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On Sufficient Conditions for Starlikeness

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Abstract. In this paper, we give some sufficient conditions for analytic functions defined on $|z| < 1$ to have positive real part and in general to satisfy the subordination $p(z) \prec q(z)$. Also some applications of these results are discussed.

Keywords: Starlikeness; Caratheodory function; Differential subordination.

1. Introduction

Let $\mathcal{A}_0$ be the class of all functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ which are analytic in $\Delta = \{z; |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ which are analytic in Δ. The class $\mathcal{P}$ of Caratheodory functions consists of functions $p(z) \in \mathcal{A}$ having positive real part. Recently Nunokawa *et. al.* [2] gave some sufficient conditions for analytic functions in Δ to have positive real part. In this paper, we generalized the results by finding some conditions on $\alpha, \beta, \gamma, \delta$ and $w(z)$ such that each of the following differential subordination implies $p(z) \in \mathcal{P}$:

- $\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta zp'(z) \prec w(z)$,
- $\alpha p(z)^2 + \delta zp(z)p'(z) \prec w(z)$,
- $\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta \frac{zp'(z)}{p(z)} \prec w(z)$,
- $\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta \frac{zp'(z)}{p^2(z)} \prec w(z)$.

Note that the conclusion $p(z) \in \mathcal{P}$ can be written as $p(z) \prec (1+z)/(1-z)$. In this paper, we find sufficient conditions for the subordination $p(z) \prec q(z)$ to hold.

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Our results include the results obtained by Nunokawa et. al.[2]. We also give some application of our results to obtain sufficient conditions for starlikeness. We need the following result of Miller and Mocanu[1] to prove our main result:

Theorem A. *Let $q(z)$ be univalent in the unit disk $\triangle$ and θ and ϕ be analytic in a domain D containing $q(\triangle)$ with $\phi(w) \neq 0$ when $w \in q(\triangle)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
(1) $Q(z)$ is starlike univalent in $\triangle$, and
(2) $\operatorname{Re} \frac{zh'(z)}{Q(z)} > 0$ for $z \in \triangle$.

If $q(z)$ is a analytic in $\triangle$ with $p(0) = q(0)$, $p(D) \in D$ and $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$and $q(z)$ is the best dominant.

2. Caratheodory Functions

We begin with the following:

Theorem 2.1. *Let α, β, γ and δ be complex numbers, $\delta \neq 0$. Let $0 \neq q(z) \in \mathcal{A}$ be univalent in $\triangle$ and satisfy the following conditions for $z \in \triangle$:*
(1) $zq'(z)$ is starlike,
(2) $\operatorname{Re} \left\{ \frac{\alpha}{\delta} + \frac{2\beta}{\delta} q(z) - \frac{\gamma}{\delta q(z)^2} + \left(1 + \frac{zq''(z)}{q'(z)}\right) \right\} > 0$.

If $p(z) \in \mathcal{A}$ satisfies

$$\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta zp'(z) \prec \alpha q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \delta zq'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Let $\theta(w) = \alpha w + \beta w^2 + \frac{\gamma}{w}$ and $\phi(w) = \delta$. Then $\phi(w) \neq 0$ and $\theta(w), \phi(w)$ are analytic in $\mathbb{C} - \{0\}$. Let the function $Q(z)$ and $h(z)$ be defined by

$$Q(z) = zq'(z)\phi(q(z)) = \delta zq'(z),$$

$$h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \delta zq'(z).$$

Clearly $Q(z)$ is starlike and

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ \frac{\alpha}{\delta} + \frac{2\beta}{\delta} q(z) - \frac{\gamma}{\delta q(z)^2} + \left(1 + \frac{zq''(z)}{q'(z)}\right) \right\} > 0.$$

The differential subordination

$$\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta zp'(z) \prec \alpha q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \delta zq'(z)$$

becomes

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)).$$

The result follows by an application of Theorem A. ∎

By taking α, β, γ and δ to be real, we have the following result:

Corollary 2.2. *Let α, γ and δ be positive numbers. If $0 \neq p(z) \in \mathcal{A}$ and*

$$\alpha p(z)+\beta p(z)^2+\frac{\gamma}{q(z)}+\delta zp'(z) \prec \alpha\left(\frac{1+z}{1-z}\right)+\beta\left(\frac{1+z}{1-z}\right)^2+\gamma\left(\frac{1-z}{1+z}\right)+\frac{2\delta z}{(1-z)^2},$$

then $\operatorname{Re} p(z) > 0$.

