Chapter 1

INTRODUCTION

1.1 Preamble

Research on stability of structural element was initiated by Loenhard Euler in 1744. He showed that, a slender column under an axial compressive load will fail due to lateral deflection rather than crushing of material. Wood and stone were the major construction materials during that period and so the structures were huge and massive; stability was not a prime consideration. Steel was used in a big way in the construction industry in the later half of nineteenth century. Steel structural elements (beams, plates and shell panels) used in aircrafts, spacecrafts, ships, bridges and offshore structures are subjected to a variety of static and dynamic loads during their life span and may undergo static and dynamic instabilities. The structural instability may lead to large deflections or large amplitude vibrations of the structural elements leading to global or local failures. Hence, buckling and dynamic instability characteristics have become important design considerations.

Geometric effects cause structural instability failure in elastic structures. Nonlinearities are introduced by the geometry of deformation which amplify the stresses calculated based on the initial undeformed configuration of the structure (Bazant and Cedolin, 2003). Different definitions of stability are used for different problems. However, dynamic stability definition is applicable to all structural stability problems. The structures appear stable from static buckling analysis when subjected to dynamic inplane loads but actually they fail due to ever increasing amplitude of vibration. This is correctly detected by dynamic analysis.

Columns are often, a part of complex structural system and hence load coming on it may not be always uniform. In the case of I-beam or wide flanged beam for example, subjected to bending moment at the ends or lateral loads on the flange, the web of the beam is subjected to non-uniform in-plane loads. The load exerted on the stiffened plate, or on the aircraft wings in the ship structures or on the slabs of a multi-storey building by the adjoining structures usually is non uniform. The type of distribution in an actual structure depends on the relative stiffnesses of the adjoining elements. The non-uniform edge loading introduces all 3-

components of stress (σ_{xx} ,σ_{yy} and σ_{xy}) in the plates of column which may considerably influence the static stability behaviour of column.

Columns subjected to dynamic in-plane loading may undergo unstable transverse vibrations, for certain combinations of the load amplitude, disturbing frequency of transverse vibration. This type of dynamic instability is called parametric instability or parametric resonance because of the fact that the applied load is parametric with respect to the motion of the plate. The dynamic axial loading is known as parametric excitation and the range of the values of parameters causing dynamic instability is called the dynamic instability region or parametric resonance region. Parametric instability zones are generally defined by the boundaries which separate stable and unstable regions in the parametric space defined above. Unlike in the case of resonances occurring under transverse forced vibration, parametric resonance will occur over a wide range of excitation frequencies, particularly when the amplitude of the dynamic load is large. In general, a significant portion of the total spectrum of the values of parameters may be occupied by parametric resonance zones.

The transverse vibration of structures are result of dynamic instability with ever increasing amplitude. Linear strain-displacement relations are erroneous at higher amplitude of vibration and hence geometric nonlinearities need to be considered Nonlinear theory results in new phenomena (Shivamoggi, 1977), which have no place in the corresponding linear problem. The new phenomena are existence of solution for all values of the frequency rather than only for a set of characteristic values and dependence of amplitude on frequency

1.2 Objective of the Present Investigation

The objective of the present investigation is plot different zones of instability of column under periodic axial loading and to investigate the instability of different points on amplitude frequency curve through phase portraits.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

Columns are often used as structural components are subjected to severe mechanical, thermal and dynamic axial loads under operating conditions. The structure loses its stability at critical value of static axial load and become dynamically unstable for certain combinations of load amplitude, disturbing frequency and frequency of transverse vibration of the structure. In the last few decades their bending and vibration behaviour have been extensively studied. Much work has been reported in literature in the recent past on the buckling and dynamic instability of columns subjected to uniform static and dynamic axial loads. However, as discussed in the previous chapter, structural elements are often subjected to non-uniform static and dynamic in-plane loads. This has necessitated reconsidering buckling, and dynamic buckling of columns subjected to different sets of axial loads. Also, it is required to understand the changes in the vibration behaviour of the structural components in the presence of loads for the design of these components. The parametric vibration behaviour is affected by the large amplitude of vibration. If the nonlinearity is considered, then dynamic instability region changes. Hence, for the proper design of different structural members under dynamic in-plane loads the knowledge of nonlinear dynamic instability analysis is required.

Static Buckling

More than two centuries ago the instability of an axially compressed column was studied by Euler. However, Euler’s work didn’t have any practical importance during his time as structures were massive and constructed with wood and stone masonry and there was no further interest in the buckling problems till mid-nineteenth century. The introduction of steel and later composites as structural materials prompted further interest in the buckling problem as the structures became slender.

Dynamic Buckling

In elastic systems, structural members such as columns, plates and shell panels under time dependent compressive axial load problem of dynamic instability arises. When the loading is time dependent, the systems are called parametrically excited systems and the instability is referred as parametric instability or dynamic instability. Dynamic instability occurs in structural components (beams, plates and shells) for certain combinations of load amplitude, disturbing frequency and frequency of transverse vibration of plates. It is essential to have the knowledge of static and dynamic stability behaviour of composite structural members subjected to various types of dynamic loads in the design of these components. Most of the early development in this field is due to Russian researchers. Bolotin (1964) has reviewed literature on the dynamic instability of isotropic beams, plates and shells in his book. The first observation of parametric resonance is attributed to Faraday in 1831 and first mathematical explanation of phenomenon is given by Rayleigh in 1883. The mathematical formulation of the problem leads to Mathieu-Hill type of linear differential equation with periodic coefficients. Boundedness of the solution corresponding to stability or instability is Hill's infinite determinant method originally developed by Hill in 1886 for a single degree of freedom system. This method has been further extended for a multiple degree of freedom system by Bolotin (1964), Valeev (1961).

Buckling of rectangular plates with various boundary conditions loaded by non-uniform inplane loads and Dynamic instability of composite plates subjected to non-uniform in-planeloads have been studied by L.S. Ramachandra and Sarat Kumar Panda

Chapter 3

Mathematical Formulations

3.1 Introduction

The transverse vibration of simply supported elastic column of uniform cross section subjected to dynamic axial load *P (t)* as shown in figure 1. Before deflection due the axial load, the column is straight with length *L*.

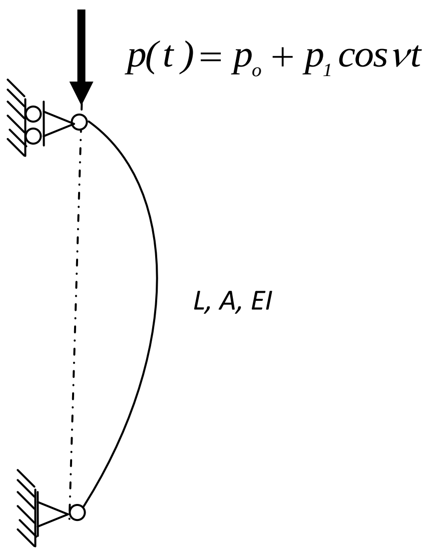

Figure 1.

3.2 Formulation

3.2.1 Mathieu Equation

Consider the problem of transverse vibrations of a straight rod loaded by a periodic longitudinal force. The rod is assumed to be simply supported and of uniform cross section along its length. The usual assumptions in the field of strength of materials are made i.e., that Hooke's law holds and plane sections remain plane.

The basic equation of static bending of a rod is

$$EI\frac{d^2y}{dx^2} + Py = 0 \qquad (3.1)$$

Where y(x) is deflection of the rod, EI is its bending stiffness, and P is the longitudinal force. After two differentiations, the equation takes the form

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0 \qquad (3.2)$$

This gives the condition that the sum of the y components of all the forces per unit length acting on the rod is equal to zero.

To arrive at the equation for the transverse vibrations of a rod loaded by the periodic longitudinal force

$$P(t) = P_0 + Pcos\theta t \qquad (3.3)$$

It is necessary to introduce additional terms into above fourth order equation that take into account the inertial forces.

Inertia forces associated with the rotation of the cross sections of the rod with respect to its own principal axes are not included as in the case of the applied theory of vibrations. Note that longitudinal inertia forces can substantially influence the dynamic stability of a rod only in the case where the frequency of the external force is near the longitudinal natural frequencies of the rod, i.e., when the longitudinal vibrations has a resonance character. In the following discussion, we will consider that the system is not close to the resonance of the longitudinal vibrations.

The inertia forces acting on the rod can be reduced to a distributed loading with these limiting assumptions, whose magnitude is

$$-m\frac{d^2y}{dt^2} \tag{3.4}$$

Where m is the mass per unit length of the rod. Thus, we arrive at the differential equation

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} + m\frac{d^2y}{dt^2} = 0 \tag{3.5}$$

i.e.,

$$EI\frac{d^4y}{dx^4} + (P_0 + Pcos\theta t)\frac{d^2y}{dx^2} + m\frac{d^2y}{dt^2} = 0 \tag{3.6}$$

for the dynamic deflections y(x,t) of the rod at any arbitrary instant of time.

We will seek the solution of above equation in the form

$$y(x,t) = q(t)sin\frac{k\pi x}{l} \tag{3.7}$$

Where q(t) are unknown functions of time and length of the rod. It is easily seen that above expression satisfies boundary conditions of the problem, requiring in the given case that the deflection, together with its second derivative, vanish at the ends of the rod. We remind that the “coordinate functions”

$$\varphi_k(x) = \sin\frac{k\pi x}{l} \tag{3.8}$$

Are of the same form as that of the free vibrations and the buckling of a simply supported rod.

Substitution of $y(x,t) = q(t)sin\frac{k\pi x}{l}$ in to our differential equation gives

$$\left[m\frac{d^2q}{dt^2} + EI\frac{k^4\pi^4q}{l^4} - (P_0 + Pcos\theta t)\frac{k^2\pi^2q}{l^2}\right]\sin\frac{k\pi x}{l} = 0 \tag{3.9}$$

It is necessary and sufficient that for the chosen solution to satisfy the quantity in brackets should vanish at any t. In other words, the functions q must satisfy differential equation

$$\frac{d^2q}{dt^2} + {\omega_k}^2\left(1 - \frac{P_0 + Pcos\theta t}{P_{*k}}\right)q = 0,\ \ (k = 1,2,3, \dots.) \tag{3.11}$$

Where

$$\omega_k = \frac{k^2\pi^2}{l^2}\sqrt{\frac{EI}{m}} \tag{3.12}$$

Is the k^{th} frequency of the free vibrations of an unloaded rod, and

$$P_{*k} = \frac{k^2\pi^2 EI}{l^2} \tag{3.13}$$

Is the k^{th} Euler buckling load (the asterisk denotes this quantity in the problems).

For convenience, we represent above differential equation in the form

$$\frac{d^2q}{dt^2} + \Omega_k^2[1 - 2\mu_k cos\theta t]q = 0 \quad\quad (k = 1,2,3, \dots.) \tag{3.14}$$

Where Ω_k is the frequency of the free vibrations of the rod loaded by a constant longitudinal force P_0,

$$\Omega_k = \omega_k\sqrt{1 - \frac{P_0}{P_{*k}}}\ , \tag{3.15}$$

And

$$\mu_k = \frac{P_t}{2(P_{*k} - P_0)}\ . \tag{3.16}$$

Since above differential equation is identical for all the forms of vibrations, i.e., it is identical for all k, we will in the future omit the indices of Ω_k and μ_k and write this equation in the form

$$q'' + \Omega^2[1 - 2\mu cos\theta t]q = 0\ . \tag{3.17}$$

The prime denotes differentiation with respect to time.

The above equation is well known Mathieu equation. For the more general case of the longitudinal force given by

$$P(t) = P_0 + P_t\phi(t) \tag{3.18}$$

Where $\phi(t)$ is a periodic function with a period T,

$$q'' + \Omega^2[1 - 2\mu\phi(t)]q = 0 \tag{3.19}$$

Such an equation, more general than the Mathieu equation, is usually called the Hill equation.

In various areas of physics and engineering Mathieu-Hill equations are encountered. Similar equations are obtained in certain problems in theoretical physics, particularly the problem of the propagation of electromagnetic waves in a medium with a periodic structure. In the quantum theory of metals, the problem of the motion of electrons in a crystal lattice reduces to the Mathieu Hill equation. The Mathieu-Hill equation is also encountered in the investigations of the stability of the oscillatory processes in nonlinear systems, in the theory of the parametric excitation of electrical oscillations, and other branches of the theory of oscillations. Certain problems of celestial mechanics and cosmogony also lead to the Hill equation, particularly the theory of the motion of the moon.

