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GRACEFUL SIGNED GRAPHS : V THE CASE OF UNION OF SIGNED CYCLES OF LENGTH THREE WITH ONE VERTEX IN COMMON

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ABSTRACT

In this paper, we give a necessary condition for signed graphs on union of n copies of signed cycles of length three with one vertex in common to be graceful and also prove the sufficiency part of the theorem partially in certain special cases.

1. Introduction

For standard terminology and notation in graph theory we follow D.B. West [14] and for that of signed graphs (abbreviated henceforth as sigraphs) we follow G.T. Chartrand [10] and T. Zaslavsky [15,16]. Additional terms will be defined as and when necessary.

A sigraph is an ordered pair S=(G,s), where G=(V,E) is a (p,q)-graph called its underlying graph and s:E—{+,–} is a function from the set of edges to the set [+,–} called a signing of G; hence an edge receiving '+' ('–') in the singing is said to be positive (negative) . Let $E^{+}(S)$ and $E^{-}(S)$ denote respectively, the sets of positive and negative edges of S. By a (p,m,n)- sigraph we mean a sigraph S having p vertices, m positive edges and n negative edges, so that m+n=q. We shall regard graph as sigraph in which all the edges are positive and call it all-positive sigraph (all-negative sigraph is defined similarly) A sigraph is said to be homogeneous if it is either all-positive or all-negative and heterogeneous otherwise.

If an injection f assigns distinct labels to the vertices of a (p,m,n) - sigraph S from the set {0,1,...,q=m+n} such that when each edge $uv \in E(S) := E^{+}(S) \cup E^{-}(S)$ is

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assigned $gf(uv) = s(uv)|f(u) - f(v)|$ the q edges receive the labels form the set {1,2,...,m,–1,–2,.....,–n}; such a function f is called a graceful labeling of S. The sigraph which admits such a labeling is called a graceful sigraph [1-6]. Note, that if n=0 then the above definition coincides with that of a graceful graph given by S.W. Golomb [11] and A. Rosa [13].

K.M. Koh et al [12] had shown the condition when a graph consisting of n copies of either $K_{p,q}$ or C_{4p} with a common vertex is graceful. They also discussed the following problem: For what values of n and p is the graph C_p^n (also called friendship graph when p=3 see[7] also see [14]; p. 433) consisting of n copies of C_p with a common vertex graceful?

The following well known results for graphs are useful in our investigation and they have been cited in order of their appearance in the literature :

Theorem 1 [7,8] : The graph C_3^t is graceful if and only if $t \equiv 0$ or 1 (mod 4.)

Theorem 2 [9]: One point union of any two cycles (i.e., union of two cycles with one vertex in common) is graceful when the number of eges $q \equiv 0$ or 3 (mod 4).

Theorem 3 [12]: The graph C_n^t is graceful if and only if nt $\equiv 0$ or 3 (mod 4).

Theorem 4 [1]: If a (p,m,n)-sigraph S=(G,s) whose underlying graph G is eulerian is graceful then $m^2 + n^2 + m + n \equiv 0$ (mod 4).

Here and henceforth Z_3^k, $k \geq 2$ denotes any sigraph on C_3^k. A collection of independent edges in a graph G is called a matching of G. Our main aim in this paper is to set up a condition for values of p,m, n and k so that the sigraph Z_3^k is graceful.

2. The Main Results

Theorem 5: If a sigraph Z_3^k is graceful then any one of the following condition holds:

(i) $k \equiv 0$ (mod 4) and n is even,
(ii) $k \equiv 1$ (mod 4)
(iii) $k \equiv 2$ (mod 4) and n is odd,

Proof: The number of vertices in Z_3^k is 2k +1 and the number of edges q=m+n=3k. By Theorem 4 above, we have

$$m^2 + n^2 + m + n \equiv 0 \text{ (mod 4)} \quad (1)$$

and this when applied to Z_3^k yields

$$(3k-n)^2 + n^2 + 3k \equiv 0 \text{ (mod 4)}$$

$$\Rightarrow 2n^2 - 6nk + 3k(3k+1) \equiv 0 \text{ (mod 4)}$$

$$\Rightarrow n^2 - 3nk + \frac{3k(3k+1)}{2} \equiv 0(mod\,2)$$

$$\Rightarrow n(n-3k) + \frac{3k(3k+1}{2} \equiv 0(mod\,2). \quad (2)$$

Suppose k=4r +3 where r is a positive integer. Then the above expression will become

