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GRACEFUL SIGNED GRAPHS: III. THE CASE OF SIGNED CYCLES IN WHICH THE NEGATIVE SECTIONS FORM A MAXIMUM MATCHING

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Abstract

In a previous paper generalizing the well known notion of graceful graphs, we define a (p, m, n)-signed graph S of order p, with m positive edges and n negative edges, to be graceful if there exists an injective function f that assigns integers $0, 1, \ldots, q = m+n$ to its vertices such that when to each edge uv of S one assigns the absolute difference $|f(u) - f(v)|$, the positive edges of S are mapped to the set $\{1, 2, \ldots, m\}$ and the negative edges of S are mapped to the set $\{1, 2, \ldots, n\}$. A result in that paper showed that if a (p, m, n)-signed graph having an Eulerian underlying graph is graceful then its size q must be congruent to 0, 2, or 3 (mod4). It was conjectured that all such signed cycles Z_k on the cycle C_k, $k \geq 3$ with an admissible number of negative sections are graceful. In this paper, we establish the truth of this conjecture for all signed cycles Z_k, $3 \leq k \equiv 0, 2$, or $3 \pmod 4$, in which the negative sections form a maximum matching.

1. Introduction

For terminology in graph theory we follow **[1]**. Additional terms are defined as needed.

A *signed graph* (or *sigraph* in short) is an ordered pair $S = (S^u, s)$ where $S^u = (V, E)$ is a graph, called the *underlying graph* of S, and $s: E \to \{+, -\}$ is a function from the edge set E to the set $\{+, -\}$. This notion was first introduced by Harary **[2]** in the context of modelling a sociopsychologic phenomenon.

Let $E^+(S) = \{e \in E : s(e) = +\}$ and $E^-(S) = \{e \in E : s(e) = -\}$. The set $E(S) = E^+(S) \cup E^-(S)$ is called the *edge set* of S. The elements of $E^+(S)$ and $E^-(S)$, respectively, are called *positive* and *negative edges* of S. An *all-positive sigraph* S is one for which $E^+(S) = E(S)$; similarly, S is *all-negative* if $E^-(S) = E(S)$. Hence, we consider a graph as an all-positive sigraph.

A sigraph is said to be *homogeneous* if it is either all-positive or all-negative and to be *heterogeneous* otherwise. Given a subsigraph H of S, by a *negative* (*positive*) *section* of H we mean a maximal connected all-negative (all-positive) *subsigraph* of H.

By *(p,m,n)-sigraph* we mean a sigraph $S = (S^u, s)$ where $S^u = (V, E)$ is a (p,q)-graph (that is, a graph of order p and size q, as defined in **[1]**), $|E^+(S)| = m$ and $|E^-(S)| = n$ so that $m + n = q$. Let f be a function that assigns distinct labels to the vertices of S from the set $\{0, 1, 2, \ldots, q\}$. Define a labeling g_f of the edges of S induced by f as follows: for each edge $uv \in E$, $g_f(uv) = s(uv)|f(u) - f(v)|$. If the q edges of S each have a unique label $g_f(uv)$ from the set $\{1, 2, \ldots, m, -1, -2, \ldots, -n\}$, then the labeling f is called a *graceful labelling* of S. A sigraph that admits such a labelling is called a *graceful sigraph* (see **[3]**). Note that if $n = 0$ (that is, when S is an all-positive sigraph) then this notion coincides with that of a graceful graphs in the Rosa and Golomb sense **[4][5]**).

Graceful labellings of sigraphs may provide insight into the more general problem of finding a unified model for automatic continuous coding of monochromatic factors in an edge packing of a graph, as described in **[6]**.

Theorem 1 [3]: Let $S = (S^u, s)$ be a (p,m,n)-sigraph such that S^u is a Eulerian. If S is graceful, then $m^2 + n^2 + m + n \equiv 0 \pmod 4$. ■

By a *signed cycle*, Z_k, we mean any signed graph on the cycle C_k of length $k \geq 3$ such that $Z_k = C_k$ if and only if Z_k is all-positive.

