm{A}_m\left(\mathrm{A}_m+2\mathrm{A}_{0m}\right)}{a^2b^2}\left(\cos^2\frac{m\pi x}{a}\cos^2\frac{\pi y}{b}-\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}\right)$$

$$=\frac{\mathrm{E}m^2\pi^4\mathrm{A}_m\left(\mathrm{A}_m+2\mathrm{A}_{0m}\right)}{a^2b^2}\left[\left(1-\sin^2\frac{m\pi x}{a}\right)\left(1-\sin^2\frac{\pi y}{b}\right)-\sin^2\frac{m\pi x}{a}\sin^2\frac{\pi y}{b}\right]$$

$$=\frac{\mathrm{E}m^2\pi^4\mathrm{A}_m\left(\mathrm{A}_m+2\mathrm{A}_{0m}\right)}{a^2b^2}\left[1-\sin^2\frac{m\pi x}{a}-\sin^2\frac{\pi y}{b}\right]$$

$$=\frac{\mathrm{E}m^2\pi^4\mathrm{A}_m\left(\mathrm{A}_m+2\mathrm{A}_{0m}\right)}{2a^2b^2}\left[1-2\sin^2\frac{m\pi x}{a}+1-2\sin^2\frac{\pi y}{b}\right]$$

(Multiplying and dividing by 2)

$$= \frac{\mathrm{E}m^2\pi^4 \mathrm{A}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{2a^2b^2}\left[\cos\frac{2m\pi x}{a} + \cos\frac{2\pi y}{b}\right]$$

$$\therefore \frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = \frac{\mathrm{E}m^2\pi^4 \mathrm{A}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{2a^2b^2}\left(\cos\frac{2m\pi x}{a} + \cos\frac{2\pi y}{b}\right)$$

Let the particular solution, F_{P}, of the stress function, F, be given by

$$F_{\mathrm{P}} = C_1 \cos\frac{2m\pi x}{a} + C_2 \cos\frac{2\pi y}{b}$$

On substituting this value of F_{P} in the above relation, we get:

$$\frac{\partial^4 F_{\mathrm{P}}}{\partial x^4} = \frac{16m^4\pi^4}{a^4} C_1 \cos\frac{2m\pi x}{a}$$

$$\frac{\partial^4 F_{\mathrm{P}}}{\partial y^4} = \frac{16\pi^4}{b^4} C_2 \cos\frac{2\pi y}{b}$$

$$2\frac{\partial^4 F_{\mathrm{P}}}{\partial x^2 \partial y^2} = 0$$

$$\therefore \frac{\partial^4 F_{\mathrm{P}}}{\partial x^4} + 2\frac{\partial^4 F_{\mathrm{P}}}{\partial x^2 \partial y^2} + \frac{\partial^4 F_{\mathrm{P}}}{\partial y^4} = \frac{16m^4\pi^4}{a^4} C_1 \cos\frac{2m\pi x}{a} + \frac{16\pi^4}{b^4} C_2 \cos\frac{2\pi y}{b}$$

Now equating the coefficients of $\cos\frac{2m\pi x}{a}$ and $\cos\frac{2\pi y}{b}$, we get:

$$\frac{16m^4\pi^4}{a^4} C_1 = \frac{\mathrm{E}m^2\pi^4 \mathrm{A}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{2a^2b^2} \quad and$$

$$\frac{16\pi^4}{b^4} C_2 = \frac{\mathrm{E}m^2\pi^4 \mathrm{A}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{2a^2b^2}$$

$$\therefore\ C_1 = \frac{\mathrm{EA}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{32}\left(\frac{a^2}{m^2b^2}\right)$$

and $$C_2 = \frac{\mathrm{EA}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{32}\left(\frac{m^2b^2}{a^2}\right)$$

$$\therefore\ F_{\mathrm{P}} = \frac{\mathrm{EA}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{32}\left(\frac{a^2}{m^2b^2}\cos\frac{2m\pi x}{a} + \frac{m^2b^2}{a^2}\cos\frac{2\pi y}{b}\right)$$

The homogeneous solution, F_{H}, of the stress function ,F, is obtained by treating the welding-induced residual stress as an initial stress parameter:

Let $\mathrm{F_H}$ be given by

$$F_{\mathrm{H}} = A\frac{x^2}{2} + B\frac{y^2}{2} + Cxy$$

We know, $\dfrac{\partial^2 F_{\mathrm{H}}}{\partial x^2} = \sigma_{ry} = A$

and $\dfrac{\partial^2 F_{\mathrm{H}}}{\partial y^2} = \sigma_{rx} + \sigma_{xav} = B$, therefore

$$F_{\mathrm{H}} = \left(\sigma_{xav} + \sigma_{rx}\right)\frac{y^2}{2} + \sigma_{ry}\,\frac{x^2}{2}$$

where σ_{rx} , σ_{ry} = welding induced residual stresses.

The applicable stress function, F, may then be expressed as a sum of the particular solution and the homogeneous solution as follows:

$$F = \left(\sigma_{xav} + \sigma_{rx}\right)\frac{y^2}{2} + \sigma_{ry}\,\frac{x^2}{2} + \frac{\mathrm{EA}_m\left(\mathrm{A}_m + 2\mathrm{A}_{0m}\right)}{32}\left(\frac{a^2}{m^2b^2}\cos\frac{2m\pi x}{a} + \frac{m^2b^2}{a^2}\cos\frac{2\pi y}{b}\right)$$

By substituting the values of w, w_0 and F in the equilibrium equation (i.e. eqn (2)) and applying the Galerkin method, the following equation is obtained:

$$\int_0^a\int_0^b\left\{D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-t\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^4(w+w_0)}{\partial x^2}+\frac{\partial^2 F}{\partial x^2}\frac{\partial^2(w+w_0)}{\partial y^2}-2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(w+w_0)}{\partial x\partial y}+\frac{p}{t}\right]\right\}$$
$$\times\sin\frac{m\pi x}{a}\sin\frac{\pi y}{b}dxdy=0$$

By performing the integration of the above equation over the entire plate, a third-order equation with respect to the unknown variable A_m is obtained. For the integration of the above equation it may be assumed that the contribution of lateral pressure to non-linear membrane stresses arises only from the deflection component of m = 1 and it is linearly superposed to those by in-plane loads. This results in

$$C_1A_m^3+C_2A_m^2+C_3A_m+C_4=0$$

Where

$$C_1=\frac{\pi^2\mathrm{E}}{16}\left(\frac{m^4 b}{a^3}+\frac{a}{b^3}\right)$$
$$C_2=\frac{3\pi^2\mathrm{E}A_{0m}}{16}\left(\frac{m^4 b}{a^3}+\frac{a}{b^3}\right)$$
$$C_3=\frac{\pi^2\mathrm{E}A_{0m}^2}{16}\left(\frac{m^4 b}{a^3}+\frac{a}{b^3}\right)+\frac{m^2 b}{a}\left(\sigma_{xav}+\sigma_{rex}\right)+\frac{a}{b}\sigma_{rey}$$
$$+\frac{\pi^2 D}{t}\frac{m^2}{ab}\left(\frac{mb}{a}+\frac{a}{mb}\right)^2$$
$$C_4=A_{0m}\left[\frac{m2b}{a}\left(\sigma_{xav}+\sigma_{rex}\right)+\frac{a}{b}\sigma_{rey}\right]-\frac{16ab}{\pi^4 t}p$$

6.1 Examples of the Analysis (using MATLAB)

- *6.1.1 Square Plate subjected to lateral Load*

  A square plate subjected to uniformly distributed lateral load Q is analyzed. Fig 4 and Fig 5 show a comparison between the load deflection relationship at the centre of the plate obtained by the traditional approach and that given by FEM. It may be seen that the traditional method is quite accurate in case of lateral load.