As shown above the governing differential equation derived is as

$$\ddot{q}_n + \omega_n^2 \left[1 - \frac{P(t)}{p_n} \right] q_n = 0 \text{ , where n = 1,2,3…} \tag{3.20}$$

$$\upsilon(x,t) = \sum_{i=1}^{\infty} q_i(t) \Phi_i(x); \qquad \Phi_i(x) = \sin\frac{i\pi x}{L} \tag{3.21}$$

ω_n is the nth natural frequency of a simply supported column with $P(t)=0$, and p_n is the nth Euler buckling load given by

$$\omega_n = \left(\frac{n\pi}{L} \right)^2 \sqrt{\frac{EI}{\rho A}}, \qquad p_n = EI\left(\frac{n\pi}{L} \right)^2 \tag{3.22}$$

The dynamic load $P(t)$ appears as a coefficient or parameter in the equation of motion, yielding a parametrically excited system. If $P(t)$ is a periodic function of period T , i.e $P(t) = P(t+T)$, then the governing differential equation with a periodic coefficient is known as Hill equation. If $P(t)$ is a sinusoidal function of period T of the form $P(t) = P_0 + P_1 \cos \nu t$, in which $\nu = \frac{2\pi}{T}$ is the circular excitation frequency than the governing differential equation becomes

$$\ddot{q}_n + \Omega_n^2 \left[1 - 2\lambda_n \cos \nu t \right] q_n = 0, \qquad n = 1,2,3... \tag{3.23}$$

where, $\Omega_n^2 = \omega_n^2 \left[1 - \frac{P_0}{p_n} \right], \;\; 2\lambda_n = \frac{p_1}{p_n - P_0}$ (3.24)

Equation (23) is known as Mathieu equation.

3.2.2 Stability of the Mathieu-Hill Equations

A Second order differential equation with a periodic coefficient of the form

$$\ddot{q} + \omega^2[1 - 2\mu\Phi(t)]q = 0 \tag{3.25}$$

Is called a Mathieu-Hill equation, where $\Phi(t + T) = \Phi(t)$ is a periodic function of period T, which may be expressed as a Fourier series,

$$\Phi(t) = \sum_{i=1}^{\infty}(a_i \cos ivt + b_i \sin ivt)\,, \quad T = \frac{2\pi}{v} \tag{3.26}$$

If $\Phi(t)$ is a sinusoidal function, i.e. $\Phi(\mathrm{t}) = \mathrm{asinvt}$ or $\Phi(\mathrm{t}) = \mathrm{acosvt}$, then the above differential equation is called Mathieu equation. Examples of Mathieu-Hill equations can be found in the study of dynamics of elastic systems, such as columns under axial loads.

If the time parameter is shifted from t to t+T, the above differential equation (3.25) remains unchaged. Hence, if q(t) is a solution, so is q(t+T). Since equation is a linear ordinary differential equation of order 2, if $q_1(t)$ and $q_2(t)$ are two linearly independent solutions, then $q_1(t+T)$ and $q_2(t+T)$ are also two solutions. For example, one can take $q_1(t) = q(t)$ and $q_2(t) = \dot{q}(t)$.

The solutions at time t+T and t have the following relationship

$$q_1(t + T) = a_{11}q_1(t) + a_{12}q_2(t),$$

$$q_2(t + T) = a_{21}q_1(t) + a_{22}q_2(t), \tag{3.27}$$

Or, in the matrix form,

$$\boldsymbol{q}(t + T) = \boldsymbol{A}\boldsymbol{q}(t) \tag{3.28}$$

Where $\mathbf{q} = \{q_1,q_2\}^T$ and **A** is a 2X2 matrix with constant elements.

Matrix **A** is in genera a full matrix so that it is a challenge to deal with above equation (3.28). In order to simplify the analysis, let $\boldsymbol{q}^*(t) = \{q_1^*(t), q_2^*(t)\}^T$ be another set of linearly independent solutions to be determined later. Then there is a constant non-singular matrix **B,** i.e., det(B) not equal to zero, such that the following relations are satisfied.

$$\boldsymbol{q}(t) = \boldsymbol{B}\boldsymbol{q}^*(t), \text{ and } \boldsymbol{q}(t + T) = \boldsymbol{B}\boldsymbol{q}^*(t + T). \tag{3.29}$$

From above two equations (28) and (29), one has

$$\boldsymbol{q}(t + T) = \boldsymbol{A}\boldsymbol{q}(t) = \boldsymbol{A}\boldsymbol{B}\boldsymbol{q}^*(t) \tag{3.30}$$

And hence

$$\boldsymbol{q}^*(t + T) = \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}\boldsymbol{q}^*(t) \tag{3.31}$$

From these equations it is desirable that $\mathbf{B}^{-1}\mathbf{AB}$ have the simplest from possible so that above equation can be easily studied.

If ρ_1 and ρ_2 are the eigen values of the matrix **A**, then matrix B diagonalizes matrix **A** i.e., $B^{-1}AB = \text{diag}\{\rho_1, \rho_2\}$, Therefore,

$$\det(\boldsymbol{A} - \rho\boldsymbol{I}) = 0, \tag{3.32}$$

And the columns of B are eigenvectors of matrix A, then matrix B diagonalizes matrix A, i.e., $B^{-1}AB = \text{diag}\{\rho_1, \rho_2\}$. Therefore

$$\boldsymbol{q}_{\boldsymbol{1}}^*(t+T) = \rho_1 \boldsymbol{q}^*(t)$$

$$\boldsymbol{q}_{\boldsymbol{2}}^*(t+T) = \rho_2 \boldsymbol{q}^*(t) \tag{3.33}$$

It is seen that by constructing matrix B using the eigenvectors of matrix A, one obtains a set of above equation (3.33), which are much simpler than the before. Since solutions q(t) and $q^*(t)$ are linearly related, they have the same stability behaviour.

To determine matrix A, consider $q_1(t)$ and $q_2(t)$ as two linearly independent solutions satisfying the initial conditions

$$q_1(0) = 1,\ \dot{q}_1(0) = 0,\ q_2(0) = 0,\ \dot{q}_2(0) = 1. \tag{3.34}$$

From these we get,

$$q_1(T) = a_{11}.1 + a_{12}.0 = a_{11},\ \dot{q}_1(T) = a_{11}.0 + a_{12}.1 = a_{12}$$

$$q_2(T) = a_{21}.1 + a_{22}.0 = a_{21},\ \dot{q}_2(T) = a_{21}.0 + a_{12}.1 = a_{22} \tag{3.35}$$

The Characteristic equation becomes,

$$\begin{vmatrix} q_1(T) - \rho & \dot{q}_1(T) \\ q_2(T) & \dot{q}_2(T) - \rho \end{vmatrix} = 0, \tag{3.36}$$

Which, after expanding the determinant, yields a quadratic equation in ρ,

$$\rho^2 - 2b\rho + c = 0, \tag{3.37}$$

Where

$$2b = q_1(T) + q_2(T) = tr(\boldsymbol{A}) = \rho_1 + \rho_2,$$

$$c = q_1(T).\dot{q}_2(T) - \dot{q}_1(T).q_2(T) = \det(\boldsymbol{A}) = \rho_1\rho_2 \tag{3.39}$$

Since $q_1(t)$ and $q_2(t)$ are solutions of our basic Mathieu equation (3.25), it is easy to show that c=1 as follows.

$$\ddot{q}_1(t) + \omega^2[1 - 2\mu\Phi(t)]q_1(t) = 0,$$

$$\ddot{q}_2(t) + \omega^2[1 - 2\mu\Phi(t)]q_2(t) = 0, \tag{3.40}$$

Multiplying the first equation by $q_2(t)$, multiplying the second equation by $q_1(t)$, and subtracting the two resulting equations yield

$$\ddot{q}_1(t)q_2(t) - q_2(t)\ddot{q}_2(t) = 0, \tag{3.41}$$

i.e.,

$$\frac{d}{dx}[\dot{q}_1(t)q_2(t) - q_2(t)\dot{q}_2(t)] = 0, \tag{3.42}$$

Which leads to

$$c = [\dot{q}_1(t)q_2(t) - q_2(t)\dot{q}_2(t)] = constant$$

$$= [\dot{q}_1(0)q_2(0) - q_2(0)\dot{q}_2(0)] = 1.1 - 0.0 = 1, for\ all\ time\ t$$

Hence the characteristic equation becomes

$$\rho^2 - 2b\rho + 1 = 0, \tag{3.43}$$

And the characteristic roots are given by $\rho_{1,2} = b \pm \sqrt{b^2 - 1}$. Depending on the values of b, there are three possibilities for ρ_1 and ρ_2.

Case I. $|b| > 1$, Distinct Real Roots

If $b^2 > 1$or $|b| > 1$ then ρ_1 and ρ_2 are real and have the same sign. Furthermore, if $|\rho_1| < 1$, then $|\rho_2| > 1$ and vice versa.

Case II |b| = 1, Equal real Root

If $|b| = 1$, $\rho_1 = \rho_2 = \pm 1$.

Case III. |b| < 1, Complex Roots

If $|b| < 1$, ρ_1 and ρ_2 are a pair of complex conjugate numbers, i.e., $\rho_{1,2} = \alpha e^{\pm i\theta}$. Since $\rho_1\rho_2 = 1$, one must have $\alpha^2 = 1\ or\ \alpha = 1$. Hence $\rho_{1,2} = \alpha e^{\pm i\theta}$, i.e., ρ_1 and ρ_2 lie on the unit circle $|\rho_{1,2}| = 1$.

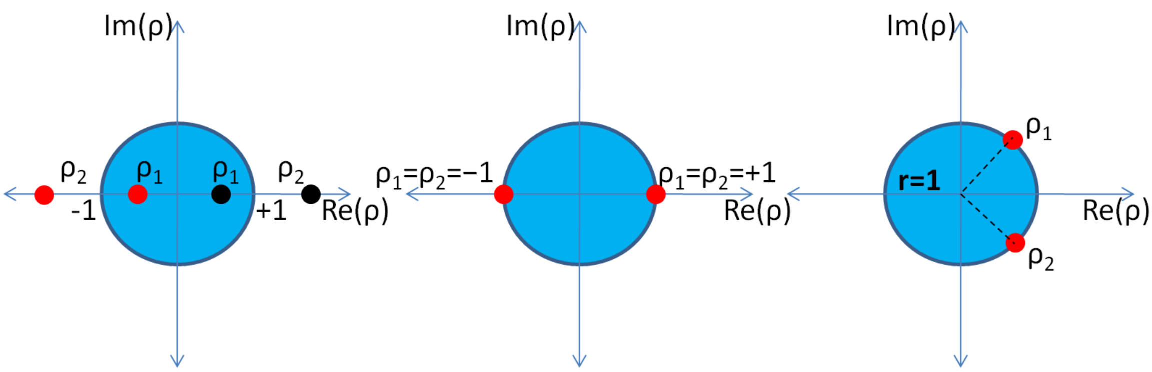

Figure 2

$$\boldsymbol{q}_1^*(t+T) = \rho_1 \boldsymbol{q}^*(t)$$

$$\boldsymbol{q}_2^*(t+T) = \rho_2 \boldsymbol{q}^*(t) \qquad (3.44)$$

The above equations (3.44) (derived previously) are functional equations. An equation of the form f(x,y,...)=0 is called a functional equation if f contains a finite number of independent variables, known functions, and unknown functions for which they must be solved. From the theory of functional equations, it can be shown that the general solutions of above equations (3.44) are of the form

$$q_k^*(t) = \Psi_k(t) e^{\lambda_k t}, \ k = 1,2, \qquad (3.45)$$

Where $\lambda_k = (1/T)\log \rho_k$, k = 1,2, are called the characteristic exponents.

Substituting these equations in the functional equations yields

$$\Psi_k(t+T) e^{\lambda_k (t+T)} = \rho \Psi_k(t) e^{\lambda_k t} \qquad (3.46)$$

Since $\exp(\lambda_k T) = \exp(\log\rho_k) = \rho_k$, one has $\Psi_k(t+T) = \Psi_k(t)$, implying that $\Psi_k(t)$ is a periodic function of period T.

Expressing the characteristic exponents ρ_k in the polar form of the complex numbers as

$$\rho_k = |\rho_k|[\cos(arg\rho_k) + i\sin(arg\rho_k)] = |\rho_k| e^{i \arg \rho_k}, \qquad (3.47)$$

Where

$$|\rho_k| = \sqrt{[Re(\rho_k)]^2 + [Im(\rho_k)]^2}, \ arg\rho_k = \tan^{-1}\frac{Im(\rho_k)}{Re(\rho_k)} \qquad (3.48)$$

Hence

$$\lambda_k = \frac{1}{T} log\ \rho_k = \frac{1}{T}(\log|\rho_k| + i \arg \rho_k), \qquad (3.49)$$

And hence

$$q_k^*(t) = \Psi_k(t) \exp\left(\frac{t}{T}\log|\rho_k|\right).\exp\left(i\frac{t}{T} i \arg \rho_k\right) \qquad (3.50)$$

From the above equation it can be seen that the stability of solution $q_k^*(t)$ is determined by the sign of $\log|\rho_k|$. Hence, depending on the values of b, there are three stability regions for the solutions given by above equation (3.50).