Corollary 1.1 [3]: If a signed cycle Z_k, $k \geq 3$, is graceful then $k \equiv 0, 2$, or $3 \pmod 4$. ■

It was conjectured in **[3]** that the converse of Corollary 1.1 also holds for all $k \geq 7$ under certain conditions. In fact, the following results are known.

Theorem 2 [3]: If a signed cycle Z_k of length $k \equiv 0 \pmod 4$ is graceful then the number of negative sections of odd length in Z_k is even. ■

Theorem 3 [7]: If a signed cycle Z_k, $k \equiv 3 \pmod 4$ contains exactly one negative section the Z_k is graceful. ■

Theorem 4 [7]: If a signed cycle Z_k, $k \equiv 2 \pmod 4$ is graceful then the number of negative sections of odd length in Z_k is odd. ■

In this paper, we establish the following result as partial progress in settling the converse of Corollary 1.1.

Theorem 5: If $6 \leq k \equiv 0, 2$, or $3 \pmod 4$ then any signed cycle Z_k in which the negative sections form a maximum matching is graceful.

2. Results

In this section we complete the proof of Theorem 5 by establishing a series of lemmas.

First, we establish the following partial result toward the sufficiency of Theorem 2. In this case, we consider negative sections of unit length (copies of K_2) that form a maximum matching of the signed cycle.

Lemma 1: If Z_k, $8 \leq k \equiv 0 \pmod 4$, is a signed cycle in which the negative sections constitute a maximum matching, then Z_k is graceful.

Proof: It is sufficient to provide a graceful labelling of Z_k whose sign structure is as stated in the hypothesis, with m and n denoting, respectively, the sum of lengths of positive and negative sections in Z_k. Furthermore, since $k \equiv 0 \pmod 4$, in this case $m = n = k/2$ is even. Accordingly, we define a graceful labelling, ψ, of Z_k as follows. Label the vertices of Z_k consecutively as $u_1, u_2, \ldots, u_k$, with u_1u_k a positive edge. Define the vertex numbering ψ in this case as follows:

$$\psi(u_1) = 0,$$

$$\psi(u_i) = n + \tfrac{i}{2} - 1, \text{ for } i \in \{2, 4, \ldots, n+2\},$$

$$\psi(u_i) = 2n - \tfrac{i-3}{2}, \text{ for } i \in \{3, 5, \ldots, n+1\},$$

$$\psi(u_i) = \tfrac{2n-i}{2} + 1, \text{ for } i \in \{n+4, n+6, \ldots, 2n\}, \text{ and}$$

$$\psi(u_i) = \left\lfloor \tfrac{i}{2} \right\rfloor, \text{ for } i \in \{n+3, n+5, \ldots, 2n-1\}.$$

Then, the induced edge function g_ψ yields the edge labels

$$\{g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| = s(u_iu_{i+1})|n-i+2| : i \in \{2, 3, 4, \ldots, n+1\}\}$$
$$= \{-1, -3, \ldots, -(n-1)\} \cup \{2, 4, \ldots, n\};$$

$$\{g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})\left|\left\lfloor \tfrac{2n-i-1}{2} \right\rfloor + 1 - \left\lfloor \tfrac{i}{2} \right\rfloor\right| : i \in \{n+3, n+4, \ldots, 2n-1\}\}$$
$$= \{3, 5, \ldots, n-3\} \cup \{-2, -4, \ldots, -(n-2)\};$$

$$g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})\left|n + \tfrac{i}{2} - 1 - \left\lfloor \tfrac{i+1}{2} \right\rfloor\right|, \text{ when } i = n+2;$$

$$g_\psi(u_1u_2) = -n; \text{ and}$$

$$g_\psi(u_1u_k) = 1.$$

The injectivity of ψ is straightforward to see from its definition in this case. We have shown that the induced edge labelling g_ψ is also injective. This completes the proof of Lemma 1. ■

Lemma 1 is illustrated in Figure 1.