  Details of the Square Plate:

$$a = 1000 \ , \ b = 1000 \, , \, t = 10$$

$$\mathrm{E} = 210000 \, \mathrm{N} / mm^2 \ , \ \nu = 0.3$$

$$w = \mathrm{A} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \ , \ w_0 = 0$$

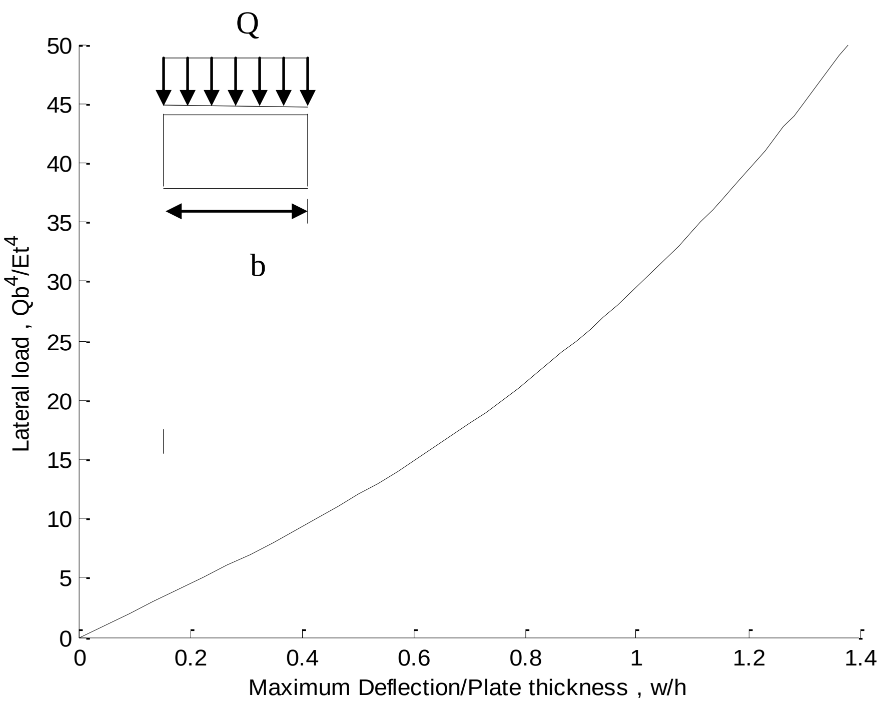


Fig. 4. Square Plate subjected to uniform lateral Load (traditional method)

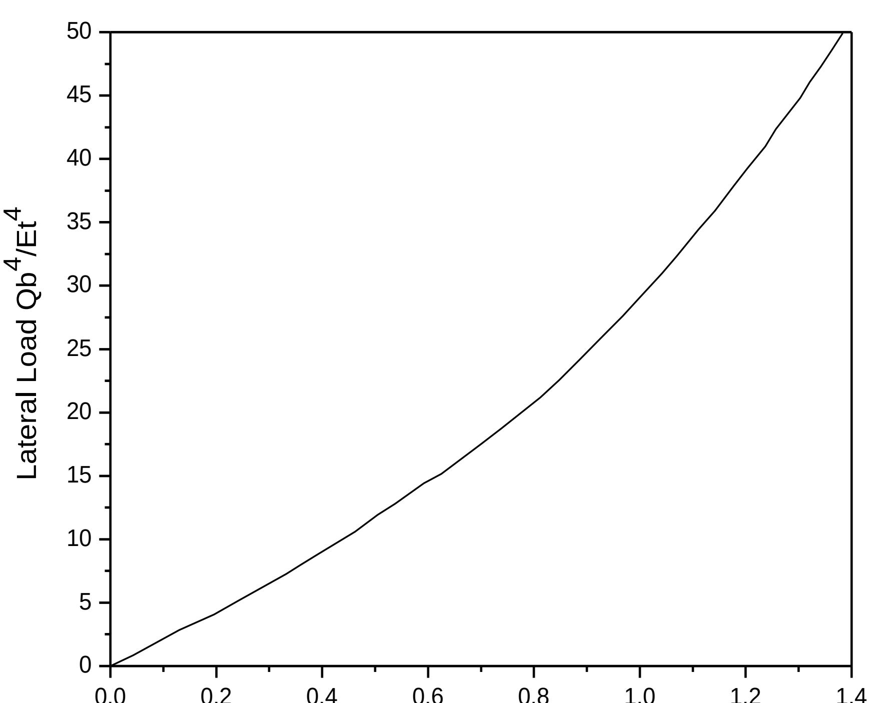


Fig. 5. Square Plate subjected to uniform lateral Load (FEM)

- *6.1.2 Simply supported Square plate under uniaxial Compression*

Fig 6 shows the Load deflection Relationship at the centre of a simply supported square plate obtained by subjecting it to a uniaxial compressive load. This graph is further validated by the Incremental Galerkin technique in the next section.

Details of the Square Plate:

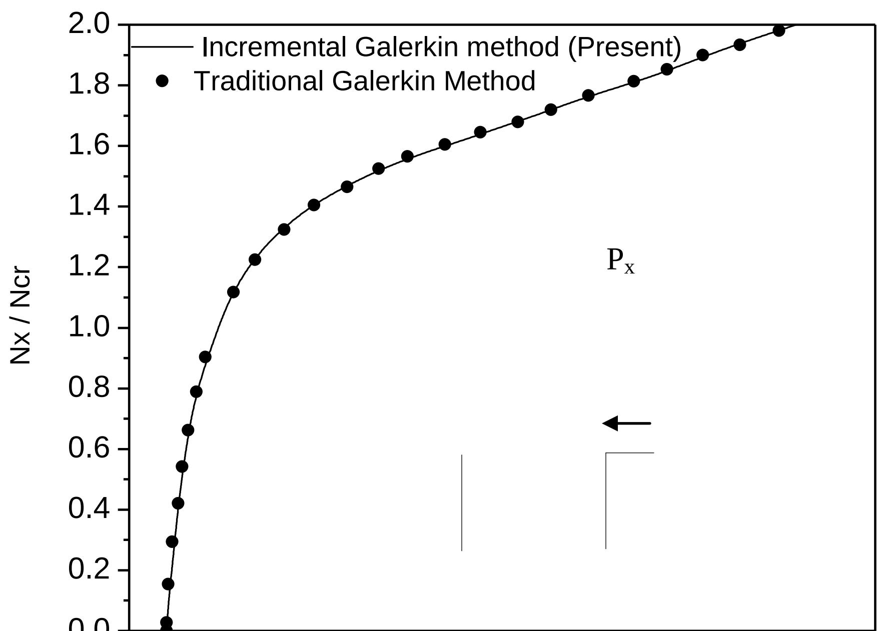
$$a = 1000 \ , \ b = 1000 \ , \ t = 10$$

$$\mathrm{E} = 210000 \ \mathrm{N}/mm^2 \ , \ \nu = 0.3$$

$$w = \mathrm{A} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$w_0 = 0.45 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$P_{xcr} = 442740 \ \mathrm{N}/mm^2$$



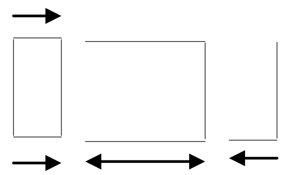


Fig. 6. Square Plate subjected to uniaxial compression.

7. The incremental approach

To solve the non linear governing differential equations for plating subjected to combined in-plane and lateral loads and to analyze the large deflection associated with it, in an efficient and easier way, we devise an incremental form of the governing differential equations. This method eliminates the following anomalies in the traditional approach:

- Linear first order equations for the unknown deflection coefficients are obtained by applying the incremental Galerkin technique as against the cubic equations obtained in the case of traditional method.

- The computational effort is reduced drastically in this method.

- As the equations are linear, a unique solution is obtained for each variable.

Derivation of the incremental forms of governing equations

Let us assume that the load is applied incrementally. Also let the deflection and stress function be denoted by w_{i-1} and F_{i-1}, respectively, at the end of the $(i-1)^{th}$ load increment step. Similarly, the deflection and stress function at the end of the i^{th} load increment step be denoted by w_i and F_i, respectively.

Therefore, the equilibrium equation and the compatibility equation at the end of the $(i-1)^{th}$ load increment step are written as follows:

$$\Phi_{i-1} = D\left(\frac{\partial^4 w_{i-1}}{\partial x^4} + 2\frac{\partial^4 w_{i-1}}{\partial x^2 \partial y^2} + \frac{\partial^4 w_{i-1}}{\partial y^4}\right) - t\left[\frac{\partial^2 F_{i-1}}{\partial y^2}\frac{\partial^4 (w_{i-1}+w_0)}{\partial x^2}\right.$$
$$\left. + \frac{\partial^2 F_{i-1}}{\partial x^2}\frac{\partial^2 (w_{i-1}+w_0)}{\partial y^2} - 2\frac{\partial^2 F_{i-1}}{\partial x \partial y}\frac{\partial^2 (w_{i-1}+w_0)}{\partial x \partial y} + \frac{p_{i-1}}{t}\right] = 0$$

$$\frac{\partial^4 F_{i-1}}{\partial x^4} + 2\frac{\partial^4 F_{i-1}}{\partial x^2 \partial y^2} + \frac{\partial^4 F_{i-1}}{\partial y^4} - E\left[\left(\frac{\partial^2 w_{i-1}}{\partial x \partial y}\right)^2 - \frac{\partial^2 w_{i-1}}{\partial x^2}\frac{\partial^2 w_{i-1}}{\partial y^2}\right.$$
$$\left. +2\frac{\partial^2 w_0}{\partial x \partial y}\frac{\partial^2 w_{i-1}}{\partial x \partial y} - \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_{i-1}}{\partial y^2} - \frac{\partial^2 w_{i-1}}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right] = 0$$