Case I. $|b| > 1$ i.e., $|\rho_1| < 1$ and $|\rho_2| > 1$ or $|\rho_1| > 1$ and $|\rho_2| < 1$

When $\rho_k > 1$, $\log|\rho_k| > 0$. One of the solutions $q_k^* \to \infty$ when $t \to \infty$, leading to instability of the solution of Mathieu equation.

Case II. |b| = 1 i.e., $\rho_1 = \rho_2 = \pm 1$

If $\rho_1 = \rho_2 = 1$, arg $\rho_k = 0$ for k = 1,2, and the equation become

$$q_k^*(t) = \Psi_k(t), \tag{3.51}$$

i.e. periodic solutions of period T.

If $\rho_1 = \rho_2 = -1$, arg $\rho_k = \pi$ for k = 1,2 and one obtains.

$$q_k^*(0) = \Psi_k(0)$$

$$q_k^*(T) = \Psi_k(T)e^{i\pi} = \Psi_k(T) = -\Psi_k(0)$$

$$q_k^*(2T) = \Psi_k(2T)e^{i2\pi} = \Psi_k(2T) = \Psi_k(0)$$

i.e., $q_k^*(t)$ are periodic solutions of period 2T.

Case III. |b| < 1 i.e., ρ_1 and ρ_2 complex, $|\rho_{1,2}| = 1$

The equation (3.50) become

$$q_k^*(t) = \Psi_k(t).\exp\left(i\frac{t}{T}i \arg\ \rho_k\right),\ k = 1,2 \tag{3.52}$$

Which represent bounded, almost-periodic solutions.

Therefore, the stability boundaries are given by |b| = 1, i.e. $|q_1(T) + \dot{q}_1(T)| = 2$.

3.2.3 Periodic Solutions of Period 2T:

Periodic solution of period $2T$, $T = \dfrac{2\pi}{\nu}$ can be expressed as a fourier series of the form

$$q(t) = \sum_{k\,odd}\left(a_k \sin\frac{k\pi t}{T} + b_k \cos\frac{k\pi t}{T}\right) = \sum_{k\,odd}\left(a_k \sin\frac{k\nu t}{2} + b_k \cos\frac{k\nu t}{2}\right) \tag{3.53}$$

Substituting equation (3.53) into equation (3.52) and expanding the resulting equation results in

$$\left[(1+\mu-r^2)a_1 - \mu a_3\right]\sin\frac{\nu t}{2} + \left[(1-9r^2)a_3 - \mu(a_1+a_5)\right]\sin\frac{3\nu t}{2} + \ldots$$
$$+\left[(1-\mu-r^2)b_1 - \mu b_3\right]\cos\frac{\nu t}{2} + \left[(1-9r^2)b_3 - \mu(b_1+b_5)\right]\cos\frac{3\nu t}{2} + \ldots = 0 \tag{3.54}$$

Where $r = \frac{\nu}{2\omega}$. Since

$$sin\frac{\nu t}{2}, sin\frac{3\nu t}{2}, sin\frac{5\nu t}{2},..., cos\frac{\nu t}{2}, cos\frac{3\nu t}{2}, cos\frac{5\nu t}{2},...$$

are linearly independent, it is required that the coefficient of each sine and cosine term be zero. Hence, setting the coefficient of $sin\frac{k\nu t}{2}$ to zero results in equations for the coefficients

$$\begin{aligned} k = 1: &\quad (1+\mu-r^2)a_1 - \mu a_3 = 0, \\ k = 3,5,...: &\quad (1-k^2r^2)a_k - \mu(a_{k-2}+a_{k+2}) = 0, \end{aligned} \tag{3.55}$$

And equating coefficient of $sin\frac{k\nu t}{2}$ to zero results in equations for the coefficients

$$\begin{aligned} k = 1: &\quad (1-\mu-r^2)b_1 - \mu b_3 = 0, \\ k = 3,5,...: &\quad (1-k^2r^2)b_k - \mu(b_{k-2}+b_{k+2}) = 0, \end{aligned} \tag{3.56}$$

Equation (3.55) and (3.56) are systems of homogeneous linear algebraic equations for the coefficients $a_1, a_3, a_5, ..., and\ b_1, b_3, b_5, ...,$ respectively. The conditions for equations (3.55) and (3.66) to have non-trivial solutions are that the determinants of the coefficient matrices, which are tridiagonal of infinite dimensions, are zero, i.e

$$\begin{vmatrix} 1\pm\mu-\mathbf{r}^2 & -\mu & 0 & ... & ... \\ -\mu & 1-9\mathbf{r}^2 & -\mu & 0 & ... \\ 0 & -\mu & 1-25\mathbf{r}^2 & -\mu & ... \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix} = 0 \tag{3.57}$$

As $\mu \to 0$, the coefficient matrices become diagonal and equations (3.57) reduce to

$$\left|\mathbf{diag}\{1-\mathbf{r}^2, 1-9\mathbf{r}^2, 1-25\mathbf{r}^2, \ldots\}\right| = 0,$$

Which yields

$$1-(2k-1)^2 r^2 = 0, \quad k = 1,2,3,\ldots,$$

Namely, the stability boundaries when $\mu \to 0$ are given by

$$r = \frac{\nu}{2\omega} = \frac{1}{2k-1}.$$

3.2.4 Periodic Solutions of Period T:

Similarly, periodic solutions of period T can be expressed as a Fourier series of the form

$$q(t) = b_0 + \sum_{k\,even}\left(a_k \sin\frac{k\pi t}{T} + b_k \cos\frac{k\pi t}{T}\right) = \sum_{k\,even}\left(a_k \sin\frac{k\nu t}{2} + b_k \cos\frac{k\nu t}{2}\right) \tag{3.58}$$

From the same procedure as for periodic solutions of period 2T, substitution equation (3.58) into (3.52) and equating the coefficients of sin(kν t/2) to zero yield

$$\begin{aligned} k=2: &\quad (1-4r^2)a_2 - \mu a_4 = 0, \\ k=4,6,\ldots: &\quad (1-k^2r^2)a_k - \mu(a_{k-2} + a_{k+2}) = 0, \end{aligned} \tag{3.59}$$

And setting the coefficients of cos(kν t/2) to zero gives

$$\begin{aligned} k=0: &\quad b_0 - \mu b_2 = 0, \\ k=2: &\quad (1-4r^2)b_2 - \mu(2b_0 + b_4) = 0, \\ k=4,6,\ldots: &\quad (1-k^2r^2)b_k - \mu(b_{k-2} + b_{k+2}) = 0. \end{aligned} \tag{3.60}$$

For non-trivial solutions of a_k and b_k, the determinants of the tridiagonal coefficient matrices must be zero, i.e.

$$\begin{vmatrix} 1-4r^2 & -\mu & 0 & \dots & \dots \\ -\mu & 1-16r^2 & -\mu & 0 & \dots \\ 0 & -\mu & 1-36r^2 & -\mu & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix} = 0, \qquad (3.61)$$

$$\begin{vmatrix} 1 & -\mu & 0 & \dots & \dots & \dots \\ -2\mu & 1-4r^2 & -\mu & 0 & \dots & \dots \\ 0 & -\mu & 1-16r^2 & -\mu & 0 & \dots \\ 0 & 0 & -\mu & 1-36r^2 & -\mu & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{vmatrix} \qquad (3.62)$$

Again, as $\mu \to 0$, the stability boundaries are reduced to points given by $r = 1/(2k)$, $k = 1,2,\dots$, which are shown.

It is obvious that the first, the third, the fifth, .., stability boundaries are given by periodic solutions of period 2T, and the second, the fourth, the sixth,…, stability boundaries are obtained from periodic solutions of period T.

In general, it is impossible to solve the determinantal equations (3.57), (3.61), and (3.62) of infinite dimension to determine the stability boundaries. Various orders of approximations can be applied to obtain approximate results.

3.3 Instability Boundaries

3.3.1 Boundaries of First Instability Region:

The first stability boundaries are given by periodic solutions of period 2T.

First-Order Approximation

First element (1,1) of the matrix in equation (3.57), the first-order approximation of the first stability boundaries is given by $1 \pm \mu - r^2 = 0$, which leads to

$$\frac{\nu}{2\omega} = r = \sqrt{1 \pm \mu} \approx 1 \pm \frac{1}{2}\mu. \tag{3.63}$$

The corresponding periodic solution of period 2T is

$$q(t) = a_1 \sin\frac{\nu t}{2} + b_1 \cos\frac{\nu t}{2} = a_1 \sin\frac{\pi t}{T} + b_1 \cos\frac{\pi t}{T}, \tag{3.64}$$

Where T = $2\pi/\nu$. The stability boundaries (3.63) shown in figure could also have been obtained by substituting this solution directly into the Mathieu equation and equation (3.53) coefficients of various harmonics to zero.

It should be emphasized that by setting the determinant (3.57) to zero, the conditions for non-trivial periodic solutions of period 2T are obtained, which give the stability boundaries. However, this analysis does not give the stability behaviour, i.e. it does not provide the information on whether the region inside the V shaped region is in figure is determined by some other methods, such as the method of averaging. This remark also applies to the stability boundaries of the second, third, …. Stability regions obtained in this section.

Second Order Approximation:

The second order approximation of the first stability bondaries is obtained from the first 2 X 2 submatrix, i.e.

$$\begin{vmatrix} 1 \pm \mu - \mathbf{r}^2 & -\mu \\ -\mu & 1 - 9\mathbf{r}^2 \end{vmatrix} = 0, \tag{3.65}$$

Which gives ($(1 \pm \mu - \mathbf{r}^2)(1 - 9\mathbf{r}^2) - \mu^2 = 0$. Solving for the r in $(1 \pm \mu - \mathbf{r}^2)$ yields

$$\frac{\nu}{2\omega} = r = \sqrt{1 \pm \mu - \frac{\mu^2}{1 = 9r^2}}. \tag{3.66}$$

Substituting from the first-order approximation (3.63) yields

$$\frac{\nu}{2\omega} \approx \sqrt{1 \pm \mu - \frac{\mu^2}{1 = 9(1 \pm \mu)}}, \tag{3.67}$$

i.e.

$$\frac{\nu}{2\omega} \approx 1 \pm \frac{1}{2}\mu - \frac{1}{16}\mu^2 ., \tag{3.68}$$

In the second-order approximation, the stability boundaries are not symmetric, as shown in Figure. The width of the stability region is

$$\Delta(1) = \left| \left(1 + \frac{1}{2}\mu - \frac{1}{16}\mu^2 \right) - \left(1 - \frac{1}{2}\mu - \frac{1}{16}\mu^2 \right) \right| = \mu. \tag{3.69}$$

3.3.2 Boundaries of Second Instability Region:

The second stability boundaries are given by periodic solutions of period T.