$$n(n-12r-9) + \frac{(12r+9)(12r+10)}{2} \equiv 0(mod\,2)$$

$$\Rightarrow n(n-12r-9) + (12r+9)(6r+5) \equiv 0(mod\,2) \quad (3)$$

If $r = 2x\text{-}1$ for a positive integer x, then

$$n(n-24x+3) + (24x-3)(12x-1) \equiv 0\,(mod\,2)$$

which is a contradiction for any value of n. Therefore, r can't be odd. Hence, let r=2x for some positive integer x. Then, from (3) we have

$$n(n-24x-9) + (24x+9)(12x+5) \equiv 0\,(mod\,2)$$

which is again a contradiction. Thus, we conclude that $k \not\equiv 3(mod\,4)$. Now let k=4r where r is a positive integer. Then from (2) we have

$$n(n-12r) + 6r(12r+1) \equiv 0(mod\,2)$$

which implies that n must be even . Thus $k \equiv 0(mod\,4)$. If $k \equiv 1$ (mod 4) then from (2) it is clear that n can be even or odd. Similarly , if $k \equiv 2(mod\,4)$ then (2) implies that n is odd. Thus, completing the proof.

Sufficiency of Theorem 5 appears to be a hard problem in general ; yet it seems to be true. Here we consider the sigraph Z_3^k on C_3^k in which the adjacent pairs of vertices which are adjacent to the central vertex (i.e., common vertex) of C_3^k called the rim edges are joined by negative edges and prove the sufficiency of Theorem 5 for such sigraphs.

Theorem 6: If $4 \leq k \equiv 0 \pmod 4$ and the sigraph Z_3^k in which the set of rim edges are negative (i.e., they form the maximum matching) then Z_3^k is graceful.

Proof: It is enough to provide a graceful labeling of Z_3^k whose signed structure is as laid down in the hypothesis, with m and n denoting respectively the number of positive and negative edges. Let the central vertex be labelled c can we assign c the number 0 (zero). Accordingly, we define a graceful labeling as follows:
For *k=4* we label the vertices of Z_3^k as shown in Figure 1 (a).

Hence, we let *k=4x*, where $x \geq 2$ be some positive integer. Then we let the end vertices of the negative edges in Z_3^k be labelled by the following pairs of integers in a one-to-one manner:
(i) $(8x, 8x-1)$
(ii) $(6x+r, 6x-r-1),\ 1 \leq r \leq x-1$
(iii) $(3x+r-1, x-r),\ 1 \leq r \leq x-1$
(iv) $(5x-1, x)$
(v) $(2x+r, 2x-r), 1 \leq r \leq x-1$
(vi) $(6x-1, 4x-1)$
(vii) $(7x+r-1, 5x-r-1), 1 \leq r \leq x-1$
(viii) $(6x, 2x)$.

Then, the induced edge labels on the positive edges of Z_3^k constitute the following disjoint sets.
(i) $\{x, 2x, 4x-1, 5x-1, 6x-1, 6x, 8x-1, 8x\}$

(ii) $\{x-r, 2x-r, 2x+r, 3x-r, 5x-r-1, 6x-r-1, 6x+r, 7x+r-1\},\ 1 \leq r \leq r-1$
and the numbers induced on the negative edges of $z^k{}_3$ constitute the following disjoint sets of integers:

(i) $\{1, 3x-1, 2x, 4x\}$

(ii) $\{2r, 2r+1, 2x+2r-1, 2x+2r\},\ 1 \leq r \leq X-1.$

Clearly, the labels on positive and negative edges respectively are all distinct. Thus, the proof is complete (e.g., see Figure 1)