Similarly, the equilibrium equation and the compatibility equation at the end of the i^{th} load increment step are written as follows:

$$\Phi_{i} = D\left(\frac{\partial^4 w_{i}}{\partial x^4} + 2\frac{\partial^4 w_{i}}{\partial x^2 \partial y^2} + \frac{\partial^4 w_{i}}{\partial y^4}\right) - t\left[\frac{\partial^2 F_{i}}{\partial y^2}\frac{\partial^4 (w_{i}+w_0)}{\partial x^2}\right.$$
$$\left. + \frac{\partial^2 F_{i}}{\partial x^2}\frac{\partial^2 (w_{i}+w_0)}{\partial y^2} - 2\frac{\partial^2 F_{i}}{\partial x \partial y}\frac{\partial^2 (w_{i}+w_0)}{\partial x \partial y} + \frac{p_{i}}{t}\right] = 0$$

$$\frac{\partial^4 F_{i}}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} - E\left[\left(\frac{\partial^2 w_{i}}{\partial x \partial y}\right)^2 - \frac{\partial^2 w_{i}}{\partial x^2}\frac{\partial^2 w_{i}}{\partial y^2}\right.$$
$$\left. +2\frac{\partial^2 w_0}{\partial x \partial y}\frac{\partial^2 w_{i}}{\partial x \partial y} - \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_{i}}{\partial y^2} - \frac{\partial^2 w_{i}}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right] = 0$$

It is assumed that the accumulated (total) deflection w_i and stress function F_i at the end of the i^{th} load increment step are given by:

$$w_i = w_{i-1} + \Delta w \tag{7}$$

$$F_i = F_{i-1} + \Delta F \tag{8}$$

where Δw and Δ F are the increments of deflection or stress function, respectively.

Substituting equations (7) and (8) into the equilibrium Eq. (1) and the compatibility Eq. (2) at the end of the $(i-1)^{th}$ load increment step, and subtracting these equations from the equilibrium equation and the compatibility equation at the end of the i^{th} load increment step, the necessary incremental forms of the governing differential equations are obtained as follows:

$$\begin{aligned}\Delta\Phi = D\left(\frac{\partial^4 \Delta w}{\partial x^4} + 2\frac{\partial^4 \Delta w}{\partial x^2 \partial y^2} + \frac{\partial^4 \Delta w}{\partial y^4}\right) - t\left[\frac{\partial^2 F_{i-1}}{\partial y^2}\frac{\partial^4 \Delta w}{\partial x^2} + \frac{\partial^2 \Delta F}{\partial y^2}\frac{\partial^4 (w_{i-1} + w_0)}{\partial x^2}\right. \\ + \frac{\partial^2 F_{i-1}}{\partial x^2}\frac{\partial^2 \Delta w}{\partial y^2} + \frac{\partial^2 \Delta F}{\partial x^2}\frac{\partial^2 (w_{i-1} + w_0)}{\partial y^2} - 2\frac{\partial^2 F_{i-1}}{\partial x \partial y}\frac{\partial^2 \Delta w}{\partial x \partial y} \\ \left. -2\frac{\partial^2 \Delta F}{\partial x \partial y}\frac{\partial^2 (w_{i-1} + w_0)}{\partial x \partial y} + \frac{\Delta p}{t}\right] = 0\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial^4 \Delta F}{\partial x^4} + 2\frac{\partial^4 \Delta F}{\partial x^2 \partial y^2} + \frac{\partial^4 \Delta F}{\partial y^4} - E\left[2\frac{\partial^2 \left(w_{i-1} + w_0\right)}{\partial x \partial y}\frac{\partial^2 \Delta w}{\partial x \partial y}\right. \\ \left. - \frac{\partial^2 \left(w_{i-1} + w_0\right)}{\partial x^2}\frac{\partial^2 \Delta w}{\partial y^2} - \frac{\partial^2 \Delta w}{\partial x^2}\frac{\partial^2 \left(w_{i-1} + w_0\right)}{\partial y^2}\right] = 0\end{aligned} \tag{10}$$

where the terms of very small quantities with order higher than second order of the increments Δw and Δ F have been neglected.

At the end of the $(i-1)^{th}$ load increment step, the deflection w_{i-1} and the stress function F_{i-1} will have been known, as follows:

$$w_{i-1} = \sum_{i=1}^{\mathrm{M}} \sum_{j=1}^{\mathrm{N}} A_{mn}^{i-1} f_m(x) g_n(y) \tag{11}$$

$$F_{i-1} = F_{\mathrm{H}}^{i-1} + \sum_{i=1}^{\mathrm{M}} \sum_{j=1}^{\mathrm{N}} \mathrm{K}_{ij}^{i-1} p_i(x) q_j(y) \tag{12}$$

Where A_{mn}^{i-1} and K_{ij}^{i-1} = the known coefficients, and

F_{H}^{i-1} = homogeneous solution for the stress function.

The welding induced residual stresses can be included in the stress function F_{H}^{i-1} as initial stress terms.

The added deflection increment Δw associated with the load increment at the i^{th} step can be assumed to be as follows:

$$\Delta w = \sum_{k=1}^{\mathrm{M}} \sum_{l=1}^{\mathrm{N}} \Delta \mathrm{A}_{kl} f_k(x) g_l(y) \tag{13}$$

where $\Delta \mathrm{A}_{kl}$ = unknown added deflection increment.

Substituting Eqs. (3), (11) and (13) into Eq. (10), the stress function increment Δ F can be obtained by:

$$\Delta F = \Delta F_{\mathrm{H}} + \sum_{i=1}^{\mathrm{M}} \sum_{j=1}^{\mathrm{N}} \Delta \mathrm{K}_{ij} p_i(x) q_j(y) \tag{14}$$

where $\Delta \mathrm{K}_{ij}$ are linear (i.e. first order) functions in terms of unknown coefficients $\Delta \mathrm{A}_{kl}$. ΔF_{H} is a homogeneous solution for the stress function increment which satisfies the applied loading condition.

To compute the unknown coefficients $\Delta \mathrm{A}_{kl}$, the Galerkin method can then be applied to Eq. (9):

$$\iiint \Delta\Phi f_r\left(x\right) g_s\left(y\right) dvol = 0, r = 1,2,3,..., s = 1,2,3,... \quad (15)$$

Substituting Eqs. (3), (11)–(14) into Eq. (15), and performing the integration over the whole volume of plating, a set of linear simultaneous equations in terms of unknown coefficients ΔA_{kl} will be obtained. Solving these linear simultaneous equations is normally easy. Having obtained ΔA_{kl} , one can then calculate Δw (i.e. from Eq. (13)), ΔF (i.e. from Eq. (14)), $w_i\left(= w_{i-1} + \Delta w\right)$ and $F_i\left(= F_{i-1} + \Delta F\right)$ at the end of the i^{th} load increment step. By repeating the above procedure with increase in the applied loads, the elastic large deflection response for plating can be obtained. In this process, it is evident that the load increments must be small in order to get more accurate solutions. Since the computational effort required for this procedure is normally very small, using smaller load increments would not lead to any severe penalties unlike the case of the usual numerical methods.

8. Application to the elastic large deflection analysis of a simply supported plating

In the following, we apply the incremental Galerkin technique to analyze the deflection of a simply supported plate subjected to various in-plane and lateral loads. The simply supported plate should satisfy the following edge conditions:

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} = 0 \quad at \quad x = 0, a$$
$$w = 0, \quad \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0 \quad at \quad y = 0, b$$

The initial, added and incremental deflection functions satisfying the boundary conditions can be assumed to be as follows:

$$w_0 = \sum_{m=1}^{\mathrm{M}} \sum_{n=1}^{\mathrm{N}} \mathrm{A}_{0mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{16}$$

$$w_{i-1} = \sum_{m=1}^{\mathrm{M}} \sum_{n=1}^{\mathrm{N}} \mathrm{A}_{mn}^{i-1} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{17}$$

$$\Delta w = \sum_{k=1}^{\mathrm{M}} \sum_{l=1}^{\mathrm{N}} \Delta\mathrm{A}_{kl} \sin\frac{k\pi x}{a} \sin\frac{l\pi y}{b} \tag{18}$$

where

$\mathrm{A}_{0mn}\left(= \mathrm{A}_{mn}^{0}\right)$ And A_{mn}^{i-1} = known coefficients

$\Delta\mathrm{A}_{kl}$ = unknown coefficients to be calculated.