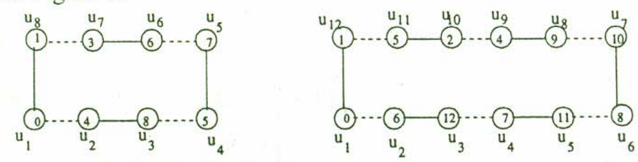

Figure 1: Examples of graceful signed cycles Z_k, $k \equiv 0 \pmod 4$.

Lemma 2: If Z_k, $7 \le k \equiv 3 \pmod 4$, is a signed cycle in which the negative sections constitute a maximum matching, then Z_k is graceful.

Proof: Again it is sufficient to provide a graceful labelling of Z_k. We construct a graceful labeling ψ as follows. Let the vertices of Z_k be labeled consecutively as $u_1, u_2, \ldots, u_k$ so that the edge u_1u_2 is positive. Note that this choice fixes the signs of other edges in Z_k due to the maximum matching condition. Define the vertex numbering ψ as follows:

$$\psi(u_1) = 0;$$

$$\psi(u_i) = \frac{k+1}{2} + \frac{i}{2} - 1, \text{ for } i \in \{2, 4, \ldots, \tfrac{k+1}{2}\};$$

$$\psi(u_i) = k - \frac{i-3}{2}, \text{ for } i \in \{3, 5, \ldots, \tfrac{k+3}{2}\};$$

$$\psi(u_i) = \frac{k-i}{2} + 1, \text{ for } i \in \{\tfrac{k+7}{2}, \tfrac{k+11}{2}, \ldots, k\}; \text{ and}$$

$$\psi(u_i) = \frac{i}{2}, \text{ for } i \in \{\tfrac{k+5}{2}, \tfrac{k+9}{2}, \ldots, k-1\}.$$

The induced edge function g_ψ yields the edge labels

$$g_\psi(u_1u_k) = 1;$$

$$g_\psi(u_1u_2) = \frac{k+1}{2};$$

$$\{g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| = s(u_iu_{i+1})\left|\tfrac{k-1}{2} - i + 2\right| : i \in \{2, 3, 4, \ldots, \tfrac{k+1}{2}\}\}$$

$$= \{-1, -3, \ldots, -(\tfrac{k-1}{2})\} \cup \{2, 4, \ldots, \tfrac{k-3}{2}\};$$

$$\{g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})\left|\tfrac{k+1}{2} - i\right| : i \in \{\tfrac{k+5}{2}, \tfrac{k+7}{2}, \ldots, k-1\}\}$$

$$= \{3, 5, \ldots, \tfrac{k-5}{2}\} \cup \{-2, -4, \ldots, -(\tfrac{k-3}{2})\}; \text{ and}$$

$$g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})\left|k - (\tfrac{i-3}{2}) - (\tfrac{i+1}{2})\right| = |k - i + 1| = \tfrac{k-1}{2} \text{ when } i = \tfrac{k+3}{2}.$$

The injectivity of ψ is straightforward to see from its definition. We have also shown that the induced edge labelling g_ψ is injective. This completing the proof of Lemma 2. ■

Lemma 2 is illustrated in Figure 2.

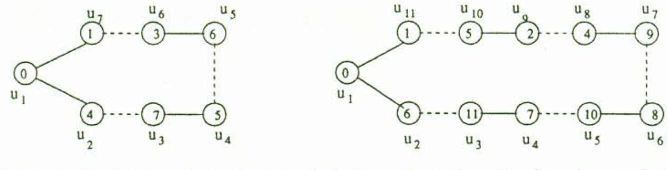

Figure 2: Examples of graceful signed cycles Z_k, $k \equiv 3 \pmod 4$.

Lemma 3: If Z_k, $6 \le k \equiv 2 \pmod 4$, is a signed cycle in which the negative sections form a maximum matching, then Z_k is graceful.

