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On a class of analytic functions involving Carlson-Shaffer linear operator (**)

1 - Introduction

Let $\mathcal{A}_0$ be the class of *analytic* functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}).$$

Let $\mathcal{A}$ be the class of all analytic functions $p(z)$ in Δ with $p(0) = 1$. Let the function $\varphi(a, c; z)$ be given by

$$\varphi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c \neq 0, -1, -2, \ldots ; z \in \Delta),$$

where $(x)_n$ is the *Pochhammer symbol* defined by

$$(x)_n := \begin{cases} 1, & n = 0 ; \\ x(x+1)(x+2)\ldots(x+n-1), & n \in \mathbb{N} := \{1, 2, 3, \ldots\}. \end{cases}$$

Corresponding to the function $\varphi(a, c; z)$, Carlson and Shaffer [1] introduced a li-

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near operator $L(a, c)$, which is defined by the following Hadamard product (or convolution):

$$L(a, c)\, f(z) := \varphi(a, c; z) * f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_n z^{n+1}.$$

We note that

$$L(a, a)\, f(z) = f(z), \quad L(2, 1)\, f(z) = zf'(z), \quad L(n+1, 1)\, f(z) = D^n f(z),$$

where $D^n f(z)$ is the *Ruscheweyh derivative* of $f(z)$.

Over the past few decades, several authors have obtained criteria for univalence and starlikeness involving the functionals $\dfrac{zf'(z)}{f(z)}$ and $1 + \dfrac{zf''(z)}{f'(z)}$ or $\dfrac{D^{n+1}f(z)}{D^n f(z)}$ and $\dfrac{D^{n+2}f(z)}{D^{n+1}f(z)}$. See Singh [**7**] and the references therein. Recently Patel and Sahoo [**6**] have studied certain classes defined by the Carlson-Shaffer linear operator $L(a, c)$. Liu and Owa [**2**] studied the operator for a class of multivalent functions. In this paper, we obtain sufficient conditions involving $\dfrac{L(a+1, c)f(z)}{L(a, c)f(z)}$ and $\dfrac{L(a+2, c)f(z)}{L(a+1, c)f(z)}$ for certain analytic function $f(z)$ to satisfy the subordination

$$\frac{L(a+1, c)\, f(z)}{L(a, c)f(z)} \prec q(z).$$

In our present investigation, we need the following Theorem of Miller and Mocanu to prove our main results:

Theorem 1.1. ([**3**], Theorem 3.4h, p. 132) *Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$ with $\varphi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) := zq'(z)\, \varphi(q(z))$, $h(z) := \vartheta(q(z)) + Q(z)$. Suppose that either*

(i) *$h(z)$ is convex, or*

(ii) *$Q(z)$ is starlike univalent in Δ. In addition, assume that*

$$\Re \frac{zh'(z)}{Q(z)} > 0 \quad (z \in \Delta).$$

If $p(z)$ is analytic in Δ, with $p(0) = q(0)$, $p(\Delta) \subset D$ and

$$\vartheta(p(z)) + zp'(z)\, \varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\, \varphi(q(z)) = h(z), \tag{1.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2 – Main results

By making use of Theorem 1.1, we first prove the following:

Theorem 2.1. *Let α, β and γ be complex numbers, $\gamma \neq 0$ and $a \neq -1$. Let $q(z) \in \mathcal{A}$ be convex univalent in Δ and*

$$\Re\left\{ \frac{\alpha(a+1)+\gamma}{\gamma} + \frac{2[\beta(a+1)+a\gamma]}{\gamma} q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 .$$

If $f(z) \in \mathcal{A}_0$ and

$$\alpha \frac{L(a+1,\, c)\, f(z)}{L(a,\, c)\, f(z)} + \beta\left(\frac{L(a+1,\, c)\, f(z)}{L(a,\, c)\, f(z)}\right)^2 + \gamma \frac{L(a+2,\, c)\, f(z)}{L(a,\, c)\, f(z)} \prec \frac{1}{a+1}\left\{[\alpha(a+1)+\gamma]\, q(z) + [\beta(a+1)+a\gamma]\, q(z)^2 + \gamma z q'(z)\right\}, \tag{2.1}$$

then

$$\frac{L(a+1,\, c)\, f(z)}{L(a,\, c)\, f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{L(a+1,\, c)\, f(z)}{L(a,\, c)\, f(z)} . \tag{2.2}$$

By taking logarithmic derivative of $p(z)$ given by (2.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{z(L(a+1,\, c)\, f(z))'}{L(a+1,\, c)\, f(z)} - \frac{z(L(a,\, c)\, f(z))'}{L(a,\, c)\, f(z)} . \tag{2.3}$$

By using the identity:

$$z(L(a,\, c)\, f(z))' = aL(a+1,\, c)\, f(z) - (a-1)\, L(a,\, c)\, f(z)$$

and (2.2) in (2.3), we obtain

$$\frac{zp'(z)}{p(z)} = (a+1)\, \frac{L(a+2,\, c)\, f(z)}{L(a+1,\, c)\, f(z)} - ap(z) - 1$$

or

$$\frac{L(a+2,\, c)\, f(z)}{L(a+1,\, c)\, f(z)} = \frac{1}{a+1}\left(\frac{zp'(z)}{p(z)} + ap(z) + 1\right). \tag{2.4}$$