Now, we know

$\mathrm{P} = \int \sigma d\mathrm{A} = \int \frac{\partial^2 F}{\partial y^2} t \partial y$ where dA = small elemental area.

and

$M = \int \mathrm{P}.\left(y - \frac{b}{2}\right)$ where $\left(y - \frac{b}{2}\right)$ = distance from the neutral axis of small element

Therefore for the given situation, the boundary conditions are:

$$\int_0^b \frac{\partial^2 F}{\partial y^2} t\partial y = \mathrm{P}_x, \quad \int_0^b \frac{\partial^2 F}{\partial y^2} t\left(y - \frac{b}{2}\right)\partial y = \mathrm{M}_x \;\; at \;\; x = 0, a$$

$$: \int_0^a \frac{\partial^2 F}{\partial x^2} t\partial x = \mathrm{P}_y, \quad \int_0^a \frac{\partial^2 F}{\partial x^2} t\left(x - \frac{a}{2}\right)\partial x = \mathrm{M}_y \;\; at \;\; y = 0, b$$

$$\frac{\partial^2 F}{\partial x \partial y} = -\tau \;\; at \; all \; boundaries$$

Where

P_x, P_y = axial loads in the x and y directions,

M_x, M_y = in-plane bending moments in the *x* and *y* directions.

To simplify the calculations we use the following substitutions:

$$sx(m) = \sin\frac{m\pi x}{a}, \quad sy(n) = \sin\frac{n\pi y}{b}, \quad cx(m) = \cos\frac{m\pi x}{a}, \quad cy(n) = \cos\frac{n\pi y}{b}$$

The stress function increment ΔF can be obtained by substituting the values of Δw, w_0 and w_{i-1} into the incremental form of compatibility equation .On substitution we get:

$$\frac{\partial^4 \Delta F}{\partial x^4}+2\frac{\partial^4 \Delta F}{\partial x^2 \partial y^2}+\frac{\partial^4 \Delta F}{\partial y^4}=\mathrm{E}\left\{\left[2\frac{\partial^2\left(\sum_m\sum_n \mathrm{A}_{mn}^{i-1} sx(m)sy(n)+\sum_m\sum_n \mathrm{A}_{0mn} sx(m)sy(n)\right)}{\partial x \partial y}\right.\right.$$

$$\left.\times\frac{\partial^2\left(\sum_k\sum_l \Delta\mathrm{A}_{kl} sx(k)sy(l)\right)}{\partial x \partial y}\right]-\left[\frac{\partial^2\left(\sum_m\sum_n \mathrm{A}_{mn}^{i-1} sx(m)sy(n)+\sum_m\sum_n \mathrm{A}_{0mn} sx(m)sy(n)\right)}{\partial x^2}\right.$$

$$\left.\times\frac{\partial^2\left(\sum_k\sum_l \Delta\mathrm{A}_{kl} sx(k)sy(l)\right)}{\partial y^2}\right]-\left[\frac{\partial^2\left(\sum_m\sum_n \mathrm{A}_{mn}^{i-1} sx(m)sy(n)+\sum_m\sum_n \mathrm{A}_{0mn} sx(m)sy(n)\right)}{\partial y^2}\right.$$

$$\left.\left.\times\frac{\partial^2\left(\sum_k\sum_l \Delta\mathrm{A}_{kl} sx(k)sy(l)\right)}{\partial x^2}\right]\right\}$$

or,

$$\frac{\partial^4 \Delta F}{\partial x^4}+2\frac{\partial^4 \Delta F}{\partial x^2 \partial y^2}+\frac{\partial^4 \Delta F}{\partial y^4}=\frac{\mathrm{E}\pi^4}{a^2b^2}\left\{\left[2mnkl\times\left(\sum_m\sum_n \mathrm{A}_{mn}^{i-1} cx(m)cy(n)+\sum_m\sum_n \mathrm{A}_{0mn} cx(m)cy(n)\right)\right.\right.$$

$$\left.\times\left(\sum_k\sum_l \Delta\mathrm{A}_{kl} cx(k)cy(l)\right)\right]-\left[m^2l^2\times\left(\sum_m\sum_n \mathrm{A}_{mn}^{i-1}\left(-sx(m)\right)sy(n)+\sum_m\sum_n \mathrm{A}_{0mn}\left(-sx(m)\right)sy(n)\right)\right.$$

$$\left.\times\left(\sum_k\sum_l \Delta\mathrm{A}_{kl} sx(k)\left(-sy(l)\right)\right)\right]-\left[n^2k^2\times\left(\sum_m\sum_n \mathrm{A}_{mn}^{i-1} sx(m)\left(-sy(n)\right)+\sum_m\sum_n \mathrm{A}_{0mn} sx(m)\left(-sy(n)\right)\right)\right.$$

$$\left.\left.\times\left(\sum_k\sum_l \Delta\mathrm{A}_{kl}\left(-sx(k)\right)sy(l)\right)\right]\right\}$$

or,

$$\frac{\partial^4 \Delta F}{\partial x^4}+2\frac{\partial^4 \Delta F}{\partial x^2 \partial y^2}+\frac{\partial^4 \Delta F}{\partial y^4}=\frac{\mathrm{E}\pi^4}{a^2b^2}\Bigg[2mnkl\times\Bigg(\sum_m\sum_n\sum_k\sum_l \Delta \mathrm{A}_{kl}\mathrm{A}_{mn}^{i-1}cx(k)cy(l)cx(m)cy(n)+$$
$$\sum_m\sum_n\sum_k\sum_l \Delta \mathrm{A}_{kl}\mathrm{A}_{0mn}cx(k)cy(l)cx(m)cy(n)\Bigg)-m^2l^2\times\Bigg(\sum_m\sum_n\sum_k\sum_l \Delta \mathrm{A}_{kl}\mathrm{A}_{mn}^{i-1}sx(k)sy(l)sx(m)sy(n)$$
$$+\sum_m\sum_n\sum_k\sum_l \Delta \mathrm{A}_{kl}\mathrm{A}_{0mn}sx(k)sy(l)sx(m)sy(n)\Bigg)-k^2n^2\times\Bigg(\sum_m\sum_n\sum_k\sum_l \Delta \mathrm{A}_{kl}\mathrm{A}_{mn}^{i-1}sx(k)sy(l)sx(m)sy(n)$$
$$+\sum_m\sum_n\sum_k\sum_l \Delta \mathrm{A}_{kl}\mathrm{A}_{0mn}sx(k)sy(l)sx(m)sy(n)\Bigg)\Bigg]$$

or,

$$\frac{\partial^4 \Delta F}{\partial x^4}+2\frac{\partial^4 \Delta F}{\partial x^2 \partial y^2}+\frac{\partial^4 \Delta F}{\partial y^4}=\frac{\mathrm{E}\pi^4}{a^2b^2}\sum_m\sum_n\sum_k\sum_l\left(\mathrm{A}_{mn}^{i-1}+\mathrm{A}_{0mn}\right)\Delta \mathrm{A}_{kl}\left[2mnkl\times cx(k)cy(l)cx(m)cy(n)\right.$$
$$\left.-\left(m^2l^2+k^2n^2\right)sx(k)sy(l)sx(m)sy(n)\right]$$

or,

$$\frac{\partial^4 \Delta F}{\partial x^4}+2\frac{\partial^4 \Delta F}{\partial x^2 \partial y^2}+\frac{\partial^4 \Delta F}{\partial y^4}=\frac{\mathrm{E}\pi^4}{4a^2b^2}\sum_m\sum_n\sum_k\sum_l\left(\mathrm{A}_{mn}^{i-1}+\mathrm{A}_{0mn}\right)\Delta \mathrm{A}_{kl}\left[8mnkl\times cx(k)cy(l)cx(m)cy(n)\right.$$
$$\left.-4\left(m^2l^2+k^2n^2\right)sx(k)sy(l)sx(m)sy(n)\right]$$

(Multiplying and dividing by 4)

or,

$$\frac{\partial^4 \Delta F}{\partial x^4} + 2\frac{\partial^4 \Delta F}{\partial x^2 \partial y^2} + \frac{\partial^4 \Delta F}{\partial y^4} = \frac{\mathrm{E}\pi^4}{4a^2b^2}\sum_m\sum_n\sum_k\sum_l \Delta \mathrm{A}_{kl}\left(\mathrm{A}_{mn}^{i-1} + \mathrm{A}_{0mn}\right)$$
$$\times\Big[-(kn-ml)^2 cx(m-k)cy(n-l) + (kn+ml)^2 cx(m-k)cy(n+l)$$
$$+(kn+ml)^2 cx(m+k)cy(n-l) - (kn-ml)^2 cx(m+k)cy(n+l)\Big] \qquad (19)$$

A particular solution, ΔF_{P} , for the stress function increment is then obtained as follows:

Let,

$$\Delta F_P = \sum_m\sum_n\sum_k\sum_l \Big[B_1(m,n,k,l)cx(m-k)cy(n-l)$$
$$+B_2(m,n,k,l)cx(m-k)cy(n+l) + B_3(m,n,k,l)cx(m+k)cy(n-l) \qquad (20)$$
$$B_4(m,n,k,l)cx(m+k)cy(n+l)\Big]$$

Then,

$$\frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial x^4} = \sum_m\sum_n\sum_k\sum_l \mathrm{B}_1(m-k)^4\frac{\pi^4}{a^4}cx(m-k)cy(n-l) + \mathrm{B}_2(m-k)^4\frac{\pi^4}{a^4}cx(m-k)cy(n+l)$$
$$\mathrm{B}_3(m+k)^4\frac{\pi^4}{a^4}cx(m+k)cy(n-l) + \mathrm{B}_4(m+k)^4\frac{\pi^4}{a^4}cx(m+k)cy(n+l)$$

$$2\frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial x^2 \partial y^2} = \sum_m\sum_n\sum_k\sum_l \frac{2\pi^4}{a^2b^2}\Big[\mathrm{B}_1(m-k)^2(n-l)^2 cx(m-k)cy(n-l) + \mathrm{B}_2(m-k)^2(n+l)^2 cx(m-k)cy(n+l)$$
$$\mathrm{B}_3(m+k)^2(n-l)^2 cx(m+k)cy(n-l) + \mathrm{B}_4(m+k)^2(n+l)^2 cx(m+k)cy(n+l)\Big]$$

$$\frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial y^4} = \sum_m\sum_n\sum_k\sum_l \mathrm{B}_1(n-l)^4\frac{\pi^4}{b^4}cx(m-k)cy(n-l) + \mathrm{B}_2(n+l)^4\frac{\pi^4}{b^4}cx(m-k)cy(n+l)$$
$$\mathrm{B}_3(n-l)^4\frac{\pi^4}{b^4}cx(m+k)cy(n-l) + \mathrm{B}_4(n+l)^4\frac{\pi^4}{b^4}cx(m+k)cy(n+l)$$

On adding the above three relations, we get:

$$\frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial x^4} + 2\frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial x^2 \partial y^2} + \frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial y^4} =$$

$$\sum_m \sum_n \sum_k \sum_l \frac{\pi^4}{a^2 b^2} \left\{ \mathrm{B}_1 \left[\frac{b^2}{a^2}(m-k)^4 + \frac{a^2}{b^2}(n-l)^4 + 2(m-k)^2(n-l)^2 \right] cx(m-k)cy(n-l) \right.$$
$$+\ \mathrm{B}_2 \left[\frac{b^2}{a^2}(m-k)^4 + \frac{a^2}{b^2}(n+l)^4 + 2(m-k)^2(n+l)^2 \right] cx(m-k)cy(n+l)$$
$$+\ \mathrm{B}_3 \left[\frac{b^2}{a^2}(m+k)^4 + \frac{a^2}{b^2}(n-l)^4 + 2(m+k)^2(n-l)^2 \right] cx(m+k)cy(n-l)$$
$$\left. +\ \mathrm{B}_4 \left[\frac{b^2}{a^2}(m+k)^4 + \frac{a^2}{b^2}(n+l)^4 + 2(m+k)^2(n+l)^2 \right] cx(m+k)cy(n+l) \right\}$$

Put $\alpha = \frac{a}{b}$, then

$$\frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial x^4} + 2\frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial x^2 \partial y^2} + \frac{\partial^4 \Delta F_{\mathrm{P}}}{\partial y^4} =$$

$$\sum_m \sum_n \sum_k \sum_l \frac{\pi^4}{a^2 b^2} \left\{ \mathrm{B}_1 \left[\frac{1}{\alpha}(m-k)^2 + \alpha(n-l)^2 \right]^2 cx(m-k)cy(n-l) \right.$$
$$+\ \mathrm{B}_2 \left[\frac{1}{\alpha}(m-k)^2 + \alpha(n+l)^2 \right]^2 cx(m-k)cy(n+l)$$
$$+\ \mathrm{B}_3 \left[\frac{1}{\alpha}(m+k)^2 + \alpha(n-l)^2 \right]^2 cx(m+k)cy(n-l)$$
$$\left. +\ \mathrm{B}_4 \left[\frac{1}{\alpha}(m+k)^2 + \alpha(n+l)^2 \right]^2 cx(m+k)cy(n+l) \right\}$$

Comparing the above relation with equation (19) , we get the values of coefficients B_1, B_2, B_3, B_4 as :

$$B_1(m,n,k,l) = \frac{E\alpha^2}{4}\Delta A_{kl}\left(A_{mn}^{i-1} + A_{0mn}\right)\frac{-(kn-ml)^2}{\left[(m-k)^2 + \alpha^2(n-l)^2\right]^2} \tag{21}$$

$$B_2(m,n,k,l) = \frac{E\alpha^2}{4}\Delta A_{kl}\left(A_{mn}^{i-1} + A_{0mn}\right)\frac{(kn+ml)^2}{\left[(m-k)^2 + \alpha^2(n+l)^2\right]^2} \tag{22}$$

$$B_3(m,n,k,l) = \frac{E\alpha^2}{4}\Delta A_{kl}\left(A_{mn}^{i-1} + A_{0mn}\right)\frac{(kn+ml)^2}{\left[(m+k)^2 + \alpha^2(n-l)^2\right]^2} \tag{23}$$

$$B_4(m,n,k,l) = \frac{E\alpha^2}{4}\Delta A_{kl}\left(A_{mn}^{i-1} + A_{0mn}\right)\frac{-(kn-ml)^2}{\left[(m+k)^2 + \alpha^2(n+l)^2\right]^2} \tag{24}$$

where α = a/b.

Substituting the above values of B_1, B_2, B_3, B_4 into the expression for equation (20),

$$\Delta F_P = \sum_m \sum_n \sum_k \sum_l \frac{E\alpha^2}{4}\Delta A_{kl}\left(A_{mn}^{i-1} + A_{0mn}\right)\left\{\frac{-(kn-ml)^2}{\left[(m-k)^2 + \alpha^2(n-l)^2\right]^2} cx(m-k)cy(n-l)\right.$$
$$+\frac{(kn+ml)^2}{\left[(m-k)^2 + \alpha^2(n+l)^2\right]^2} cx(m-k)cy(n+l) + \frac{(kn+ml)^2}{\left[(m+k)^2 + \alpha^2(n-l)^2\right]^2} cx(m+k)cy(n-l)$$
$$\left.+\frac{-(kn-ml)^2}{\left[(m+k)^2 + \alpha^2(n+l)^2\right]^2} cx(m+k)cy(n+l)\right\}$$

ΔF_P Can be written in a more simplified form as follows:

$$\Delta F_P = \frac{E\alpha^2}{4}\sum_m\sum_n\sum_k\sum_l \Delta A_{kl}\left(A_{mn}^{i-1} + A_{0mn}\right)$$
$$\times\sum_{r=1}^{2}\sum_{s=1}^{2}(-1)^{r+s+1} h_1\left[(-1)^r k,(-1)^r l\right]cx\left[m+(-1)^r k\right]cy\left[n+(-1)^s l\right] \tag{25}$$

where

$$h_1[\omega_1,\omega_2] = \frac{(-n\omega_1 + m\omega_2)^2}{\left[(m+\omega_1)^2 + \alpha^2(n+\omega_2)^2\right]^2}$$
$$h_1 = 0 \quad if \quad m+\omega_1 = 0 \quad and \quad n+\omega_2 = 0$$

Let the homogeneous solution, ΔF_H , of stress function increment ΔF , be given by

$$\Delta F_H = \frac{cx^2}{2} + dxy + \frac{ey^2}{2} + \frac{fx^3}{6} + \frac{gx^2y}{2} + \frac{hxy^2}{2} + \frac{iy^3}{6}$$

Now, at boundary, stress increment in x direction = $-\Delta P_x - \frac{12\Delta M_x}{b^3 t}\left(y - \frac{b}{2}\right)$

Similarly, stress increment in y direction = $-\Delta P_y - \frac{12\Delta M_y}{a^3 t}\left(x - \frac{a}{2}\right)$

and shear stress increment = $\Delta\tau_{xy}$

at boundary,

$$\therefore \quad \frac{\partial^2 \Delta F_H}{\partial x^2} = c + fx + gy = -\Delta \mathrm{P}_y - \frac{12\Delta \mathrm{M}_y}{a^3 t}\left(x - \frac{a}{2}\right)$$

$$\frac{\partial^2 \Delta F_H}{\partial y^2} = e + hx + iy = -\Delta \mathrm{P}_x - \frac{12\Delta \mathrm{M}_x}{b^3 t}\left(y - \frac{b}{2}\right)$$

$$\frac{\partial^2 \Delta F_H}{\partial x \partial y} = \left(d + gx + hy\right) = -\Delta \tau_{xy}$$