Proof: Clearly it is sufficient to provide a graceful labelling of Z_k. We construct a graceful labeling ψ as follows. Let the vertices of Z_k be labeled consecutively as $u_1, u_2, \ldots, u_k$. Let the edge u_1u_k be negative (however, in this proof the vertex function works independently of this choice). Define the vertex numbering ψ as follows:

$$\psi(u_1) = 0;$$

$$\psi(u_i) = \tfrac{k}{2} + \tfrac{i}{2} - 1, \text{ for } i \in \{2, 4, \ldots, \tfrac{k}{2}+1\};$$

$$\psi(u_i) = k - \tfrac{i-3}{2}, \text{ for } i \in \{3, 5, \ldots, \tfrac{k}{2}\};$$

$$\psi(u_i) = \left\lfloor \tfrac{k-i}{2} \right\rfloor + 1, \text{ for } i \in \{\tfrac{k+6}{2}, \tfrac{k+10}{2}, \ldots, k\}; \text{ and}$$

$$\psi(u_i) = \left\lfloor \tfrac{i}{2} \right\rfloor, \text{ for } i \in \{\tfrac{k+4}{2}, \tfrac{k+8}{2}, \ldots, k-1\}.$$

Then the induced edge function g_ψ yields the edge labels

$$g_\psi(u_1u_k) = -1;$$

$$g_\psi(u_1u_2) = \tfrac{k}{2};$$

$$\{g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| = s(u_iu_{i+1})\left|\tfrac{k}{2} - i + 2\right| : i \in \{2, 3, 4, \ldots, \tfrac{k}{2}\}\}$$
$$= \{-3, -5, \ldots, -(\tfrac{k}{2})\} \cup \{2, 4, \ldots, \tfrac{k}{2} - 1\};$$

$$\{g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})\left|\left\lfloor \tfrac{i}{2} \right\rfloor - \left\lfloor \tfrac{k-i-1}{2} \right\rfloor - 1\right| : i \in \{\tfrac{k+4}{2}, \tfrac{k+6}{2}, \ldots, k-1\}\}$$
$$= \{1, 3, \ldots, \tfrac{k-4}{2}\} \cup \{-2, -4, \ldots, -(\tfrac{k-2}{2})\}; \text{ and}$$

$$g_\psi(u_iu_{i+1}) = s(u_iu_{i+1})\left|\tfrac{k}{2} + \tfrac{i}{2} - 1 - \left\lfloor \tfrac{i+1}{2} \right\rfloor\right| \text{ when } i = \tfrac{k}{2} + 1.$$

The injectivity of ψ is straightforward from its definition. We have also shown that the induced edge labelling g_ψ is injective. This completing the proof of Lemma 3. ■

Lemma 3 is illustrated in Figure 3.

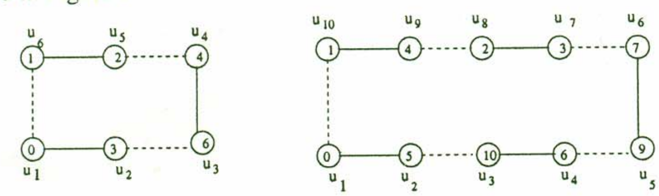

Figure 3: Examples of graceful signed cycles Z_k, $k \equiv 2 \pmod 4$.

3. Concluding Remarks

We have determined the graceful signed cycles Z_k for all integers $6 \le k \equiv 0, 2,$ and $3 \pmod 4$, in which the negative sections constitute a maximum matching. In general, the determination of graceful signed cycles in which there are more than one negative section seems to be a hard problem. For $k = 3$, that Z_3 is graceful is noted in **[7]**. For $k = 4$, that Z_4 is not graceful when the negative sections form a maximum matching is noted in **[3]**. For $k = 5$, that no signed cycle Z_5 on C_5 is graceful follows from Theorem 1 **[3]**. Thus, we have completely characterized the values $k \equiv 0, 2,$ and $3 \pmod 4$ for which Z_k is graceful when the all-negative subsigraph of Z_k forms a maximum matching.

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