Proof. This result follows from Theorem 2.1 by taking $q(z) = \frac{1+z}{1-z}$. Then the function $Q(z) = 2\delta z/(1-z)^2$ is clearly starlike. Since

$$\frac{\alpha}{\delta} + \frac{2\beta}{\delta}q(z) - \frac{\gamma}{\delta q(z)^2} + \frac{zQ'(z)}{Q(z)} = \frac{\alpha}{\delta} + \frac{2\beta}{\delta}\left(\frac{1+z}{1-z}\right) - \frac{\gamma}{\delta}\left(\frac{1-z}{1+z}\right)^2 + \frac{1+z}{1-z}$$

we have, with $z = e^{i\theta}$

$$\operatorname{Re}\left\{\frac{\alpha}{\delta} + \frac{2\beta}{\delta}q(z) - \frac{\gamma}{\delta q(z)^2} + \frac{zQ'(z)}{Q(z)}\right\} = \frac{\alpha}{\delta} + \frac{\gamma}{\delta}\tan^2\frac{\theta}{2} \geq 0.$$

∎

Let $\gamma = 0$ and $q(z)$ be the function defined by

$$q(z) = \frac{1+Az}{1+Bz}, \qquad -1 < B < A \leq 1.$$

Then we have

$$zq'(z) = \frac{(A-B)z}{(1+Bz)^2}$$

Let $g(z) = zq'(z)$. Then

$$\frac{zg'(z)}{g(z)} = \frac{1-Bz}{1+Bz}.$$

If $z = re^{i\theta}$, we have

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = \frac{1-B^2r^2}{1+B^2r^2+2Br\cos\theta} \geq 0.$$

Hence $zq'(z)$ is starlike in $\triangle$. Also it follows that

$$\begin{aligned}\frac{\alpha}{\delta} + \frac{2\beta}{\delta}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right) &= \frac{[1+(\alpha+2\beta)/\delta] + [2\beta A/\delta - (1-\alpha/\delta)B]z}{1+Bz} \\ &= \frac{u+vz}{1+Bz},\end{aligned}$$

where $u = 1+(\alpha+2\beta)/\delta$ and $v = 2\beta A/\delta-(1-\alpha/\delta)B$. The function $w(z) = \frac{u+vz}{1+Bz}$ maps $\triangle$ into the disk

$$\left| w - \frac{\overline{u} - B\overline{v}}{1-B^2} \right| \leq \frac{|v - B\overline{u}|}{1-B^2}.$$

Therefore

$$\mathrm{Re}\left\{ \frac{\alpha}{\delta} + \frac{2\beta}{\delta} q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right) \right\} = \frac{\mathrm{Re}(\overline{u} - B\overline{v}) - |v - B\overline{u}|}{1-B^2} \geq 0$$

provided

$$\mathrm{Re}(\overline{u} - B\overline{v}) \geq |v - B\overline{u}|$$

or

$$\mathrm{Re}(u - Bv) \geq |v - B\overline{u}|.$$

Hence we have the following result:

Corollary 2.3. *Let $-1 < B < A \leq 1$. Let α, β and δ satisfy $Re(u-Bv) \geq |v-B\overline{u}|$ where $u = 1 + (\alpha+2\beta)/\delta$ and $v = 2\beta A/\delta - (1-\alpha/\delta)B$. If $p(z) \in \mathcal{A}$ and*

$$\alpha p(z) + \beta p(z)^2 + \delta z p'(z) \prec \alpha\left(\frac{1+Az}{1+Bz}\right) + \beta\left(\frac{1+Az}{1+Bz}\right)^2 + \delta\frac{(A-B)z}{(1+Bz)^2},$$

then $p(z) \prec \frac{1+Az}{1+Bz}$.

By taking $\alpha = 0, A = 1, B = -1$ and β and δ to be real, then we have the following result of Nunokawa *et. al.*[2]:

Corollary 2.4. *Let $1 + 2\beta/\delta > 0$. If $p(z) \in \mathcal{A}$ and*

$$\beta p(z)^2 + \delta z p'(z) \prec \beta\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2},$$

then $\mathrm{Re}\, p(z) > 0$.