On solving, we get:

$$c = -\frac{\Delta \mathrm{P}_y}{at} - \frac{12\Delta M_y}{a^3 t} \times \frac{a}{2} \quad , \quad d = -\Delta \tau_{xy}$$

$$e = -\frac{\Delta \mathrm{P}_x}{bt} - \frac{12\Delta M_x}{b^3 t} \times \frac{b}{2} \quad , \quad f = -\frac{12\Delta M_y}{a^3 t} x$$

$$g = 0 \ , \ h = 0 \ , \ i = -\frac{12\Delta M_x}{b^3 t} y$$

$$\therefore \ \Delta F_H = -\frac{\Delta \mathrm{P}_x}{2bt} y^2 - \frac{\Delta \mathrm{P}_y}{2at} x^2 - \frac{\Delta \mathrm{M}_x}{b^3 t} y^2 \left(2y - 3b\right) - \frac{\Delta \mathrm{M}_y}{a^3 t} x^2 \left(2x - 3a\right) - \Delta \tau_{xy} xy \qquad (26)$$

Since the overall stress increment is equal to the sum of the particular and homogeneous solutions, therefore:

$$\Delta F = \Delta F_H + \Delta F_P$$

Similarly , to get the stress function , F_P^{i-1} , at the end of the $(i-1)^{th}$ load increment step , we substitute the values of w, w_0 and w_{i-1} into the incremental form of the compatibility equation at the end of the $(i-1)^{th}$ load increment step, resulting in the following equation :

$$\frac{\partial^4 F_{i-1}}{\partial x^4}+2\frac{\partial^4 F_{i-1}}{\partial x^2 \partial y^2}+\frac{\partial^4 F_{i-1}}{\partial y^4}=\frac{\mathrm{E}\pi^4}{4a^2b^2}\sum_m\sum_n\sum_k\sum_l\left(\mathrm{A}_{mn}^{i-1}\mathrm{A}_{kl}^{i-1}-\mathrm{A}_{mn}^{0}\mathrm{A}_{kl}^{0}\right)$$
$$\times\left[ml\left(kn-ml\right)cx\left(m-k\right)cy\left(n-l\right)+ml\left(kn+ml\right)cx\left(m-k\right)cy\left(n+l\right)\right. \qquad (27)$$
$$\left.+ml\left(kn+ml\right)cx\left(m+k\right)cy\left(n-l\right)+ml\left(kn-ml\right)cx\left(m+k\right)cy\left(n+l\right)\right]$$

A particular solution, F_P^{i-1}, for the stress function increment , F_{i-1} is then obtained as follows: Let,

$$F_P^{i-1}=\sum_m\sum_n\sum_k\sum_l\left[C_1\left(m,n,k,l\right)cx\left(m-k\right)cy\left(n-l\right)\right.$$
$$+C_2\left(m,n,k,l\right)cx\left(m-k\right)cy\left(n+l\right)+C_3\left(m,n,k,l\right)cx\left(m+k\right)cy\left(n-l\right) \qquad (28)$$
$$\left.C_4\left(m,n,k,l\right)cx\left(m+k\right)cy\left(n+l\right)\right]$$

The coefficients C_1, C_2, C_3, C_4 can then be determined as follows:

$$C_1\left(m,n,k,l\right)=\frac{\mathrm{E}\alpha^2}{4}\left(\mathrm{A}_{mn}^{i-1}\mathrm{A}_{kl}^{i-1}-\mathrm{A}_{mn}^{0}\mathrm{A}_{kl}^{0}\right)\frac{ml\left(kn-ml\right)}{\left[\left(m-k\right)^2+\left(n-l\right)^2/\alpha^2\right]^2}$$
$$C_2\left(m,n,k,l\right)=\frac{\mathrm{E}\alpha^2}{4}\left(\mathrm{A}_{mn}^{i-1}\mathrm{A}_{kl}^{i-1}-\mathrm{A}_{mn}^{0}\mathrm{A}_{kl}^{0}\right)\frac{ml\left(kn+ml\right)}{\left[\left(m-k\right)^2+\left(n+l\right)^2/\alpha^2\right]^2}$$
$$C_3\left(m,n,k,l\right)=\frac{\mathrm{E}\alpha^2}{4}\left(\mathrm{A}_{mn}^{i-1}\mathrm{A}_{kl}^{i-1}-\mathrm{A}_{mn}^{0}\mathrm{A}_{kl}^{0}\right)\frac{ml\left(kn+ml\right)}{\left[\left(m+k\right)^2+\left(n-l\right)^2/\alpha^2\right]^2}$$
$$C_4\left(m,n,k,l\right)=\frac{\mathrm{E}\alpha^2}{4}\left(\mathrm{A}_{mn}^{i-1}\mathrm{A}_{kl}^{i-1}-\mathrm{A}_{mn}^{0}\mathrm{A}_{kl}^{0}\right)\frac{ml\left(kn-ml\right)}{\left[\left(m+k\right)^2+\left(n+l\right)^2/\alpha^2\right]^2}$$

Substituting the above values of C_1, C_2, C_3, C_4 into the expression for F_P^{i-1},

$$F_{\mathrm{P}}^{i-1} = \sum_m \sum_n \sum_k \sum_l \frac{\mathrm{E}\alpha^2}{4}\left(\mathrm{A}_{mn}^{i-1}\mathrm{A}_{kl}^{i-1} - \mathrm{A}_{mn}^{0}\mathrm{A}_{kl}^{0}\right)\left\{\frac{ml\left(kn-ml\right)}{\left[\left(m-k\right)^2+\alpha^2\left(n-l\right)^2\right]^2}cx\left(m-k\right)cy\left(n-l\right)\right.$$

$$+\frac{ml\left(kn+ml\right)}{\left[\left(m-k\right)^2+\alpha^2\left(n+l\right)^2\right]^2}cx\left(m-k\right)cy\left(n+l\right)+\frac{ml\left(kn+ml\right)}{\left[\left(m+k\right)^2+\alpha^2\left(n-l\right)^2\right]^2}cx\left(m+k\right)cy\left(n-l\right)$$

$$\left.+\frac{ml\left(kn-ml\right)}{\left[\left(m+k\right)^2+\alpha^2\left(n+l\right)^2\right]^2}cx\left(m+k\right)cy\left(n+l\right)\right\}$$

F_P^{i-1} can be written in a more simplified form as follows:

$$F_P^{i-1} = \frac{\mathrm{E}}{4\alpha^2}\sum_m \sum_n \sum_k \sum_l \left(\mathrm{A}_{mn}^{i-1}\mathrm{A}_{kl}^{i-1} - \mathrm{A}_{mn}^{0}\mathrm{A}_{kl}^{0}\right)$$
$$\times\sum_{r=1}^{2}\sum_{s=1}^{2}(-1)^{r+s} h_2\left[(-1)^r k,(-1)^r l\right]cx\left[m+(-1)^r k\right]cy\left[n+(-1)^s l\right] \quad (29)$$

where

$$h_2\left[\omega_1,\omega_2\right] = \frac{m\omega_2\left(n\omega_1 - m\omega_2\right)}{\left[\left(m+\omega_1\right)^2+\left(n+\omega_2\right)^2/\alpha^2\right]^2}$$
$$h_2 = 0 \quad if \quad m+\omega_1 = 0 \quad and \quad n+\omega_2 = 0$$

The homogeneous solution F_H^{i-1} , at the end of $(i-1)^{th}$ load increment step, being the summation of load steps from 1 to (i-1) , is given by:

$$F_{\mathrm{H}}^{i-1} = -\frac{\mathrm{P}_x^{i-1}}{2bt}y^2 - \frac{\sigma_{rx}}{2}y^2 - \frac{\mathrm{P}_y^{i-1}}{2at}x^2 - \frac{\sigma_{ry}}{2}x^2 - \frac{\mathrm{M}_x^{i-1}}{b^3t}y^2\left(2y-3b\right) - \frac{\mathrm{M}_y^{i-1}}{a^3t}x^2\left(2x-3a\right) - \tau_{xy}^{i-1}xy \quad (30)$$

Therefore, the stress function, F^{i-1} at the end of the $(i-1)^{th}$ load increment step can be obtained by the sum of equations (29) and (30) as follows:

$$F^{i-1} = F_P^{i-1} + F_H^{i-1}$$

The values of $\Delta w, w_{i-1}, w_0, \Delta F \text{ and } F^{i-1}$ are then substituted into the incremental form of the governing equilibrium equation. We then apply the Galerkin method to compute the unknown coefficients ΔA_{kl} .