First-Order Approximation

The first 2x2 sub-matrix in equation (3.57) gives

$$\begin{vmatrix} 1 - 4r^2 & -\mu \\ -\mu & 1 - 16r^2 \end{vmatrix} = 0. \tag{3.70}$$

Solving for the r in ($1-4r^2$)yields

$$\frac{\nu}{2\omega}=r=\frac{1}{2}\left(1-\frac{\mu^2}{1-16r^2}\right)^{1/2}. \tag{3.71}$$

Substituting in the stability boundary $r=\frac{1}{2}$ obtained for $\mu\rightarrow 0$ results in

$$\begin{aligned}\frac{\nu}{2\omega} &\approx \frac{1}{2}\left[1-\frac{\mu^2}{1-16\left(\frac{1}{2}\right)^2}\right]^{1/2}=\frac{1}{2}\left(1+\frac{1}{3}\mu^2\right)^{1/2}.\\ &\approx \frac{1}{2}+\frac{1}{12}\mu^2, \text{ for small } \mu.\end{aligned} \tag{3.72}$$

Similarly, the first 2x2 submatrix of equation (3.62) yields

$$\begin{vmatrix} 1 & -\mu \\ -2\mu & 1-4r^2 \end{vmatrix}=0, \tag{3.73}$$

And solving for the r in ($1-4r^2$) leads to

$$\frac{\nu}{2\omega}=r=\frac{1}{2}\sqrt{1-2\mu^2}\approx\frac{1}{2}-\frac{1}{2}\mu^2. \tag{3.74}$$

The corresponding periodic solution of periodic T, $T=2\pi/\nu$, is

$$q(t)=b_0+a_2\sin\frac{2\pi t}{T}+b_2\cos\frac{2\pi t}{T}.$$

Second Order approximation

The first 3x3 submatrix o f equation (3.61) gives

$$\begin{vmatrix} 1-4r^2 & -\mu & 0 \\ -\mu & 1-16r^2 & -\mu \\ 0 & -\mu & 1-36r^2 \end{vmatrix}=0, \tag{3.75}$$

Which leads to

$$(1-4r^2)\left[(1-16r^2)(1-36r^2)-\mu^2\right]-(-\mu)(-\mu)(1-36r^2)=0. \tag{3.76}$$

Solving for the r in (1-4r^2) yields

$$r=\frac{1}{2}\left[1-\frac{\mu^2(1-36r^2)}{(1-16r^2)(1-36r^2)-\mu^2}\right]^{1/2}. \tag{3.77}$$

Substituting in the first-order approximation (3.74) results in

$$\frac{\nu}{2\omega}=r\approx\frac{1}{2}\left\{1-\frac{\mu^2\left[1-36\left(\frac{1}{2}+\frac{1}{12}\mu^2\right)^2\right]}{\left[1-16\left(\frac{1}{2}+\frac{1}{12}\mu^2\right)^2\right]\left[1-36\left(\frac{1}{2}+\frac{1}{12}\mu^2\right)^2\right]-\mu^2}\right\}^{1/2}$$

$$\approx\frac{1}{2}+\frac{1}{12}\mu^2-\frac{35}{864}\mu^4. \tag{3.78}$$

The first 3x3 sub-matrix of equation (3.62) yields

$$\begin{vmatrix}1 & -\mu & 0\\ -2\mu & 1-4r^2 & -\mu\\ 0 & -\mu & 1-16r^2\end{vmatrix}=0,$$

i.e.

$$1.\left[\left(1-4r^2\right)\left(1-16r^2\right)-\mu^2\right]-\left[(-\mu)(-2\mu)\left(1-16r^2\right)\right]=0.$$

Solving for the r in (1-4r^2) results in

$$\frac{\nu}{2\omega}=r=\frac{1}{2}\left\{1-\frac{\mu^2\left[1+2\left(1-16r^2\right)\right]}{1-16r}\right\}^{1/2}.$$

Substituting in the first order approximation (3.74) leads to

$$\frac{\nu}{2\omega} \approx \frac{1}{2}\left\{1-\frac{\mu^2\left[1+2\left(1-16\left(\frac{1}{2}-\frac{1}{2}\mu^2\right)^2\right)\right]}{1-16\left(\frac{1}{2}-\frac{1}{2}\mu^2\right)^2}\right\}^{1/2}. \tag{3.79}$$

Or, after expanding in series of μ,

$$\frac{\nu}{2\omega} = \frac{1}{2} - \frac{5}{12}\mu^2 + \frac{7}{144}\mu^4. \tag{3.80}$$

3.3.3 Boundaries of Third Instability Region:

For the third stability boundaries, consider periodic solutions of period 2T.

First Order Approximation

The first 2x2 submatrix in equation (3.57) gives

$$\begin{vmatrix} 1\pm\mu-r^2 & -\mu \\ -\mu & 1-9r^2 \end{vmatrix} = 0, \tag{3.81}$$

Which results in $\left(1\pm\mu-r^2\right)\left(1-9r^2\right)-\mu^2=0.$ solving for the r in $\left(1-9r^2\right)$ leads to

$$\frac{\nu}{2\omega} = r = \frac{1}{3}\left(1-\frac{\mu^2}{1\pm\mu-\left(\frac{1}{3}\right)^2}\right)^{1/2}$$
$$\approx \frac{1}{3} - \frac{3}{16}\mu^2 \pm \frac{27}{128}\mu^3. \tag{3.82}$$

Second Order Approximation

The first 3x3 submatrix of equation (3.57) yields

$$\begin{vmatrix} 1 \pm \mu - \mathbf{r}^2 & -\mu & 0 \\ -\mu & 1-9\mathbf{r}^2 & -\mu \\ 0 & -\mu & 1-25\mathbf{r}^2 \end{vmatrix} = 0, \tag{3.83}$$

Which leads to

$$\left(1 \pm \mu - r^2\right)\left[\left(1-9r^2\right)\left(1-25r^2\right) - \mu^2\right] - (-\mu)(-\mu)\left(1-25r^2\right) = 0. \tag{3.84}$$

Solving for the r in $\left(1-9r^2\right)$ results in

$$r = \frac{1}{3}\left[1 - \mu^2\left(1 + \frac{1-25r^2}{1 \pm \mu - r^2}\right)\left(1-25r^2\right)^{-1}\right]^{1/2}. \tag{3.85}$$

Substituting in the first-order approximation for r given by equation (3.82) results in

$$\frac{\nu}{2\omega} = r \approx \frac{1}{3} - \frac{3}{32}\mu^2 \pm \frac{27}{128}\mu^3 - \frac{243}{4096}\mu^4. \tag{3.86}$$

The stability boundaries of the third stability region are shown in above figure. The width of the stability region is

$$\Delta\left(\frac{1}{3}\right) = \left|\left(\frac{1}{3} - \frac{3}{32}\mu^2 + \frac{27}{128}\mu^3 - \frac{243}{4096}\mu^4\right) - \left(\frac{1}{3} - \frac{3}{32}\mu^2 - \frac{27}{128}\mu^3 - \frac{243}{4096}\mu^4\right)\right| = \frac{27}{64}\mu^3. \tag{3.88}$$

3.4 Stability of the Damped Mathieu Equations

Consider the damped Mathieu equations of the form

$$\ddot{q} + 2\beta\dot{q} + \omega^2(1 - 2\mu\cos\nu t)q = 0. \tag{3.89}$$

To study the first, the third, the fifth ..., stability regions, periodic solutions of period 2T are considered, whereas to determine the second, the fourth, the sixth, .., , stability boundaries, periodic solutions of periodic solutions of period T must be investigated.

3.4.1 Periodic Solutions of Period 2T

Substituting the periodic solution of period 2T given by equation (3.53) into the damped Mathieu equation (3.89) and setting the coefficients of $\sin(\frac{k\nu t}{2})$ and $\cos(\frac{k\nu t}{2})$ to zero lead to

$$k = 1;\ \begin{cases}(1+\mu-r^2)a_1 - \mu a_3 - (2\beta r/\omega)b_1 = 0,\\ (1-\mu-r^2)b_1 - \mu b_3 + (2\beta r/\omega)a_1 = 0,\end{cases}$$

$$k = 3,5,\dots;\ \begin{cases}(1-k^2r^2)a_k - \mu(a_{k-2}+a_{k+2}) - (2k\beta r/\omega)b_k = 0,\\ (1-k^2r^2)b_k - \mu(b_{k-2}+b_{k+2}) - (2k\beta r/\omega)a_k = 0,\end{cases} \tag{3.90}$$

Where $r = \nu/2\omega$. This is a system of homogeneous linear algebraic equations for the coefficients $a_1, a_3 \dots, b_1, b_3 \dots$

In order to have non-trivial solutions, the determinant of the coefficient matrix of vector $\{\dots, a_3, a_1, b_1, b_3 \dots\}^T$ is set to zero, i.e.

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1-9r^2 & -\mu & 0 & -\frac{6\beta r}{\omega} & \cdots \\ \cdots & -\mu & 1+\mu-r^2 & -\frac{2\beta r}{\omega} & 0 & \cdots \\ \cdots & 0 & \frac{2\beta r}{\omega} & 1-\mu-r^2 & -\mu & \cdots \\ \cdots & \frac{6\beta r}{\omega} & 0 & -\mu & 1-9r^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{3.91}$$

As $\mu \to 0$, equation (3.91) becomes $\Delta_{2T} = \Delta_1\Delta_3 \dots .\Delta_k \dots .$, where

$$\Delta_k = \begin{vmatrix} 1-k^2r^2 & -\frac{2k\beta r}{\omega} \\ \frac{2k\beta r}{\omega} & 1-k^2r^2 \end{vmatrix} = (1-k^2r^2)^2 + \left(\frac{2k\beta r}{\omega}\right)^2 > 0, \tag{3.92}$$

For k odd, which implies that the stability boundaries do not intersect the $r = \nu/2\omega$ axis.

Boundaries of the First Instability region

For the first-order approximation, the 2x2 submatrix in equation (3.91) that corresponds to the coefficients $\{a_1,b_1\}^T$ yields

$$\begin{vmatrix} \mathbf{1}+\boldsymbol{\mu}-\boldsymbol{r}^{\mathbf{2}} & -2\beta r/\omega \\ 2\beta r/\omega & \mathbf{1}-\boldsymbol{\mu}-\boldsymbol{r}^{\mathbf{2}} \end{vmatrix} = 0. \qquad (3.93)$$

Which leads to $[(\mathbf{1}-\boldsymbol{r}^{\mathbf{2}})-\boldsymbol{\mu}^{\mathbf{2}}]+(2\beta r/\omega)^2=0$. Solving for the r in the bolded yields, for small μ and β/ω,

$$\frac{\nu}{2\omega}=r=\sqrt{1\pm\sqrt{\mu^2-\left(\frac{2\beta r}{\omega}\right)^2}}. \qquad (3.94)$$

Substituting in the stability boundary $r=1$ obtained for $\mu\rightarrow 0$ leads to

$$\frac{\nu}{2\omega}\approx\sqrt{1\pm\sqrt{\mu^2-\left(\frac{2\beta}{\omega}\right)^2}}. \qquad (3.95)$$

The minimal value of the amplitude of forcing for instability is given by $\mu_{cr}=2\beta/\omega$ The stability boundaries given by equation (3.95) for the damped Mathieu equation are shown in figure in the shaded regions as compared with those of the undamped Mathieu equation.

3.4.2 Periodic Solutions of Period T

Substituting the periodic solution of period T given by equation (3.58) into (3.89) and equating the coefficients of various harmonics $\sin\frac{kvt}{2}$ and $\cos\frac{kvt}{2}$ to zero lead to

$$b_0-\mu b_2=0,$$

$$(4\beta r/\omega)a_2 - 2\mu b_0 + (1-4r^2)b_2 - \mu b_4 = 0,$$

$$-\mu a_{k+2} + (1-k^2r^2)a_k - \mu a_{k-2} - (2k\beta r/\omega)b_k = 0,$$

$$(2k\beta r/\omega)a_k - \mu b_{k-2} + (1-k^2r^2)b_k - \mu b_{k+2} = 0,$$

Where $r = \nu/2\omega$ and $a_0 = 0$.

This is a system of homogeneous linear algebraic equations for the coefficients $a_{2,}\ a_{4,}\ \dots\dots, b_{0,}b_{2,}b_{4,}$ .... For non-trivial solutions, setting the determinant of the coefficient matrix of the vector $\{\dots.\,a_{2,}\ a_{4,}\ \dots\dots, b_{0,}b_{2,}b_{4,}\ \dots.\}^T$ to zero yields

$$\Delta_T = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1-16r^2 & -\mu & 0 & 0 & -8\beta r/\omega & 0 & \cdots \\ \cdots & -\mu & 1-4r^2 & 0 & -4\beta r/\omega & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -\mu & 0 & 0 & \cdots \\ \cdots & 0 & 4\beta r/\omega & -2\mu & 1-4r^2 & -\mu & 0 & \cdots \\ \cdots & 8\beta r/\omega & 0 & 0 & -\mu & 1-16r^2 & -\mu & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad (3.96)$$

Boundaries of the second stability region

For the first-order approximation, the 3x3 submatrix corresponding to the coefficients $\{a_2, b_0, b_2\}^T$ in equation (3.96) yields

$$\begin{vmatrix} \mathbf{1-4r^2} & 0 & -4\beta r/\omega \\ 0 & \mathbf{1} & -\mu \\ 4\beta r/\omega & -2\mu & \mathbf{1-4r^2} \end{vmatrix} = 0, \qquad (3.97)$$

Which leads to, after expanding the determinant,

$$(1-4r^2)^2 - 2\mu^2(1-4r^2) + \left(\frac{4\beta r}{\omega}\right)^2 = 0. \qquad (3.98)$$

This may be regarded as a quadratic equation in $(1-4r^2)$; solving for r in it yields

$$\frac{\nu}{2\omega} = r = \frac{1}{2}\sqrt{1-\mu^2 \pm \sqrt{\mu^4 - \left(\frac{4\beta r}{\omega}\right)^2}} \qquad (3.99)$$

$$\approx \frac{1}{2}\sqrt{1-\mu^2 \pm \sqrt{\mu^4-\left(\frac{2\beta}{\omega}\right)^2}}$$

In which the stability boundary $r = \frac{1}{2}$ for $\mu \to 0$ is substituted. The stability boundaries are shown.

From equation (3.99), the minimal value of the amplitude of forcing for stability $\mu_{k,cr} = (2\beta/\omega)^{1/2}$.