If $q(z)$ is a convex function that maps $\triangle$ onto a region in the right half plane, then the conditions of Theorem 2.1 are satisfied by $q(z)$ whenever $\alpha\delta > 0,\ \beta\delta > 0$, and $\gamma = 0$. By taking

$$q(z) = \left(\frac{1+z}{1-z}\right)^\lambda, \quad 0 < \lambda \leq 1,$$

we have the following:

Corollary 2.5. *Let $\alpha\delta,\ \beta\delta > 0$. If $p(z) \in \mathcal{A}$ and*

$$\alpha p(z) + \beta p(z)^2 + \delta z p'(z) \prec \alpha\left(\frac{1+z}{1-z}\right)^\lambda + \beta\left(\frac{1+z}{1-z}\right)^{2\lambda} + \frac{2\delta\lambda z}{1-z^2}\left(\frac{1+z}{1-z}\right)^\lambda,$$

then $|\text{Arg}\, p(z)| \le \lambda\pi/2$.

Theorem 2.6. *Let* α *and* δ *be complex numbers,* $\delta \neq 0$. *Let* $q(z) \in \mathcal{A}$ *be univalent in* $\triangle$ *and satisfy the following conditions for* $z \in \triangle$:

(1) *Let* $Q(z) = \delta z q(z) q'(z)$ *be starlike,*

(2) $\text{Re}\,\{\frac{2\alpha}{\delta} + zQ'(z)/Q(z)\} > 0$.

If $p(z) \in \mathcal{A}$ *satisfies*

$$\alpha p(z)^2 + \delta z p(z) p'(z) \prec \alpha q(z)^2 + \delta z q(z) q'(z),$$

then $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

The proof of this theorem is similar to that of Theorem 2.1 and therefore omitted.

Let $q(z)$ be the function

$$q(z) = \left(\frac{1+Az}{1+Bz}\right)^{\frac{1}{2}}, \qquad -1 < B < A \le 1, \qquad 0 < \lambda \le 1.$$

Then we have

$$zq(z)q'(z) = \frac{(A-B)z}{2(1+Bz)^2}.$$

Let $Q(z) = zq(z)q'(z)$. Then

$$\frac{zQ'(z)}{Q(z)} = \frac{1-Bz}{1+Bz}.$$

If $z = re^{i\theta}$, we have

$$\text{Re}\frac{zQ'(z)}{Q(z)} = \frac{1-B^2r^2}{1+B^2r^2+2Br\cos\theta} \ge 0.$$

Hence $zq(z)q'(z)$ is starlike in $\triangle$. Thus, we have

$$\begin{aligned}\frac{2\alpha}{\delta} + \frac{zQ'(z)}{Q(z)} &= \frac{(2\alpha/\delta+1)+(2\alpha/\delta-1)Bz}{1+Bz}\\ &= \frac{u+vz}{1+Bz},\end{aligned}$$

where $u = (2\alpha/\delta + 1)$ and $v = (2\alpha/\delta - 1)B$. The function $w(z) = \frac{u+vz}{1+Bz}$ maps $\triangle$ into the disk

$$\left|w - \frac{\overline{u} - B\overline{v}}{1-B^2}\right| \le \frac{|v - B\overline{u}|}{1-B^2}.$$

Therefore

$$\text{Re}\left[\frac{2\alpha}{\delta} + \frac{zQ'(z)}{Q(z)}\right] = \frac{\text{Re}(\overline{u} - B\overline{v}) - |v - B\overline{u}|}{1-B^2} \ge 0$$

provided that

$$\mathrm{Re}(\overline{u} - B\overline{v}) \geq |v - B\overline{u}|$$

or

$$\mathrm{Re}(u - Bv) \geq |v - B\overline{u}|.$$

Therefore we have the following result:

Corollary 2.7. *Let $-1 < B < A \leq 1$. Let α, β and δ satisfy $Re(u-Bv) \geq |v-B\overline{u}|$ where $u = (2\alpha/\delta + 1)$ and $v = (2\alpha/\delta - 1)B$. If $p(z) \in \mathcal{A}$ and*

$$\alpha p(z)^2 + \delta z p(z) p'(z) \prec \alpha \left(\frac{1+Az}{1+Bz}\right) + \frac{\delta}{2} \frac{(A-B)z}{(1+Bz)^2},$$

then $p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\frac{1}{2}}$.