The integration of equilibrium equation eventually results in a set of linear first order simultaneous equations for the unknown coefficients ΔA_{kl} .The equation can be written in a matrix form as follows:

$$\{\Delta F\} = ([F_0] + [B] + [M])\{\Delta A\} \tag{31}$$

where $\{\Delta F\}$= external load increments, $[F]$= stiffness matrix associated with initial stress (including weight -induced residual stresses), $[B]$= bending stiffness matrix, $[M]$= stiffness matrix due to membrane action. $\{\Delta A\}$ = unknown coefficients of deflection amplitudes.

Having obtained ΔA_{kl}, we can then calculate Δw , ΔF , $w_i(= w_{i-1} + \Delta w)$ and $F_i(= F_{i-1} + \Delta F)$ at the end of the i^{th} load increment step. By repeating the above procedure with increase in the applied loads, the elastic large deflection response for simply supported plate can be obtained.

8.1 Examples of the Analysis

- *8.1.1 Simply Supported square plate under uniaxial compression*

The square plate in example 7.1.2 is now analyzed using the incremental technique. The dimensions of the plate are same as in example 7.1.2. The initial and the added deflection functions are also assumed to be the same. A comparison of Fig 6 and Fig 7 shows that the new method is quite accurate for this case.

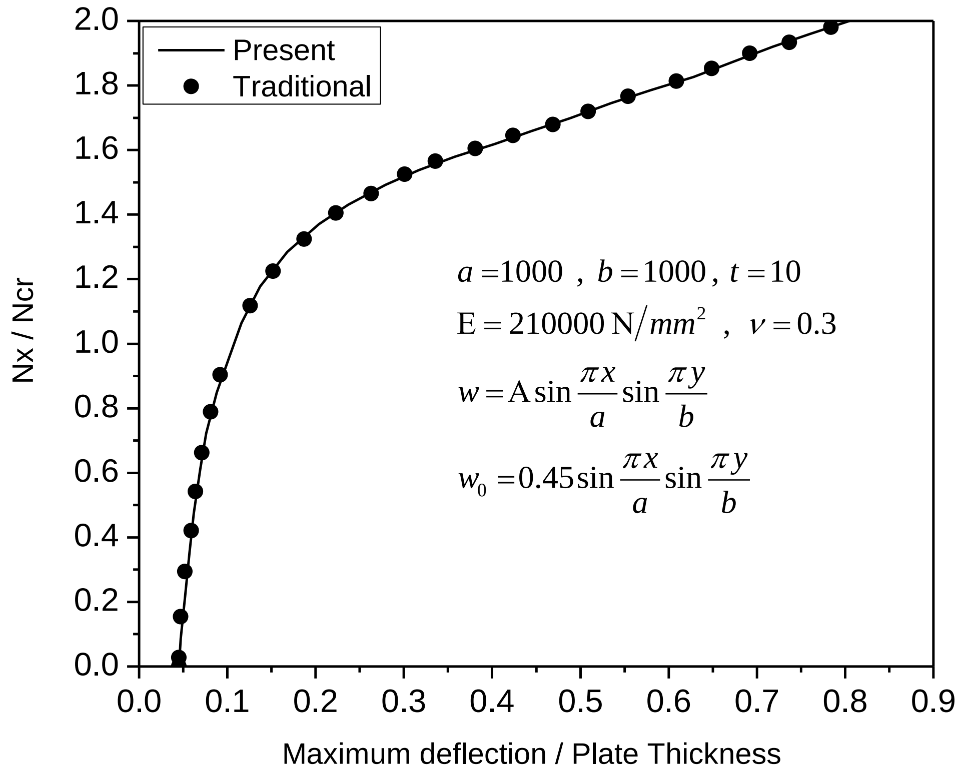


Fig.7. Simply Supported Plate subjected to uniaxial compression (incremental method)

- *8.1.2 Simply Supported square plate under uniaxial compression*

A simply supported square plate subjected to uniaxial compression is analyzed and the Load deflection relationship is obtained at the centre for different values of initial deflection. The dimensions of the plate are the same as in example 9.1.1. The initial and the added deflection functions are also assumed to be the same. The critical load however, is 81994 N/mm^2 in this case.

$$w = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$w_0 = 0.45\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$P_{xcr} = 81994 \ \mathrm{N}/mm^2$$

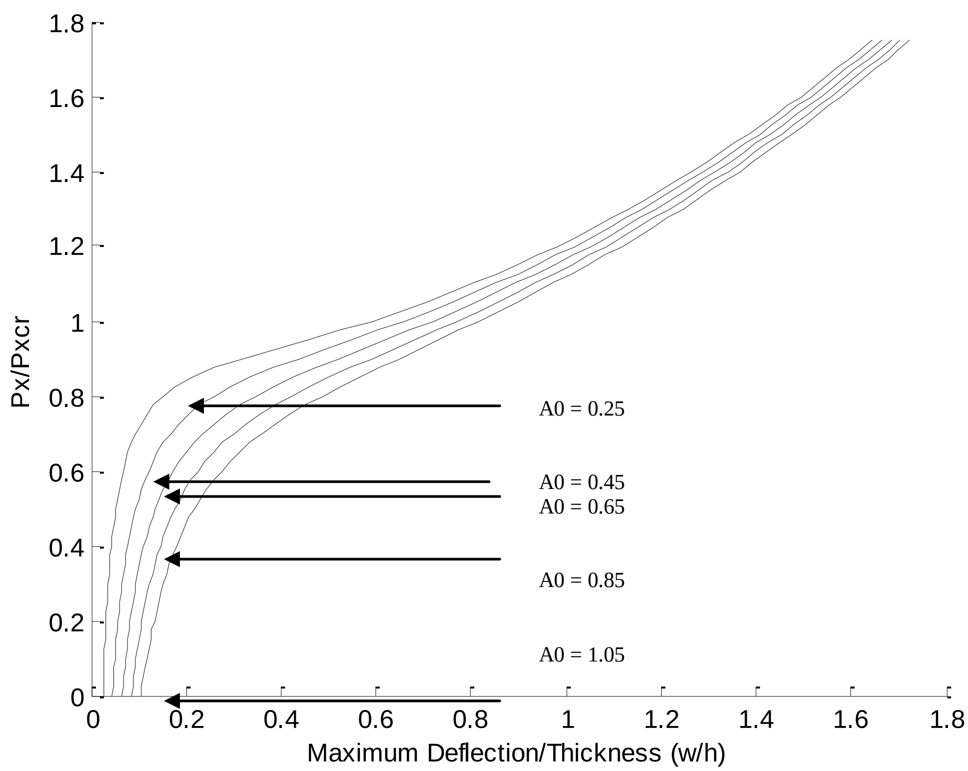


Fig.8. Load deflection relationship for a Simply supported Plate subjected to uniaxial compression (incremental method)

- *Square Plate subjected to Lateral Loading*

The square plate in example 7.1.1 is now analyzed using the incremental technique. A comparison of Fig 10 with Fig 4 and Fig 5 shows that the present method is quite accurate for a square plate subjected to lateral loading.

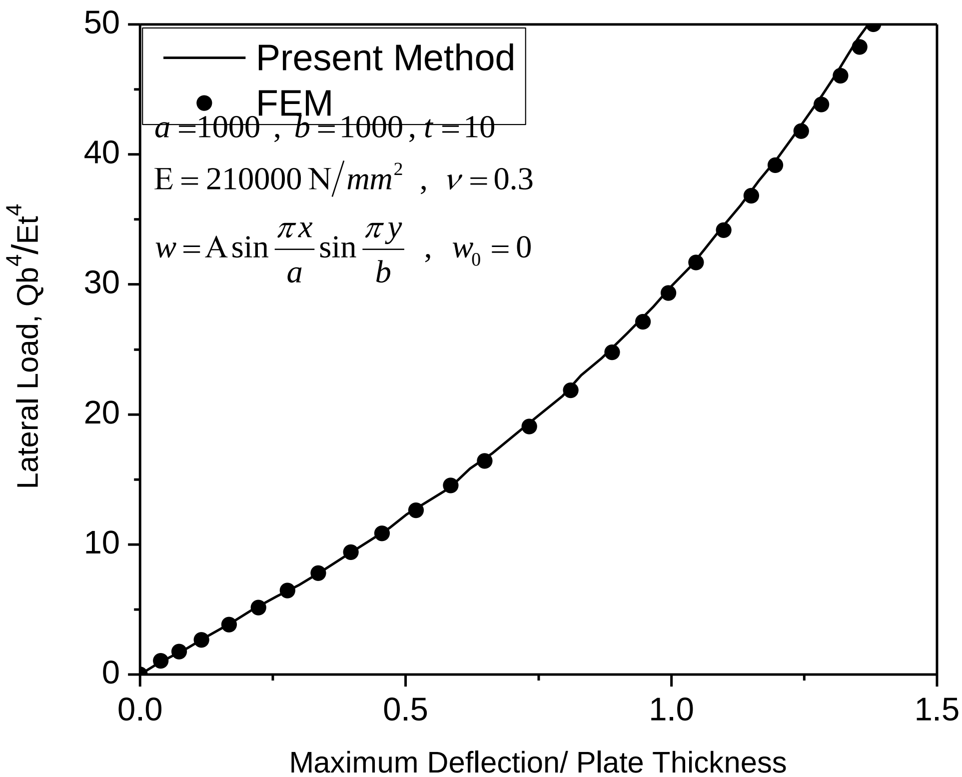


Fig.10 Load deflection relationship for square plate subjected to lateral loading (present method)

- *8.1.4 Rectangular Plate subjected to in-plane compression in longitudinal*

Direction

A simply supported rectangular plate as shown in figure 11 is now analyzed taking two terms of the deflection function for different values of the initial deflection. The plate is subjected to uniaxial in plane compression in longitudinal direction. The analysis is carried out by the present method for two different points A and B on the plate. The load is applied incrementally from 0 to an average stress of 16 kgf/mm^2.