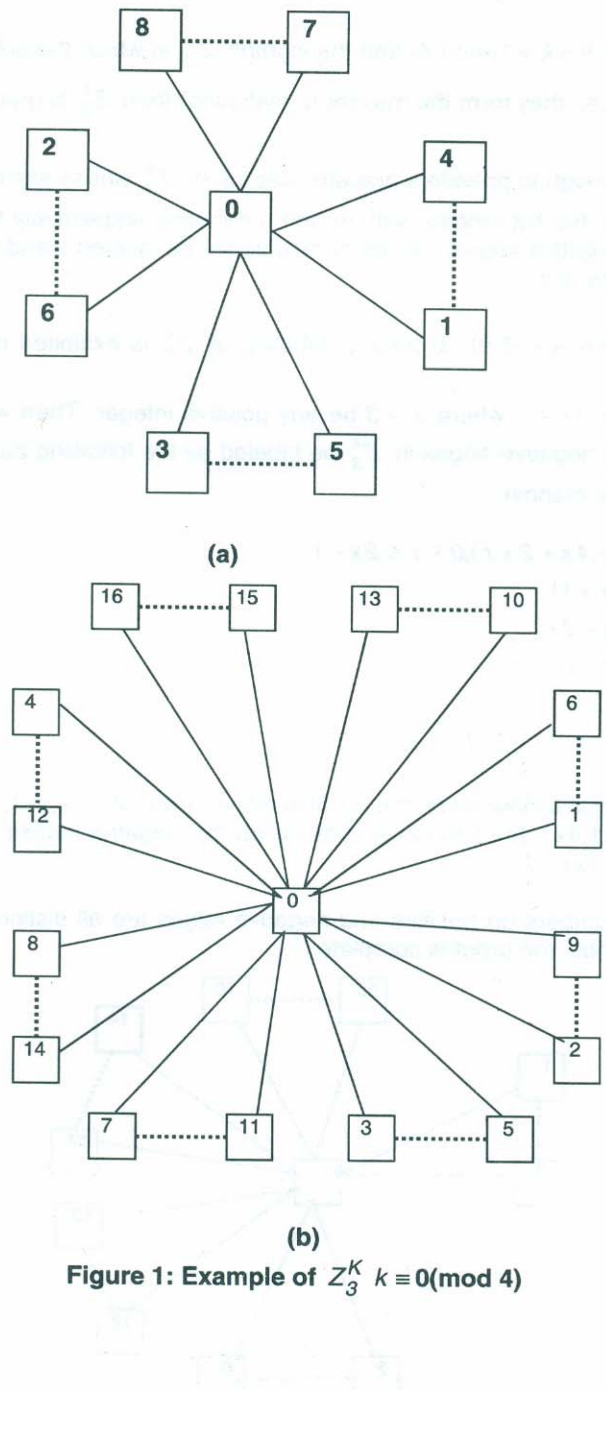


Figure 1: Example of Z_3^K $k \equiv 0 (\text{mod } 4)$

Theorem 7: If $5 \leq k \equiv 1(\text{mod } 4)$ and the sigraph Z_3^k in which the set of rim edges are negative (i.e., they form the maximum matching) then Z_3^k is graceful.

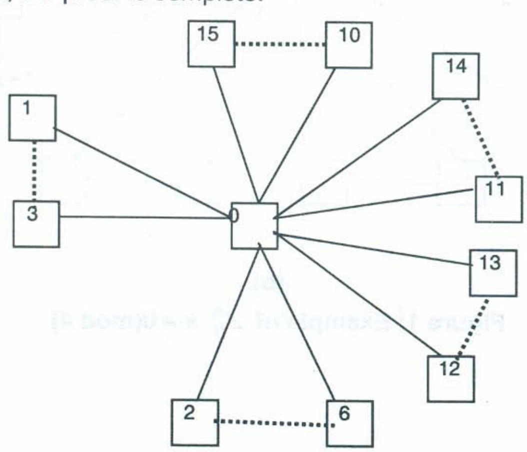
Proof: It is enough to provide a graceful labeling of Z_3^k whose signed structure is as laid down in the hypothesis, with m and n denoting respectively the number of positive and negative edges. Let the central vertex be labeled c and we assign the number 0 (zero) to it.

For each choice $k \in \{5,9\}$, a graceful labeling of Z_3^k is exhibited in Figure 2(a), (b).
Hence, let $k = 4x + 1$, where $x \geq 3$ be any positive integer. Then we let the end vertices of the negative edges in Z_3^k be labeled by the following pairs of integers in a one-to-one manner:

(i) $(8x+2-r, 4x+2+r), 0 \leq r \leq 2x-1$
(ii) $(6x+2, 2x+1)$
(iii) $(4x+1, 2x+2)$
(iv) $(4x+1-r, r), 1 \leq r \leq x$
(v) $(x+1, x+2)$
(vi) $(3x+1-r, x+r+2), 1 \leq r \leq x-2$

Then, the numbers induced on the positive edges are {1,2.....,x,x+1,...,2x,2x+1,....., 3x-1,3x,.....4x-1,4x...,8x+1,8x+2} and those on the negative edges are (1,3...,2x-3,2x-1,...4x-1, 4x+1}.

Clearly, the numbers on positive and negative edges are all distinct as shown in Figure 2(c). Thus, the proof is complete.