3.5 Stability of the Undamped Mathieu Equations

Consider the undamped Mathieu equation

$$\bar{q}(t) + \omega^2(1 - 2\varepsilon\mu cos\nu t)q(t) = 0, \qquad (3.100)$$

For small amplitude of parametric excitation with $0 < \varepsilon \ll 1$ being a small parameter. Apply the time scaling $\tau = \nu t$ and denote differentiation with respect to τ using a prime. Above equation (3.100) becomes

$$q''(\tau) + \frac{\omega^2}{\nu^2}(1 - 2\varepsilon\mu\ cos\tau)q(\tau) = 0. \qquad (3.101)$$

Let the parametric excitation frequency ν vary around a reference frequency ω_0i.e. $\nu = \omega_0(1 - \varepsilon\Delta)$, wehre Δ is detuning parameter. Above equation (3.101) can be written as

$$q''(\tau) + \frac{\omega^2}{\omega_0^2(1-\varepsilon\Delta)^2}(1 - 2\varepsilon\mu\ cos\tau)q(\tau) = 0. \qquad (3.102)$$

Denoting $k = \omega/\omega_0$ and because $(1 - \varepsilon\Delta)^{-2} \approx 1 + 2\varepsilon\Delta$, one obtains, after dropping terms of higher order.

$$q''(\tau) + k^2q(\tau) = -2\varepsilon k^2(\Delta - \mu\ cos\tau)q(\tau). \qquad (3.103)$$

The complementary solution of equation (3.103) when setting the right side to zero is given by

$$q_C(\tau) = a\ cos\Phi(\tau),\ q'_C(\tau) = -ak\ sin\Phi(\tau),\ \Phi(\tau) = k\tau + \varphi, \qquad (3.104)$$

Where a and φ are constants. The solution $q(\tau)$ is a sinusoidal function with frequency k or period $2\pi/k$.

Equation (102) can be solved by applying the method of variation of parameters. In the complementary solution (3.104), a and φ are constants or parameters. Varying the parameters, a and φ in equation (3.104) and making them functions of τ instead of constants, leads to

$$q(\tau) = a(\tau)\cos\Phi(\tau),\ \Phi(\tau) = k\tau + \varphi(\tau)$$

$$q'(\tau) = -a(\tau)k\sin\Phi(\tau). \qquad (3.105)$$

Differentiating equation $q(\tau) = a(\tau)\cos\Phi(\tau)$ (3.105) yields

$$q^{'}(\tau) = a^{'}(\tau)\cos\Phi(\tau) - a(\tau)[k + \varphi^{'}(\tau)]\sin\Phi(\tau). \tag{3.106}$$

Comparing with $q^{'}(\tau)$given by $q^{'}(\tau) = -a(\tau)k\sin\Phi(\tau)$ (3.105) one must have

$$a^{'}(\tau)\cos\Phi(\tau) - a(\tau)\varphi^{'}(\tau)\sin\Phi(\tau) = 0. \tag{3.107}$$

Differentiating equation $q^{'}(\tau) = -a(\tau)k\sin\Phi(\tau)$ (3.105) leads to

$$q^{''}(\tau) = -a^{'}(\tau)k\sin\Phi(\tau) - a(\tau)k^2\cos\Phi(\tau) - a(\tau)k\varphi^{'}(\tau)\cos\Phi(\tau) \tag{3.109}$$

and substituting it along with equation (3.105) into (3.103) yields

$$a^{'}(\tau)\sin\Phi(\tau) + a(\tau)\varphi^{'}(\tau)\cos\Phi(\tau) = 2\varepsilon k(\Delta - \mu\, cos\tau)a(\tau)\cos\Phi(\tau) \tag{3.110}$$

Solving for $a^{'}(t)$ and $\varphi'(t)$ from equations (3.107) and (3.109) results in

$$a^{'}(\tau) = 2\varepsilon k a(\tau)(\Delta - \mu\, cos\tau)\sin\Phi(\tau)\cos\Phi(\tau),$$

$$\varphi^{'}(\tau) = 2\varepsilon k(\Delta - \mu\, cos\tau)cos^2\Phi(\tau) \tag{3.111}$$

From eqns (3.111), it is seen that $a^{'}(t)$ and $\varphi'(t)$are small because of the small parameter ε, which imples that $a(t)$ and $\varphi(t)$ change slowly with τ. In one period $\Delta a/a(\tau) \ll 1$ and $\Delta\varphi$ ≪1. The amplitude function $a(\tau)$changes slowly with τ while $q(\tau)$ varies fast with a frequency approximately equal to k as given by eqn (3.105).

As an approximation, the right sides of equations (3.111) can be replaced by their averaged values in one period, i.e. $a(t)$ and $\varphi(t)$ replaced by $\bar{a}(t)$ and $\bar{\varphi}(t)$, where the over-bar stands for the averaged value. The averaging operator is defined by

$$\mathcal{A}(\,) = \lim_{T\to\infty}\frac{1}{T}\int_{\tau}^{\tau+T}(\,)d\tau, \tag{3.112}$$

Where integration is performed over the explicitly appearing time parameter τ in the integrand. This approach is called method of averaging.

It is easy to evaluate the operation of the averaging operator $\mathcal{A}$ on terms of the form $\cos(k\tau + \alpha\bar{\varphi})$ and $\sin(k\tau + \alpha\bar{\varphi})$ for $k \neq 0$

$$\mathcal{A}\begin{Bmatrix} cos \\ sin \end{Bmatrix}(k\tau + \alpha\bar{\varphi}) = \lim_{T\to\infty}\frac{1}{T}\int_{\tau}^{\tau+T}\begin{Bmatrix} cos \\ sin \end{Bmatrix}(k\tau + \alpha\bar{\varphi})d\tau,$$

$$= \lim_{T\to\infty}\frac{1}{kT}\begin{Bmatrix} +sin \\ -cos \end{Bmatrix}(k\tau + \alpha\bar{\varphi})\Big|_{\tau=\tau}^{\tau+T} = 0 \tag{3.113}$$

And for k = 0

$$\mathcal{A}\begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi}) = \lim_{T\to\infty}\frac{1}{T}\int_{\tau}^{\tau+T}\begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi})d\tau,$$

$$= \begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi})\lim_{T\to\infty}\frac{1}{T}\tau\Big|_{\tau=\tau}^{\tau+T} = \begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi}) \tag{3.114}$$

Hence, when the method of averaging is applied, sine and cosine terms of the form $\cos(k\tau + \alpha\bar{\varphi})$ and $\sin(k\tau + \alpha\bar{\varphi})$ vanish in the averaged equations if τ is equal to 0.

Applying the method of averaging (discussed in Appendix A) and combining the trigonometric terms in equations (3.111) yield

$$\bar{a}' = \varepsilon k\bar{a}\mathcal{A}\left\{\Delta\ \boldsymbol{\sin(2k\tau + 2\bar{\varphi})}\ - \frac{\mu}{2}\{\boldsymbol{sin[(2k+1)\tau + 2\bar{\varphi}]} + sin[(2k-1)\tau + 2\bar{\varphi}]\}\right\},$$

$$\bar{\varphi}' =\ \varepsilon k\mathcal{A}\begin{Bmatrix} \Delta[1 + \boldsymbol{\sin(2k\tau + 2\bar{\varphi})}] - \\ \frac{\mu}{2}\{2\boldsymbol{\cos\tau} + \boldsymbol{\cos[(2k+1)\tau + 2\bar{\varphi}]} + \cos(2k-1)\tau + 2\bar{\varphi}\} \end{Bmatrix}, \tag{3.115}$$

In which the sine and cosine terms in the bolded vanish when averaged. The values of other sinusoidal terms, when averaged, depend on the values of k.

It should be emphasized that $\bar{a}$ and $\bar{\varphi}$ are treated as constants when performing the averaging.

Hence, one obtains

$$\bar{a}' = -\frac{1}{2}\varepsilon k\mu\bar{a}\mathcal{A}\{\sin[(2k-1)\tau + 2\varphi]\},$$

$$\bar{\varphi}' = \varepsilon k\Delta - \frac{1}{2}\varepsilon k\mu\mathcal{A}\{\cos[(2k-1)\tau + 2\bar{\varphi}]\} \tag{3.116}$$

When $k \neq 1/2$, i.e. $\omega_0 = 2\omega$

The averaged equation (3.116) become

$$\bar{a}' = 0, \qquad \bar{\varphi}' = \varepsilon k\Delta,$$

The solutions of which can be easily obtained as

$$\bar{a} = Constant = \bar{a}_0, \qquad \overline{\varphi}' = \varepsilon k \Delta \tau + \bar{\varphi}_0.$$

The solution of equation (3.103) is then given by, form equation (3.105)

$$q(\tau) = \bar{a}\cos\overline{\Phi} = \bar{a}_0 \cos[k(1+\varepsilon\Delta)\tau + \bar{\varphi}_0] \tag{3.117}$$

Since $k = \frac{\omega}{\omega_0} = \omega(1-\varepsilon\Delta)/\nu$, Hence $k(1+\varepsilon\Delta) = \omega(1+\varepsilon^2\Delta^2)/\nu \approx \omega/\nu$. Equation (3.117) becomes

$$q(\tau) = \bar{a}_0 \cos\left(\frac{\omega}{\nu}\tau + \bar{\varphi}_0\right),$$

$$q(t) = \bar{a}_0 \cos(\omega t + \bar{\varphi}_0), \tag{3.118}$$

Which is the same as the solution of the system (3.101) without forcing i.e,

$$\ddot{q}(t) = \omega^2 q(t) = 0.$$

Therefore, if $\omega_0 \neq 2\omega$, the parametric forcing function $2\mu\cos\nu t$ has no effect on the system in the first-order approximation

3.6 Finding Instability at Various points by Plotting Phase Portraits

Consider a simply supported column under axial load $P(t) = P_0 \cos \nu t$ as earlier. When only the fundamental mode is taken, the equation of motion is given by

$$\ddot{q} + 2\varepsilon\zeta\omega\dot{q} + \omega^2 q(1 - 2\varepsilon\mu cos\nu t + \varepsilon\gamma q^2) = 0 \qquad (3.119)$$

Where $0 < \varepsilon \ll 1$ is a small parameter

3.6.1 Steady State Solutions

Apply the time scaling $\tau = \nu t$ and $\nu = \omega_0(1 - \varepsilon\Delta)$, where ω_0 is a reference frequency and Δ is the detuning parameter. With the notation $k = \omega/\omega_0$, $\frac{d()}{d\tau} = ()'$, above equation (3.119) becomes

$$q'' + k^2 q = \varepsilon\left(-2\frac{\omega}{\nu}\zeta q' - 2\Delta k^2 q + 2\frac{\omega^2}{\nu^2}\mu q cos\tau - \gamma\frac{\omega^2}{\nu^2}q^3\right). \qquad (3.120)$$

Applying the transformation

$$q(\tau) = a(\tau)cos\Phi(\tau), \qquad q'(\tau) = -a(\tau)ksin\Phi(\tau), \qquad \Phi(\tau) = k\tau + \varphi(\tau)$$

Results in

$$a' = \varepsilon a\left\{-\frac{\zeta\omega}{\nu}(1 - \boldsymbol{cos2\Phi}) + k\Delta\boldsymbol{sin2\Phi} + \frac{\gamma\omega^2}{k\nu^2}a^2\left(\frac{1}{8}\boldsymbol{sin4\Phi} + \frac{1}{4}\boldsymbol{sin2\Phi}\right) - \frac{\mu\omega^2}{2k\nu^2}[sin(2\Phi - \tau) + \boldsymbol{sin(2\Phi + \tau)}]\right\},$$

$$\varphi' = \varepsilon\left\{\begin{matrix} -\frac{\zeta\omega}{\nu}\boldsymbol{sin2\Phi} + k\Delta(1 + \boldsymbol{cos2\Phi}) + \frac{\gamma\omega^2}{k\nu^2}a^2\left(\frac{1}{8}\boldsymbol{cos4\Phi} + \frac{1}{2}\boldsymbol{cos2\Phi} + \frac{3}{8}\right) \\ -\frac{\mu\omega^2}{k\nu^2}\left[\boldsymbol{cos\tau} + \frac{1}{2}cos(2\Phi - \tau) + \frac{1}{2}\boldsymbol{cos(2\Phi + \tau)}\right] \end{matrix}\right\} \qquad (3.121)$$

For a first-order approximation, apply the method of averaging (Appendix A) to approximate $a(\tau)$ and $\varphi(\tau)$ by the averaged values $\overline{a(\tau)}$ and $\bar{\varphi}(\tau)$, respectively, with the averaging as described above in equation (3.112).