Details of the plate:

$$a = 1500,\ b = 1000,\ \nu = 0.3$$

$$\text{E} = 21000\ kgf/mm^2,\ \text{A} \equiv \left(\frac{a}{4}, \frac{b}{2}\right),\ \text{B} \equiv \left(\frac{3a}{4}, \frac{b}{2}\right)$$

$$w_0 = \left(1.1 \sin\frac{\pi x}{a} + 0.22 \sin\frac{2\pi x}{a}\right) \sin\frac{\pi y}{b}$$

$$w = \left(\text{A}_{11} \sin\frac{\pi x}{a} + \text{A}_{21} \sin\frac{2\pi x}{a}\right) \sin\frac{\pi y}{b}$$

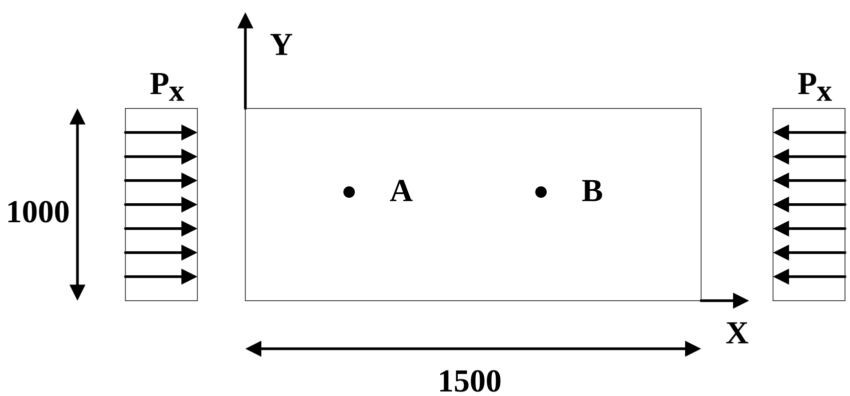


Fig. 11. All edges are simply supported and kept straight in the plane of the plate

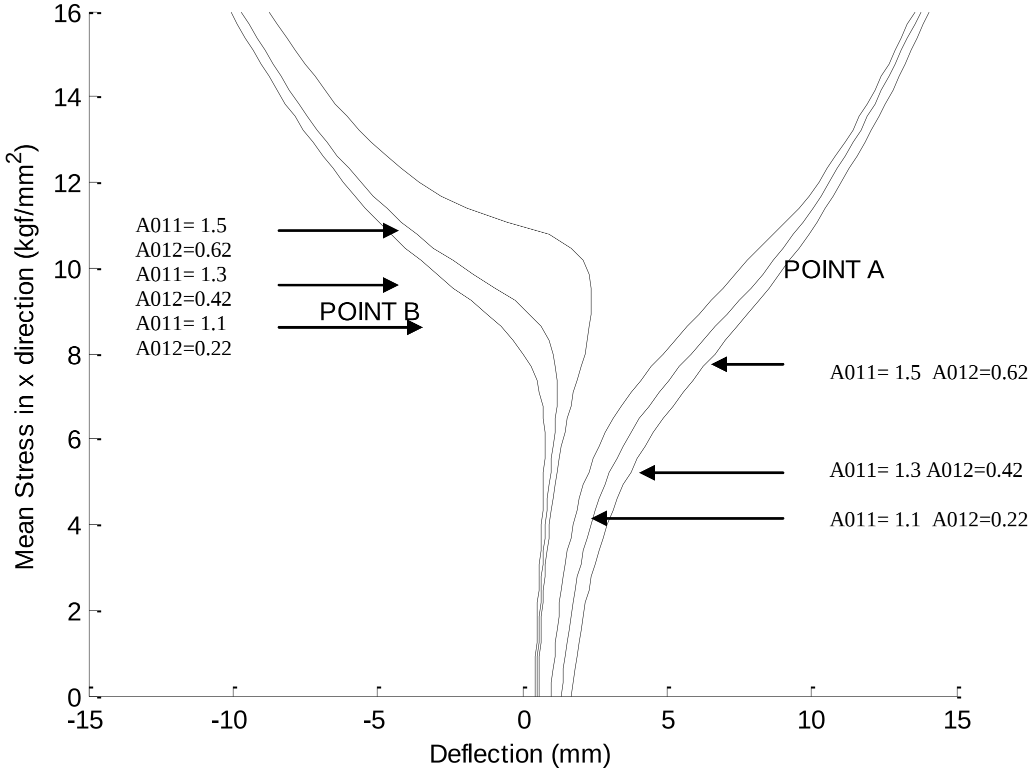


Fig.12 Load Deflection relationship of a plate subjected to in plane compression (present method)

9. Treatment of Plasticity

Till now, the differential equations governing the elastic large deflection response of plates have been formulated and are solved analytically. But the effects of plasticity have not been included. It is normally difficult to formulate governing differential equations representing both geometric and material nonlinearities for plates, although not impossible. A major source of difficulty is that an analytical treatment of plasticity with increase in the applied loads is very difficult. Even if such treatment were possible, it would not be an easy task to solve the resulting equations analytically. Hence an easier alternative is to deal with progress of the plasticity numerically. In the present method, therefore, the progress of plasticity with increase in the applied loads is treated numerically. For this purpose, as indicated in Fig. 3, the plate is subdivided into a number of mesh regions in the three dimensions similar to the conventional finite element method. The average membrane stress components for each mesh region can be calculated at every load increment step. Yielding for each mesh region is checked for the plate by using the following von Mises yield criteria:

$$\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 \geq \sigma_0^2 \quad (32)$$

As the applied loads increase, the stiffness matrices for the plate are redefined by considering the progress of plasticity. In Eq. (31), the stiffness matrix associated with external loads would be calculated for the whole volume of the plate regardless of the plasticity. However, the bending stiffness will be reduced by the plasticity if any mesh region yields. In the calculation (i.e. integration) of the bending stiffness matrix, therefore, contribution to the yielded regions is removed.

As noted above, it is assumed that the plate is composed of a number of membrane strings (or fibers) in the two (i.e. x, y) directions. Each fiber has a number of layers in the z direction. The end condition for each fiber would satisfy the plate edge condition as well. In fact, due to the membrane action of the fibers, occurrence of the additional plate deflection may to some extent be disturbed with further increase in the applied loads. However, if any local region in the fiber is yielded, the fiber (i.e. string) will be cut such that the membrane action is not available further. In calculating (or integrating) the stiffness matrix due to membrane action, therefore, the entire fibers associated with yielded regions are not included. It should be noted that a mesh region inside the plate may be common to two fibers, i.e. in the x (i.e. length) and y (i.e. breadth) directions. In this case, the contribution from the two relevant fibers (i.e. strings) should be removed in the calculation of the stiffness matrix associated with the membrane effects.

The stiffness for the plate will be progressively reduced by large deflection and local yielding. The plate can be considered to have reached the ultimate limit state when the plate stiffness eventually becomes zero (or negative). The process indicated above to include plasticity effects is carried out numerically. In this regard, the present method could perhaps be better classified as a semi-analytical approach.

10. Conclusion*

- The Incremental technique is the most efficient method for analyzing large deflection behaviour of plates.

- Plasticity can be numerically incorporated into this technique thereby saving computational effort.

- The applicability of the method has been verified to a great extent by comparing the results obtained in various cases with existing theoretical and experimental results.

- The insights and developments obtained in this study can be very efficiently used for the design of ship structures since the developed method is quick as well as accurate.

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*Remark: Although the formulation for plasticity has been carried out, but it has not been included for obtaining the results in this paper and the material of the plate is assumed to be elastic everywhere.

11. References

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