(a)

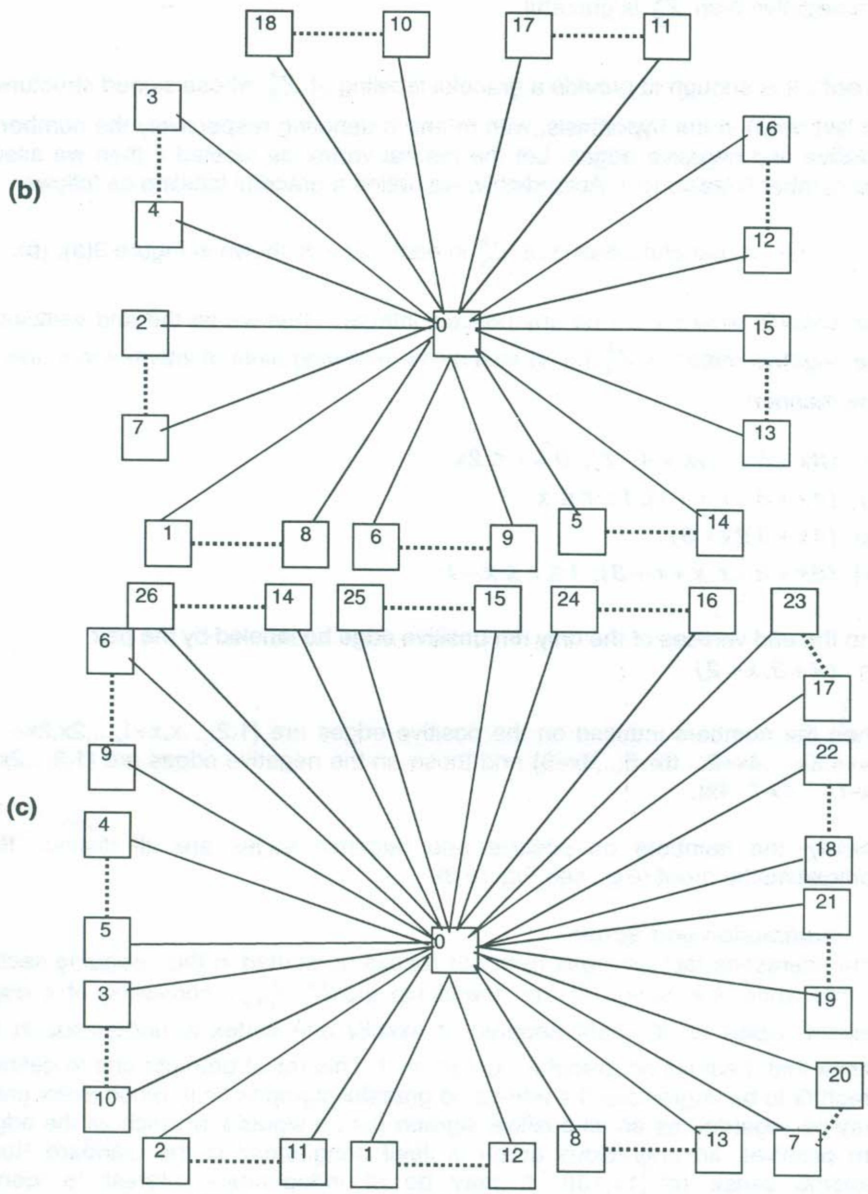


Figure 2: Example Z_3^k, $k \equiv 1 \pmod 4$

Theorem 8: If $2 \leq k \equiv 2 \pmod 4$ and the sigraph Z_3^k in which all but one rim edge are negative then Z_3^k is graceful.

Proof : It is enough to provide a graceful labeling of Z_3^k whose signed structure is as laid down in the hypothesis, with m and n denoting respectively the number of positive and negative edges. Let the central vertex be labeled c then we assign the number 0 (zero) to it. Accordingly, we define a graceful labeling as follows:

If $k \in \{2,6\}$, a graceful labeling of Z_3^k in each case is shown in Figure 3(a), (b).

Let k=4x+2, where x ≥ 2 be any positive integer. Then we let the end vertices of the negative edges in Z_3^k be labeled by the following pairs of integers in a one-to-one manner:

(i) $(8x+5-r, 4x+4+r),\ 0 \leq r \leq 2x$
(ii) $(4x+3-r, r+1),\ 1 \leq r \leq x$
(iii) $(4x+3, 2x+3)$
(iv) $(3x+3-r, x+r+3),\ 1 \leq r \leq x-1$

and the end vertices of the only rim positive edge be labeled by the pair
(v) $(x+3, x+2)$

Then the numbers induced on the positive edges are {1,2,....x,x+1,...,2x,2x+1,... 2x+4,3x,...,4x+2,...6x+5,..,8x+5} and those on the negative edges are {1,3,...2x-3, 2x-1,....,4x-1, 4x}.

Clearly, the numbers on positive and negative edges are all distinct, thus completing the proof (e.g., see Figure 3).