By taking $A = 1, B = -1$ and α, β, δ to be real, we have the following result:

Corollary 2.8. *If $p(z) \in \mathcal{A}$ and*

$$\alpha p(z)^2 + \delta z p(z) p'(z) \prec \alpha \frac{1+z}{1-z} + \frac{\delta z}{(1-z)^2},$$

then $p(z) \prec \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}$.

Theorem 2.9. *Let α, β, γ and δ be complex numbers, $\delta \neq 0$. Let $0 \neq q(z) \in \mathcal{A}$ be univalent in $\triangle$ and satisfy the following conditions for $z \in \triangle$:*

(1) Let $Q(z) = \delta z q'(z)/q(z)$ be starlike,

(2) $\mathrm{Re}\left\{\frac{\alpha}{\delta} q(z) + \frac{2\beta}{\delta} q^2(z) - \frac{\gamma}{\delta q(z)} + zQ'(z)/Q(z)\right\} > 0$.

If $p(z) \in \mathcal{A}$ satisfies

$$\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta \frac{zp'(z)}{p(z)} \prec \alpha q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \delta \frac{zq'(z)}{q(z)},$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Let $\theta(w) = \alpha w + \beta w^2 + \frac{\gamma}{w}$ and $\phi(w) = \delta/w$. Then $\phi(w) \neq 0$ and $\theta(w)$, $\phi(w)$ are analytic in $\mathbb{C} - \{0\}$ which contains $q(\triangle)$. Let the function $Q(z)$ and $h(z)$ be defined by

$$Q(z) = zq'(z)\phi(q(z)) = \delta \frac{zq'(z)}{q(z)},$$

$$h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \delta \frac{zq'(z)}{q(z)}.$$

Clearly $Q(z)$ is starlike and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left\{\frac{\alpha}{\delta}q(z) + \frac{2\beta}{\delta}q^2(z) - \frac{\gamma}{\delta q(z)} + \frac{zQ'(z)}{Q(z)}\right\} > 0.$$

The differential subordination

$$\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)} \prec \alpha q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q(z)}$$

becomes

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

and the result follows, by using Theorem A. ∎

Corollary 2.10. *Let β and δ be positive numbers. If $0 \neq p(z) \in \chi$ satisfies*

$$\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} - \delta\frac{zp'(z)}{p(z)} \prec \alpha\frac{1+z}{1-z} + \beta\left(\frac{1+z}{1-z}\right)^2 + \gamma\frac{1-z}{1+z} - \frac{2\delta z}{1-z^2}$$

then $\operatorname{Re} p(z) > 0$.

Proof. The result follows from Theorem 2.9 by taking $q(z) = \frac{1+z}{1-z}$ and replacing δ by $-\delta$. Then the function $Q(z) = -2\delta z/1 - z^2$ is clearly starlike. Since

$$-\frac{\alpha}{\delta}q(z) - \frac{2\beta}{\delta}q^2(z) + \frac{\gamma}{\delta q(z)} + \frac{zQ'(z)}{Q(z)}$$

$$= -\frac{\alpha}{\delta}\left(\frac{1+z}{1-z}\right) - \frac{2\beta}{\delta}\left(\frac{1+z}{1-z}\right)^2 + \frac{\gamma}{\delta}\left(\frac{1-z}{1+z}\right) + \frac{1+z^2}{1-z^2},$$

we have, with $z = e^{i\theta}$

$$\operatorname{Re}\left\{-\frac{\alpha}{\delta}q(z) - \frac{2\beta}{\delta}q^2(z) + \frac{\gamma}{\delta q(z)} + zQ'(z)/Q(z)\right\} = \frac{2\beta}{\delta}\cot^2\frac{\theta}{2} \geq 0.$$

∎

By taking $\alpha = 1, \beta = \gamma = 0$ and $\delta = -1$, then we have the following result of Nunokawa *et. al.*[2]:

Corollary 2.11. *If $0 \neq p(z) \in \mathcal{A}$ and*

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+4z+z^2}{1-z^2},$$

then $\operatorname{Re} p(z) > 0$.