Case 1 No Resonance $\boldsymbol{k \neq 1/2}$

For the case $k \neq 1/2$, i.e. $\omega_0 \neq 2\omega$ or ν is not in the vicinity of 2ω, all sinusoidal terms in above equations vanish when averaged and the averaged equations are

$$\bar{a}' = -\varepsilon \frac{\zeta\omega}{\nu} \bar{a}, \tag{3.122}$$

$$\bar{\varphi}' = \varepsilon k \left(\Delta + \frac{3\gamma\omega^2}{8k\nu^2} \bar{a}^2\right).$$

From above equation (3.112) one obtains

$$\bar{a}(\tau) = a_0 exp\left(-\varepsilon \frac{\zeta\omega}{\nu} \tau\right) = \exp(-\varepsilon\zeta\omega t), \tag{3.123}$$

Which approaches 0 as t→ ∞.

Case II Subharmonic Resonance $\boldsymbol{k = 1/2}$

For $k = 1/2$, i.e. $\omega_0 = 2\omega$ or ν is in the vicinity of 2ω, all sinusoidal terms in bold in above equations (3.121) vanish when averaged. Noting that

$$\frac{\omega^2}{k\nu^2} = \frac{\omega^2}{k{\omega_0}^2(1-\varepsilon\Delta)^2} \approx \frac{\omega^2}{k{\omega_0}^2} = \frac{1}{2} \tag{3.124}$$

One obtains the averaged equations as

$$a' = -\varepsilon \left(\frac{\zeta\omega}{\nu} + \frac{\mu}{4} sin2\varphi\right) \bar{a},$$

$$\varphi' = \varepsilon \frac{1}{2}\left(\Delta + \frac{3\gamma}{8} \bar{a}^2 - \frac{\mu}{2} cos2\bar{\varphi}\right). \tag{3.125}$$

The steady state solutions $\bar{a}_0$ and $\bar{\varphi}_0$ are given by $\bar{a}, = \bar{\varphi}' = 0$

If $\bar{a} = 0$ then $q(\tau) = 0$.

If $\bar{a} \neq 0$, then from equation (3.125), one obtains

$$sin2\bar{\varphi}_0 = \frac{4\zeta\omega}{\nu\mu}, \tag{3.126}$$

Which gives

$$cos2\bar{\varphi}_0 = \pm\left[1-\left(\frac{4\zeta\omega}{\nu\mu}\right)^2\right]^{1/2} \quad (3.127)$$

Equation (3.125) yields

$$\Delta = -\frac{3\gamma}{8}\overline{a_0}^2 + \frac{\mu}{2}cos2\overline{\varphi_0} = -\frac{3\gamma}{8}\overline{a_0}^2 \pm \left[\frac{\mu^2}{4} - \left(\frac{2\zeta\omega}{\nu}\right)^2\right]^{1/2} \quad (3.128)$$

Since $\varepsilon\Delta = 1 - \nu/\omega_0$, equation (3.128) becomes

$$\frac{\nu}{2\omega} - 1 = \varepsilon\left\{\frac{3\gamma}{8}\overline{a_0}^2 \pm \left[\frac{\mu^2}{4} - \left(\frac{2\zeta\omega}{\nu}\right)^2\right]^{1/2}\right\} \quad (3.129)$$

Which gives the amplitude-frequency relation.

3.6.2 Undamped Case $\boldsymbol{\zeta = 0}$

In the undamped case $\zeta = 0, \sin 2\bar{\varphi}_0 = 0, and \cos 2\bar{\varphi}_0 = \pm 1$, which imples $\bar{\varphi}_0 = 0, \pi/2$, and the response is

$$q(\tau) = \bar{a}_0 cos(k\tau + \bar{\varphi}_0) = \begin{matrix} \bar{a}_0 \cos\frac{1}{2}\tau, \\ -\bar{a}_0 \sin\frac{1}{2}\tau, \end{matrix} \quad corresponding\ to\ \bar{\varphi}_0 = \begin{matrix} 0 \\ \pi/2 \end{matrix} \quad (3.130)$$

Corresponding amplitude-frequency relation is reduced to

$$\frac{\nu}{2\omega} - 1 = \varepsilon\left(\frac{3\gamma}{8}\overline{a_0}^2 \pm \frac{\mu}{2}\right) \quad (3.131)$$

When the nonlinear term is not considered, i.e. $\gamma = 0$ in eqn (3.120) the V shaped instability region of the undamped Mathieu equation. For a given value of μ the system is unstable when $\nu/2\omega$ is between $1 - \varepsilon\mu/2$ and $1 + \varepsilon\mu/2$. In this unstable region, the amplitude of the solution grows exponentially

When the system is nonlinear, i.e. $\gamma \neq 0$ in the eqn (3.120) the solution of the system cannot grow unbounded because of the nonlinearity. The system has steady state solutions, the amplitudes of which are governed by the amplitude-frequency relation.

3.7 Stability of the Steady State Solutions

To study the stability of the steady-state solutions, let $\bar{a} = \bar{a}_0 + u$ and $\bar{\varphi} = \bar{\varphi}_0 + \nu$, where u and ν are small variations about the steady state solutions $\bar{a}_0$ and $\bar{\varphi}_0$, and substitute in to equations. Noting that, with $sin2\nu = 2\nu,\ cos2\nu = 1$,

$$sin2\bar{\varphi} = sin2\bar{\varphi}_0.cos2\nu + cos2\bar{\varphi}_0.sin2\nu \approx sin2\bar{\varphi}_0 + 2\nu.cos2\bar{\varphi}_0$$

$$cos2\bar{\varphi} = cos2\bar{\varphi}_0.cos2\nu - sin2\bar{\varphi}_0.sin2\nu \approx cos2\bar{\varphi}_0 - 2\nu.sin2\bar{\varphi}_0$$

$$\bar{a}_0^{'} = -\varepsilon\left(\frac{\zeta\omega}{\nu} + \frac{\mu}{4}sin2\bar{\varphi}_0\right)\bar{a}_0,\ \bar{\varphi}_0^{'} = \varepsilon\frac{1}{2}\left(\Delta + \frac{3\gamma}{8}\bar{a}_0{}^2 - \frac{\mu}{2}cos\bar{\varphi}_0\right), \quad (3.132)$$

One obtains

$$u^{'} = -\varepsilon\left(\frac{\zeta\omega}{\nu} + \frac{\mu}{4}sin2\bar{\varphi}_0\right)u - \varepsilon\frac{\mu}{2}\bar{a}_0 cos\bar{\varphi}_0.\nu,$$

$$\nu^{'} = \varepsilon\frac{3\gamma}{16}(2\bar{a}_0 u + u^2) + \varepsilon\frac{\mu}{2}\sin 2\bar{\varphi}_0\,.\nu \quad (3.133)$$

Linearizing the equations in u and ν and using equation (3.126) leads to

$$\begin{Bmatrix} u' \\ \nu' \end{Bmatrix} + \begin{bmatrix} 0 & \varepsilon\frac{\mu}{2}\bar{a}_0 cos\bar{\varphi}_0 \\ -\frac{3\gamma}{8}\bar{a}_0 & -\varepsilon\frac{\mu}{2}\sin 2\bar{\varphi}_0 \end{bmatrix}\begin{Bmatrix} u \\ \nu \end{Bmatrix} = 0. \quad (3.134)$$

This is a system of first-order linear ordinary differential equations that admit solutions of the form $Ce^{\rho t}$, in which the characteristic numbers ρ are the roots of

$$\begin{vmatrix} \rho & \varepsilon\frac{\mu}{2}\bar{a}_0 cos\bar{\varphi}_0 \\ -\frac{3\gamma}{8}\bar{a}_0 & \rho - \varepsilon\frac{\mu}{2}\sin 2\bar{\varphi}_0 \end{vmatrix} = 0 \quad (3.135)$$

i.e.,

$$\rho^2 + \varepsilon\frac{2\zeta\omega}{\nu}\rho + \varepsilon^2\frac{3\gamma\mu}{16}\bar{a}_0^2\cos 2\bar{\varphi}_0 = 0 \quad (3.136)$$

In which equation (3.127) has been used. The solutions of this quadratic equation are

$$\rho = \varepsilon\left\{-\frac{\zeta\omega}{\nu} \pm \left[\left(-\frac{\zeta\omega}{\nu}\right)^2 - \frac{3\gamma\mu}{16}\bar{a}_0^2\cos 2\bar{\varphi}_0\right]^{1/2}\right\}. \quad (3.137)$$

3.7.1 Non Trivial Solution

Consider first the non trivial solution $\bar{a}_0 \neq 0$

Case 1. Damped System, $\boldsymbol{\zeta} > 0$

When $\cos 2\bar{\varphi}_0 < 0$ i.e., along the lower or right curve, ρ_{max} is real and positive. The steady-state solution represented by this curve is unstable.

When $\cos 2\bar{\varphi}_0 > 0$, there are two possibilities, i.e. tow real roots that are both negative or a pair of complex conjugate roots with negative real part, provided $\zeta > 0$. The steady-state solution represented by the upper or left-hand curve is stable.

When $\cos 2\bar{\varphi}_0 = 0$, $\rho_1 = 0$, $\rho_2 = -2\varepsilon\zeta\omega/\nu$, and the steady-state solution represented by the point with vertical tangent, which bounds the upper and lower curves, is stable.

Case 2. Undamped System $\boldsymbol{\zeta = 0}$

For the undamped case of $\zeta = 0$, the characteristic numbers are

$$\rho = \pm \frac{\varepsilon\bar{a}_0}{4}\sqrt{-3\gamma\mu\cos 2\bar{\varphi}_0}. \tag{3.138}$$

$\cos 2\bar{\varphi}_0 = -1$. The characteristic numbers are real with $\rho_1 > 0 \; and \; \rho_2 < 0$. Hence the steady-state solution represented by the right curve is unstable.

$\cos 2\bar{\varphi}_0 = 1$. This corresponds to the left curve and the characteristic numbers are purely imaginary given by $\rho = \pm i\varepsilon\sqrt{3\gamma\mu}\bar{a}_0/4$. Hence stability cannot be decided by consideration of the linear terms alone in the perturbed equations.

In this case, equation (3.125) can be written as, for $\zeta = 0$,

$$\frac{d\bar{a}}{d\bar{\varphi}} = -\frac{\mu\bar{a}\sin 2\bar{\varphi}}{2\Delta + \frac{3\gamma}{4}\bar{a}^2 - \mu\cos 2\bar{\varphi}} \tag{3.139}$$

i.e.

$$\left(2\Delta + \frac{3}{4}\gamma\bar{a}^2\right)d\bar{a} - \mu(\cos 2\bar{\varphi}\, d\bar{a} - \bar{a}\sin 2\bar{\varphi}\, d\bar{\varphi}) = 0. \tag{3.140}$$

Multiplying this equation by $\bar{a}$ and integrating both sides of the equation yield

$$\Delta\bar{a}^2 + \frac{3}{16}\gamma\bar{a}^4 - \frac{\mu}{2}\bar{a}^2\cos 2\bar{\varphi} = C. \tag{3.141}$$

If one puts $x = \bar{a}\cos\bar{\varphi}$ and y= $\bar{a}\sin\bar{\varphi}$, equation (3.142) becomes

$$\left(\Delta - \frac{\mu}{2}\right)x^2 + \left(\Delta + \frac{\mu}{2}\right)y^2 + \frac{3\gamma}{16}(x^2 + y^2)^2 = C. \qquad (3.142)$$

By plotting equation (3.142) for various values of μ and γ, the stability of the steady-state solution can be determined.

1. When $\Delta = \frac{1}{2}\mu$, equation (3.142) becomes

   $$\mu y^2 + \frac{3\gamma}{16}(x^2 + y^2) = C \qquad (3.143)$$

   The phase portrait is drawn . It is seen that the steady-state solution corresponding to the singular point $x = y = 0$ is a centre and is hence stable.

2. When $\Delta = 0$, equation (3.142) becomes

   $$\frac{1}{2}\mu(y^2 - x^2) + \frac{3\gamma}{16}(x^2 + y^2)^2 = C, \qquad (3.144)$$

   With the phase portrait shown. The steady-state solution corresponding to $x = y = 0$ is a saddle point and is therefore unstable. The steady-state solutions with $\bar{a}_0 \neq 0$ are related to two centres on the x-axis and are hence stable.

3. When $\Delta = -\frac{1}{2}\mu$, equation (3.142) is reduced to

   $$-\mu x^2 + \frac{3\gamma}{16}(x^2 + y^2) = C, \qquad (3.145)$$

   And the phase portrait is shown. Similar to the case when $\Delta = 0$, the steady-state solution corresponding to $x = y = 0$ is a saddle point and is unstable. The steady-state solutions with $\bar{a}_0 = 0$ correspond to two singular points that are centres and are stable.