3 Conclusion and Scope

One interesting fact emerges from our results presented in the foregoing section is that when $k \equiv 3 \pmod 4$ the "friendship graph" F_{2k+1} consisting of k edge-disjoint copies of K_3 concatenated at exactly one vertex is ungracious in the sense that it admits no graceful sigraph on it. This result prompts one to define a graph G to be ungracious if it admits no graceful sigraphs on it. Since every graph may be regarded as an all-positive sigraph (i.e., a sigraph in which all the edges are positive), an ungracious graph is itself nongraceful in the standard Rosa-Golomb sense (cf.:[11,13]). It may be of independent interest to identify ungracious graphs.

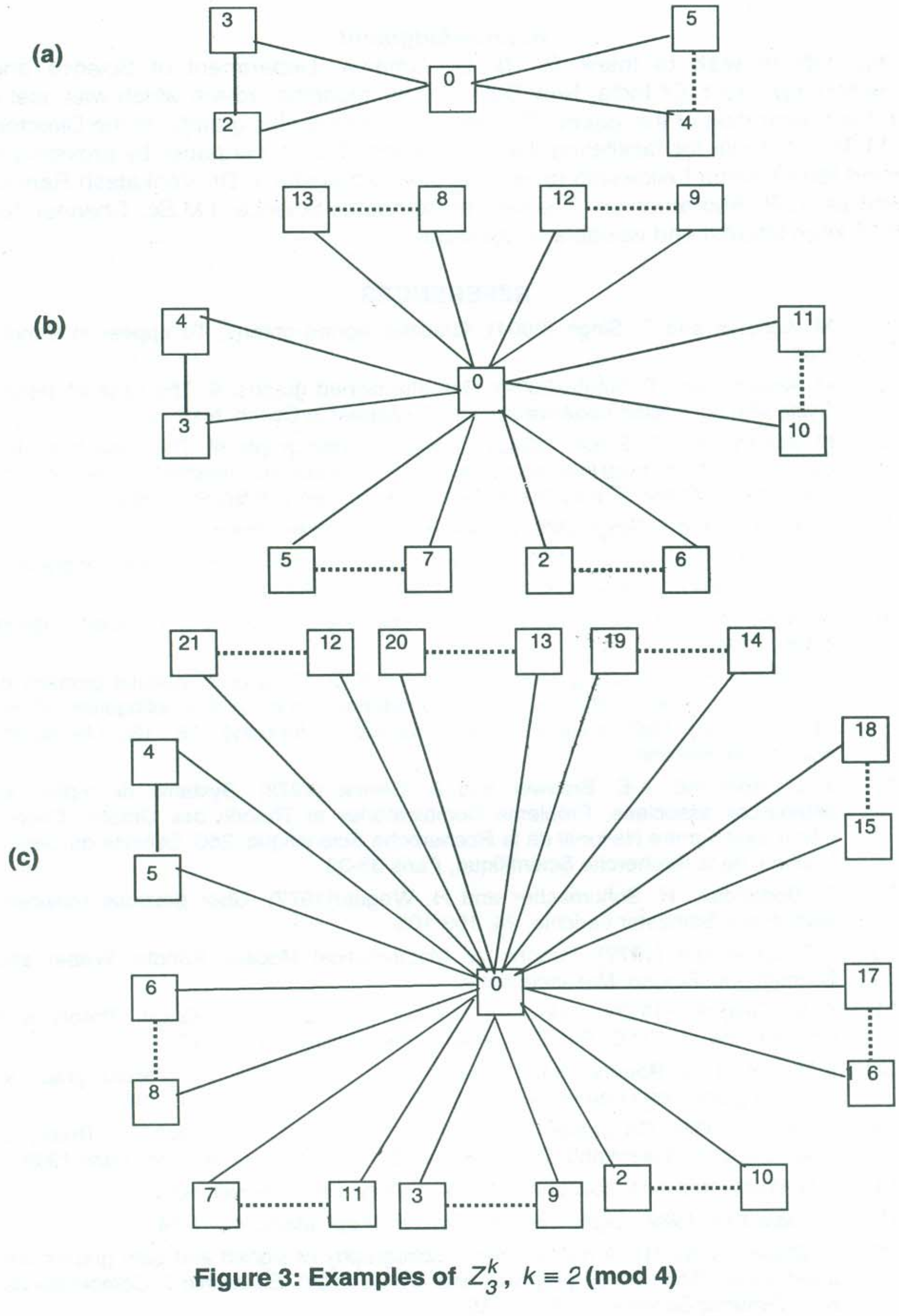


Figure 3: Examples of Z_3^k, $k \equiv 2 \pmod 4$

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