Theorem 2.12. *Let α, β and δ be complex numbers, $\delta \neq 0$. Let $0 \neq q(z) \in \mathcal{A}$ be*

(1) *Let* $Q(z) = \delta zq'(z)/q^2(z)$ *be starlike,*
(2) $\operatorname{Re}\left\{\frac{\alpha}{\delta}q^2(z) + \frac{2\beta}{\delta}q^3(z) - \frac{\gamma}{\delta} + zQ'(z)/Q(z)\right\} > 0.$

If $p(z) \in \mathcal{A}$ *satisfies*

$$\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)} \prec \alpha q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q^2(z)},$$

then $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

The proof of this theorem is similar to that of Theorem 2.9 and therefore it is omitted.

Corollary 2.13. *Let* α, γ *and* δ *be positive numbers. If* $0 \neq p(z) \in \chi$ *satisfies*

$$\alpha p(z) + \beta p(z)^2 + \frac{\gamma}{p(z)} - \delta\frac{zp'(z)}{p^2(z)} \prec \alpha\frac{1+z}{1-z} + \beta\left(\frac{1+z}{1-z}\right)^2 + \gamma\frac{1-z}{1+z} - \frac{2\delta z}{(1+z)^2},$$

then $\operatorname{Re} p(z) > 0$.

Proof. The result follows from the Theorem 2.12 by taking $q(z) = \frac{1+z}{1-z}$ and replacing δ by $-\delta$. Then the function $Q(z) = -2\delta z/(1+z)^2$ is clearly starlike. Since

$$-\frac{\alpha}{\delta}q^2(z) - \frac{2\beta}{\delta}q^3(z) + \frac{\gamma}{\delta} + \frac{zQ'(z)}{Q(z)} = -\frac{\alpha}{\delta}\left(\frac{1+z}{1-z}\right)^2 - \frac{2\beta}{\delta}\left(\frac{1+z}{1-z}\right)^3 + \frac{\gamma}{\delta} + \frac{1-z}{1+z}$$

we have, with $z = e^{i\theta}$

$$\operatorname{Re}\left\{-\frac{\alpha}{\delta}q^2(z) - \frac{2\beta}{\delta}q^3(z) + \frac{\gamma}{\delta} + \frac{zQ'(z)}{Q(z)}\right\} = \frac{\alpha}{\delta}\cot^2\frac{\theta}{2} + \frac{\gamma}{\delta} \geq 0.$$

∎

By taking $\alpha = \beta = \gamma = 0$ in Theorem 2.12, we have the following:

Corollary 2.14. *Let* $q(z) \in \mathcal{A}$ *be univalent in* Δ, $q(z) \neq 0$. *Let* $zq'(z)/q(z)^2$ *be starlike. If* $p(z) \in \mathcal{A}$ *satisfies*

$$\frac{zp'(z)}{p(z)^2} \prec \frac{zq'(z)}{q(z)^2},$$

then $p(z) \prec q(z)$. *The dominant* q *is the best dominant.*

3. Starlikeness Criteria

In this section, we give application of our results for getting sufficient conditions for starlikeness. Let $-1 < B < A \le 1$. The class $S^*[A, B]$ consists of functions $f \in \mathcal{A}_0$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \qquad z \in \triangle.$$

In particular $S^*[1, -1] = S^*$, the class of starlike functions. For the class $S^*[A, B]$, we have the following:

Theorem 3.1. *Let $-1 < B < A \le 1$. Let α, β and δ satisfy $Re(u - Bv) \ge |v - B\overline{u}|$ where $u = 1 + \alpha + 2\beta/\delta$ and $v = 2\beta A/\delta - (1-\alpha)B$. If $f(z) \in \mathcal{A}_0$ and*

$$\frac{zf'(z)}{f(z)}\left[\alpha + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]$$

$$\prec \alpha\left(\frac{1+Az}{1+Bz}\right) + \beta\left(\frac{1+Az}{1+Bz}\right)^2 + \delta\frac{(A-B)z}{(1+Bz)^2},$$

then $f(z) \in S^*[A, B]$.