4. When $\Delta < -\frac{1}{2}\mu$, a typical phase portrait is shown. The steady state solution corresponding to $x = y = 0$ is a centre and is stable. The steady-state solutions represented by the lower or right curve of the amplitude-frequency relationship are two saddle points and are unstable; whereas those represented by the upper or left-hand curve of the amplitude-frequency relationship are two centres and are stable.

3.7.2 Trivial Solution

For the trivial solution, $\bar{a}_0 = 0$. The damped and undamped cases are studied in the following.

For the damped case when $\zeta \neq 0$, equations (3.125) can be converted to rectangular coordinates by putting $x = \bar{a}\cos\bar{\varphi}$ and y= $\bar{a}\sin\bar{\varphi}$,

$$x' = \bar{a}'\cos\bar{\varphi} - \bar{a}\sin\bar{\varphi}\,.\,\bar{\varphi}'$$

$$= -\varepsilon\left(\frac{\zeta\omega}{\nu} + \frac{\mu}{4}\sin 2\bar{\varphi}\right)\bar{a}\cos\bar{\varphi} - \varepsilon\frac{1}{2}\left(\Delta + \frac{3\gamma}{8}\bar{a}^2 - \frac{\mu}{2}cos2\bar{\varphi}\right)\bar{a}\sin\bar{\varphi} \qquad (3.146)$$

Which yields

$$x' = -\varepsilon\left[\frac{\zeta\omega}{\nu}x + \left(\frac{\Delta}{2} + \frac{\mu}{4}\right)y + \frac{3\gamma}{16}(x^2 + y^2)y\right]$$

And similarly

$$y' = -\varepsilon\left[\frac{\zeta\omega}{\nu}y - \left(\frac{\Delta}{2} - \frac{\mu}{4}\right)x - \frac{3\gamma}{16}(x^2 + y^2)x\right] \qquad (3.147)$$

To investigate the stability of the trivial solution $x_0 = y_0 = 0$, consider the linearized equations

$$x' = -\varepsilon\left[\frac{\zeta\omega}{\nu}x + \left(\frac{\Delta}{2} + \frac{\mu}{4}\right)y\right],\; y' = -\varepsilon\left[\frac{\zeta\omega}{\nu}y - \left(\frac{\Delta}{2} - \frac{\mu}{4}\right)x\right] \qquad (3.148)$$

Which admit solutions of the form $x, y \propto e^{\rho\tau}$, where the characteristic numbers are given by

$$\begin{vmatrix} \rho + \varepsilon\frac{\zeta\omega}{\nu} & \varepsilon\left(\frac{\Delta}{2} + \frac{\mu}{4}\right) \\ -\varepsilon\left(\frac{\Delta}{2} - \frac{\mu}{4}\right) & \rho + \varepsilon\frac{\zeta\omega}{\nu} \end{vmatrix} = 0, \qquad (3.149)$$

i.e.

$$\left(\rho + \varepsilon\frac{\zeta\omega}{\nu}\right)^2 + \varepsilon^2\left(\frac{\Delta^2}{4} - \frac{\mu^2}{16}\right) = 0, \qquad (3.150)$$

or

$$\rho = \varepsilon\left[-\frac{\zeta\omega}{\nu} \pm \left(\frac{\mu^2}{16} - \frac{\Delta^2}{4}\right)^{1/2}\right] \qquad (3.151)$$

The trivial solution is unstable when ρ is real and $\rho_{max} > 0$, implying that

$$-\frac{\zeta\omega}{\nu}+\left(\frac{\mu^2}{16}-\frac{\Delta^2}{4}\right)^{1/2}>0, \tag{3.152}$$

Or, since $\nu \approx 2\omega$,

$$|\Delta|<\left[\frac{\mu^2}{4}-\left(\frac{2\omega}{\nu}\right)^2\zeta^2\right]^{1/2}\approx\left(\frac{\mu^2}{4}-\zeta^2\right)^{1/2} \tag{3.153}$$

Otherwise the solution is Stable.

Chapter 4

Results and Discussions

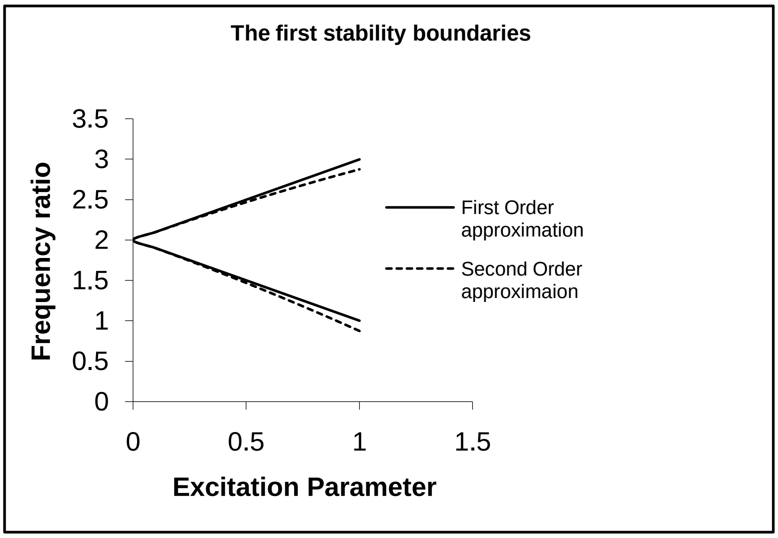

Figure 3: First and second approximation to principle instability region for column under axial excitation

The first stability boundaries are shown in figure 3. The second order approximation gives the better results as two terms has been considered in the fourier series solution. As the excitation parameters increases, the width of the instability boundaries widens. The width of the instability zone can be expressed as the expression

$$\Delta(1) = \left| \left(1 + \frac{1}{2}\mu - \frac{1}{16}\mu^2 \right) - \left(1 - \frac{1}{2}\mu - \frac{1}{16}\mu^2 \right) \right| = \mu. \tag{4.1}$$

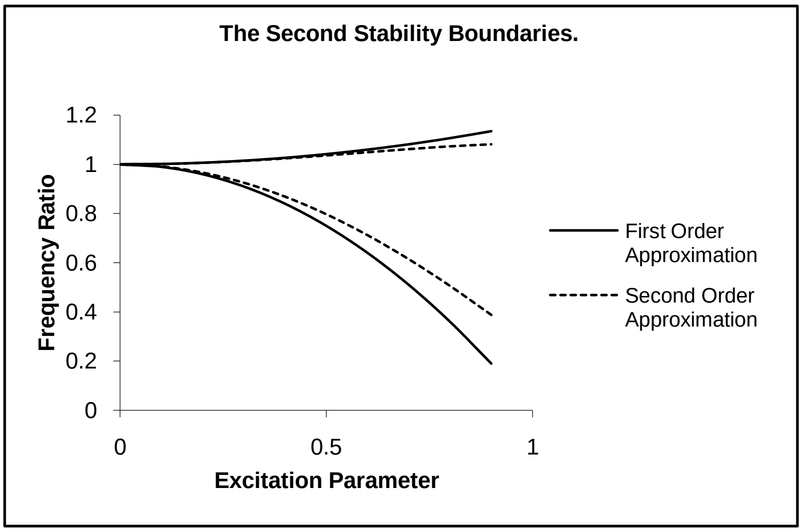

Figure 4: First and second approximation to second instability region for column under axial excitation

The boundaries of second stability region is shown in figure 4. As the value of excitation frequency increases the width of the second instability region widens. The width of the second instability region can be expressed as

$$\Delta\left(\frac{1}{2}\right)=\left|\left(\frac{1}{2}+\frac{1}{12}\mu^{2}-\frac{35}{864}\mu^{4}\right)-\left(\frac{1}{2}-\frac{5}{12}\mu^{2}-\frac{7}{144}\mu^{4}\right)\right|\approx\frac{1}{2}\mu^{2} \tag{4.2}$$

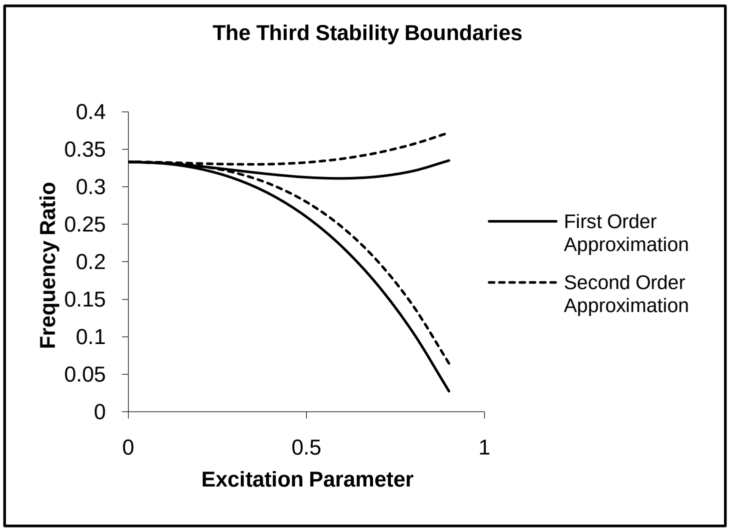

Figure 5: First and second approximation to third instability region for column under axial excitation

The third zone of instability is shown in figure 5. The width of the third zone of instability is smaller than first and second zone of instability. The instability zone widens with the increase of excitation parameter. The width of the zone at any point can be expressed as

$$\Delta\left(\frac{1}{3}\right)=\left|\left(\frac{1}{3}-\frac{3}{32}\mu^2+\frac{27}{128}\mu^3-\frac{243}{4096}\mu^4\right)-\left(\frac{1}{3}-\frac{3}{32}\mu^2-\frac{27}{128}\mu^3-\frac{243}{4096}\mu^4\right)\right|=\frac{27}{64}\mu^3. \qquad (4.3)$$

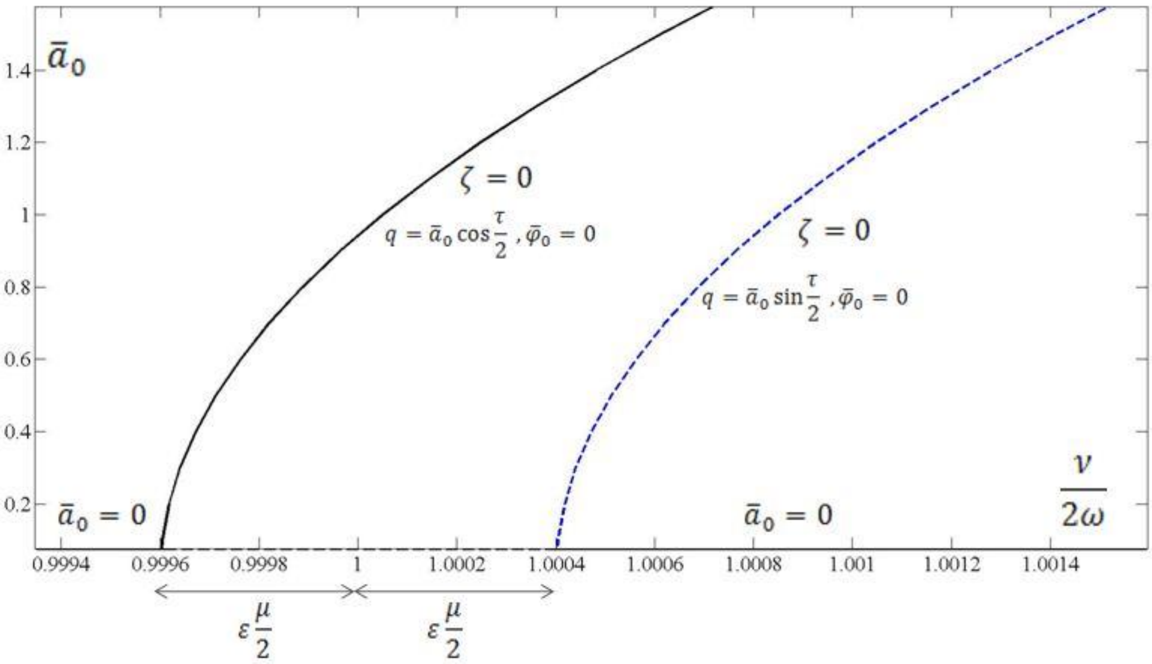

Figure 6: Amplitude Frequency relation Undamped Case.

$$\frac{\nu}{2\omega} - 1 = \varepsilon\left(\frac{3\gamma}{8}\overline{a_0}^2 \pm \frac{\mu}{2}\right) \qquad (4.4)$$

The solid curve (left) is stable and the dotted curve (left) is unstable.