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$. Then a computation shows that

$$\frac{zp'(z)}{p(z)} + p(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

Therefore, it follows that

$$\frac{zf'(z)}{f(z)}\left[\alpha + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]$$

$$\begin{aligned} &= p(z)\left[\alpha + (\beta-\delta)p(z) + \delta\left(\frac{zp'(z)}{p(z)} + p(z)\right)\right] \\ &= \alpha p(z) + \beta p(z)^2 + \delta z p'(z) \\ &\prec \alpha\left(\frac{1+Az}{1+Bz}\right) + \beta\left(\frac{1+Az}{1+Bz}\right)^2 + \delta\frac{(A-B)z}{(1+Bz)^2}. \end{aligned}$$

Using Corollary 2.3, we have the result. ∎

As a special case, we have the following:

Corollary 3.2. *Let $1 + 2\beta/\delta > 0$. If $f(z) \in \mathcal{A}_0$ and*

$$\frac{zf'(z)}{f(z)}\left[(\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \beta\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2},$$

then $f(z) \in S^*$.

The class $SS^*(\lambda)$ of strongly starlike functions of order λ consists of functions $f \in \mathcal{A}_0$ satisfying

$$\left|\text{Arg } \left(\frac{zf'(z)}{f(z)}\right)\right| \leq \frac{\lambda\pi}{2}.$$

Which is equivalent to the following:

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\lambda}, \qquad z \in \triangle .$$

For this class, we have the following :

Theorem 3.3. *Let $-1 < B < A \leq 1$. Let α, β and δ satisfy $Re(u - Bv) \geq |v - B\overline{u}|$ where $u = (2\alpha/\delta + 1)$ and $v = (2\alpha/\delta - 1)B$. If $f(z) \in \mathcal{A}_0$ and*

$$\alpha\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{\delta z}{2}\left[\left(\frac{zf'(z)}{f(z)}\right)^2\right]' \prec \alpha\frac{1+Az}{1+Bz} + \frac{\delta}{2}\frac{(A-B)z}{(1+Bz)^2},$$

then $\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\frac{1}{2}}$.

The result can be obtained by using Corollary 2.7, with

$$p(z) = zf'(z)/f(z).$$

As a special case, we have

Corollary 3.4. *If $f(z) \in \mathcal{A}_0$ and*

$$\alpha\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{\delta z}{2}\left[\left(\frac{zf'(z)}{f(z)}\right)^2\right]' \prec \alpha\frac{1+z}{1-z} + \frac{\delta z}{(1-z)^2},$$

then $\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}$ or equivalently $f(z)$ is strongly starlike of order 1/2.

Using Corollary 2.10 , with $p(z) = zf'(z)/f(z)$, we have the following result:

Theorem 3.5. *Let β and δ be positive numbers. If $f(z) \in \mathcal{A}_0$ satisfies*

$$(\alpha+\delta)\frac{zf'(z)}{f(z)} + \beta\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{\gamma f(z)}{zf'(z)} - \delta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \alpha\frac{1+z}{1-z} + \beta\left(\frac{1+z}{1-z}\right)^2 + \gamma\frac{1-z}{1+z} - \frac{2\delta z}{1-z^2},$$

then $f(z) \in S^*$.

Using Corollary 2.13 , with $p(z) = zf'(z)/f(z)$, we have the following result:

Theorem 3.6. *Let α, γ and δ be positive numbers. If $f(z) \in \mathcal{A}_0$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} + \beta\left(\frac{zf'(z)}{f(z)}\right)^2 + \gamma\frac{f(z)}{zf'(z)} - \delta\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}$$

$$\prec \alpha\frac{1+z}{1-z} + \beta\left(\frac{1+z}{1-z}\right)^2 + \gamma\frac{1-z}{1+z} - \frac{2\delta z}{(1+z)^2} - \delta,$$

then $f(z) \in S^*$.

By taking $\alpha = \beta = \gamma = 0$, we have the following result of Obradovič and Tuneski[3]:

Corollary 3.7. *If $f(z) \in \mathcal{A}_0$ satisfies*

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{2z}{(1+z)^2},$$

then $f(z) \in S^*$.

Using Corollary 2.2 , with $\alpha = \beta = 0$ and $p(z) = zf'(z)/f(z)$, we have the following result:

Theorem 3.8. *Let γ and δ be positive numbers. If $f(z) \in \mathcal{A}_0$ satisfies*

$$\frac{\gamma f(z)}{zf'(z)} + \frac{\delta zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \gamma\frac{1-z}{1+z} + \frac{2\delta z}{(1-z)^2},$$

then $f(z) \in S^*$.

References

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