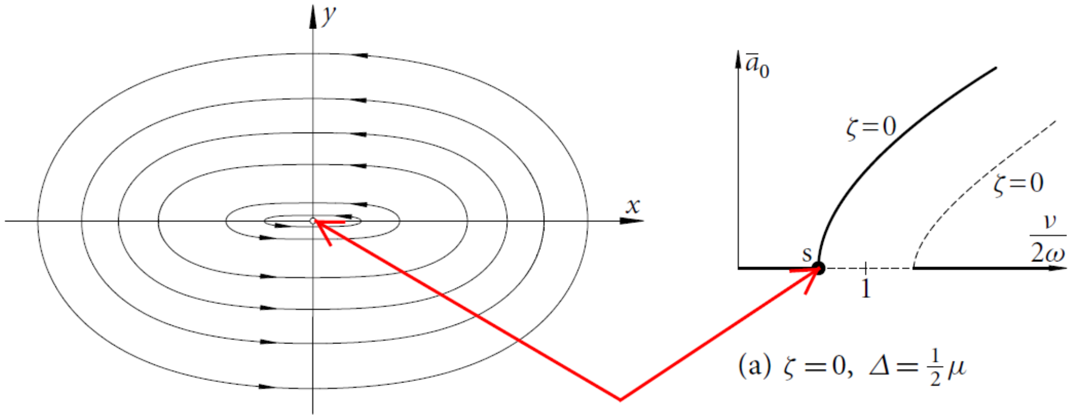

Figure 7: Phase Portrait 1

When $\Delta = \frac{1}{2}\mu$, equation (3.142) becomes

$$\mu y^2 + \frac{3\gamma}{16}(x^2 + y^2) = C. \tag{4.5}$$

The phase portrait is drawn . It is seen that the steady-state solution corresponding to the singular point $x = y = 0$ is a centre and is hence stable.

The origin is a centre and is stable, which corresponds to this point (amplitude=0, stable) on the amplitude-frequency diagram.

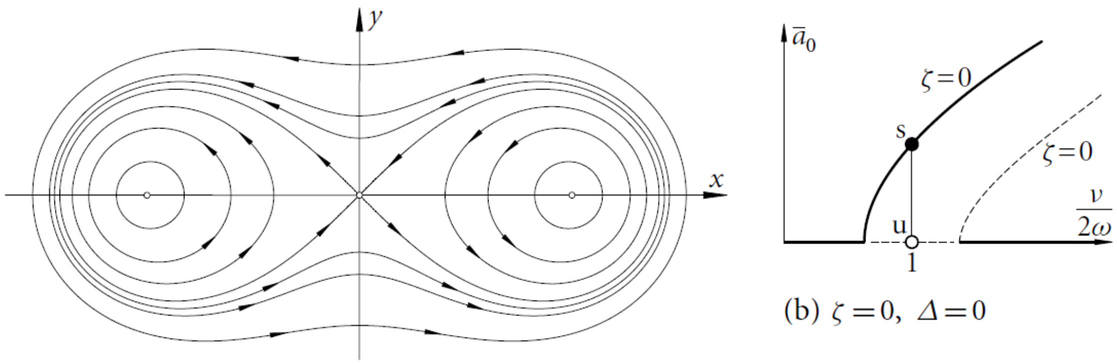

Figure 8: Phase Portrait 2

When $\Delta = 0$, equation (3.142) becomes

$$\frac{1}{2}\mu(y^2 - x^2) + \frac{3\gamma}{16}(x^2 + y^2)^2 = C, \tag{4.6}$$

With the phase portrait shown. The steady-state solution corresponding to $x = y = 0$ is a saddle point and is therefore unstable. The steady-state solutions with $\bar{a}_0 \neq 0$ are related to two centres on the x-axis and are hence stable.

The origin is a saddle point and is unstable, which corresponds to this point (zero amplitude, unstable) on the amplitude-frequency diagram.

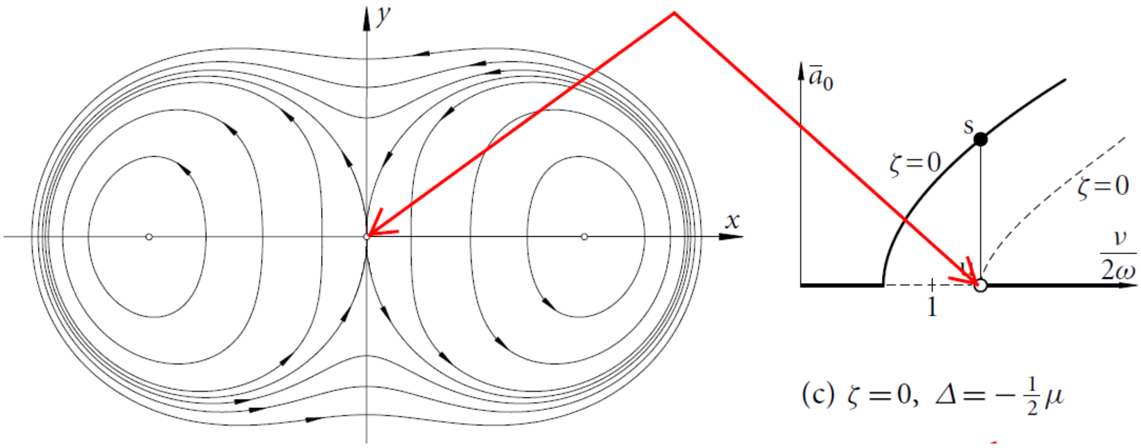

Figure 9: Phase Portrait 3

When $\Delta = -\frac{1}{2}\mu$, equation (3.142) is reduced to

$$-\mu x^2 + \frac{3\gamma}{16}(x^2 + y^2) = C, \quad (4.7)$$

And the phase portrait is shown. Similar to the case when $\Delta = 0$, the steady-state solution corresponding to $x = y = 0$ is a saddle point and is unstable. The steady-state solutions with $\bar{a}_0 = 0$ correspond to two singular points that are centres and are stable.

The origin is a saddle point and is unstable, which corresponds to this point (zero amplitude, unstable) on the amplitude-frequency diagram.

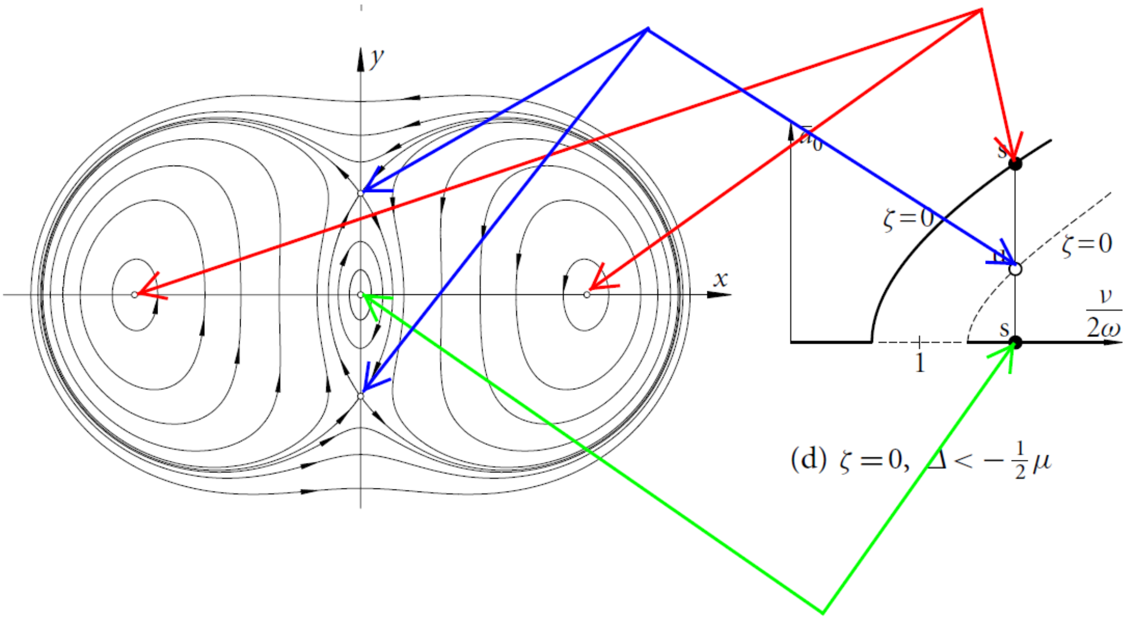

Figure 10: Phase Portrait 4

When $\Delta < -\frac{1}{2}\mu$, a typical phase portrait is shown. The steady state solution corresponding to $x = y = 0$ is a centre and is stable. The steady-state solutions represented by the lower or right curve of the amplitude-frequency relationship are two saddle points and are unstable; whereas those represented by the upper or left-hand curve of the amplitude-frequency relationship are two centres and are stable.

The origin is a centre and is stable, which corresponds to this point (amplitude=0, stable) on the amplitude-frequency diagram.

The two equilibrium points on x-axis are centres and are stable, which correspond to this point (non-zero amplitude, stable) on the amplitude-frequency diagram.

The two intersecting points (nodes) on y-axis are saddle points and are unstable, which corresponds to this point (non-zero amplitude, unstable) on the amplitude-frequency diagram.

Conclusions

1. While plotting instability regions, second order approximation gives better results as two terms has been considered in Fourier series solution. As load parameter increases, the width of instability boundary is widening.

2. From the first, second and third instability boundaries, it is found that the width of first instability region is wider than the second instability region and the width of second instability region in wider than the third instability region.

3. The frequency amplitude relation has been plotted and its stability at various points has been checked by plotting phase portraits and it has been found that the solid curve in figure 6 is stable and dotted curve is unstable.

Appendix A

Method of Averaging

As an approximation, the periodic functions can be replaced by their averaged values in one period, i.e. $a(t)$ and $\varphi(t)$ replaced by $\bar{a}(t)$ and $\bar{\varphi}(t)$, where the over-bar stands for the averaged value. The averaging operator is defined by

$$\mathcal{A}(\,) = \lim_{T\to\infty} \frac{1}{T}\int_{\tau}^{\tau+T} (\,) d\tau, \qquad \text{(A-1)}$$

Where integration is performed over the explicitly appearing time parameter τ in the integrand. This approach is called method of averaging.

It is easy to evaluate the operation of the averaging operator $\mathcal{A}$ on terms of the form $\cos(k\tau + \alpha\bar{\varphi})$ and $\sin(k\tau + \alpha\bar{\varphi})$ for $k \neq 0$

$$\mathcal{A}\begin{Bmatrix} cos \\ sin \end{Bmatrix}(k\tau + \alpha\bar{\varphi}) = \lim_{T\to\infty} \frac{1}{T}\int_{\tau}^{\tau+T} \begin{Bmatrix} cos \\ sin \end{Bmatrix}(k\tau + \alpha\bar{\varphi}) d\tau,$$

$$= \lim_{T\to\infty} \frac{1}{kT}\begin{Bmatrix} +sin \\ -cos \end{Bmatrix}(k\tau + \alpha\bar{\varphi})\Big|_{\tau=\tau}^{\tau+T} = 0 \qquad \text{(A-2)}$$

And for k = 0

$$\mathcal{A}\begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi}) = \lim_{T\to\infty} \frac{1}{T}\int_{\tau}^{\tau+T} \begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi}) d\tau,$$

$$= \begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi}) \lim_{T\to\infty} \frac{1}{T}\tau\Big|_{\tau=\tau}^{\tau+T} = \begin{Bmatrix} cos \\ sin \end{Bmatrix}(\alpha\bar{\varphi}) \qquad \text{(A-3)}$$

Hence, when the method of averaging is applied, sine and cosine terms of the form $\cos(k\tau + \alpha\bar{\varphi})$ and $\sin(k\tau + \alpha\bar{\varphi})$ vanish in the averaged equations if τ is equal to 0.

References

Timoshenko, S.P. and Gere, J.M., 1961. *Theory of elastic stability*. McGraw-Hill, New Delhi.

Bolotin, V. V. 1964. *The dynamic stability of elastic systems*. *San Francisco: holden-day*.

Sarat Kumar Panda 2010. *Static and dynamic instability analysis of plates and cylindrical panels using the Third-Order Theory*. PhD thesis, IIT Kharagpur.

Wei-Chau Xie 2006. *Dynamic Stability of Structures*. Cambridge University Press.

Sarat Kumar Panda and L.S.Ramachandra., 2010. Buckling of rectangular plates with various boundary conditions loaded by non-uniform inplane loads, *International Journal of Mechanical Sciences, 52, 819-828.*

L.S. Ramachandra and Sarat Kumar Panda., 2011. Dynamic instability of composite plates subjected to non-uniform in-planeloads, *Journal of Sound and Vibration*, *331, 53-65.*

Sarat Kumar Panda and L. S. Ramachandra., 2011. Buckling and Postbuckling Behavior of Cross-Ply Composite Plate Subjected to Non-uniform In-Plane Loads, *American Society for Civil Engineers, 137(9), 589-597.*