Therefore, it follows from (2.2) and (2.4) that

$$\alpha\,\frac{L(a+1,\, c)\, f(z)}{L(a,\, c)\, f(z)} + \beta\left(\frac{L(a+1,\, c)\, f(z)}{L(a,\, c)\, f(z)}\right)^2 + \gamma\,\frac{L(a+2,\, c)\, f(z)}{L(a,\, c)\, f(z)}$$

$$= \alpha p(z) + \beta p(z)^2 + \frac{\gamma}{a+1}\left(zp'(z) + ap^2(z) + p(z)\right)$$

$$= \frac{1}{a+1}\left\{[\alpha(a+1)+\gamma]\,p(z) + [\beta(a+1)+a\gamma]\,p(z)^2 + \gamma zp'(z)\right\},$$

and hence the subordination (2.1) becomes

$$\begin{aligned} &[\alpha(a+1)+\gamma]\,p(z) + [\beta(a+1)+a\gamma]\,p(z)^2 + \gamma zp'(z) \\ &\prec [\alpha(a+1)+\gamma]\,q(z) + [\beta(a+1)+a\gamma]\,q(z)^2 + \gamma zq'(z). \end{aligned} \tag{2.5}$$

This subordination (2.5) is same as (1.1) when the functions ϑ and φ are defined by

$$\vartheta(w) := [\alpha(a+1)+\gamma]\,w + [\beta(a+1)+a\gamma]\,w^2 \quad \text{and} \quad \varphi(w) := \gamma\,.$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $\mathbb{C}$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned} Q(z) &:= zq'(z)\,\varphi(q(z)) = \gamma zq'(z), \\ h(z) &:= \vartheta(q(z)) + Q(z) \\ &= [\alpha(a+1)+\gamma]\,q(z) + [\beta(a+1)+a\gamma]\,q(z)^2 + \gamma zq'(z). \end{aligned}$$

By our hypothesis of the Theorem 2.1, we see that $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\alpha(a+1)+\gamma}{\gamma} + \frac{2[\beta(a+1)+a\gamma]}{\gamma}\,q(z) + \left(1+\frac{zq''(z)}{q'(z)}\right)\right\} > 0\,.$$

Since ϑ and φ satisfy the conditions of Theorem 1.1, our Theorem 2.1 follows by an application of Theorem 1.1. ■

By taking $a = n + 1$ and $c = 1$ in Theorem 2.1, we have the following result:

Corollary 2.2. *Let α, β and γ be complex numbers, $\gamma \neq 0$. Let $q(z) \in \mathcal{A}$ be convex univalent in Δ and*

$$\Re\left\{\frac{\alpha(n+2)+\gamma}{\gamma} + \frac{2[\beta(n+2)+(n+1)\gamma]}{\gamma}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\frac{D^{n+1}f(z)}{D^n f(z)}\left[\alpha + \beta\frac{D^{n+1}f(z)}{D^n f(z)} + \gamma\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right]$$

$$\prec \frac{1}{n+2}\left\{[\alpha(n+2)+\gamma]\,q(z) + [\beta(n+2)+(n+1)\gamma]\,q(z)^2 + \gamma zq'(z)\right\},$$

then

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

By taking $a = c = 1$ in Theorem 2.1, we have the following result:

Corollary 2.3. *Let α, β and γ be complex numbers, $\gamma \neq 0$. Let $q(z) \in \mathcal{A}$ be convex univalent in Δ and*

$$\Re\left\{\frac{2\alpha+\gamma}{\gamma} + \frac{2[2\beta+\gamma]}{\gamma}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\frac{zf'(z)}{f(z)}\left[\left(\alpha + \frac{\gamma}{2}\right) + \beta\frac{zf'(z)}{f(z)} + \frac{\gamma}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]$$

$$\prec \left(\alpha + \frac{\gamma}{2}\right)q(z) + \left(\beta + \frac{\gamma}{2}\right)q(z)^2 + \frac{\gamma}{2}zq'(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

By choosing suitable parameters and the function $q(z)$, we get the result of Padmanabhan [5]. Our next result is a generalization in a different direction:

Theorem 2.4. *Let* α, β *and* γ *be complex numbers,* $\beta \neq 0$ *and* $a \neq -1$. *Let* $0 \neq q(z) \in \mathcal{A}$ *be convex univalent in* Δ *and*

$$\Re\left\{1 + \frac{2[a\beta + \gamma(a+1)]}{\beta} q(z) - \frac{\gamma(a+1)}{\beta q(z)^2} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 .$$

If $f(z) \in \mathcal{A}_0$ *satisfies*

$$\begin{aligned}(2.6)\quad & \alpha\frac{L(a,c)f(z)}{L(a+1,c)f(z)} + \frac{L(a+1,c)f(z)}{L(a,c)f(z)}\left(\beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \gamma\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right) \\ & \prec \frac{1}{a+1}\left\{\beta q(z) + [a\beta + \gamma(a+1)]\, q(z)^2 + \frac{\alpha(a+1)}{q(z)} + \beta z q'(z)\right\},\end{aligned}$$

then

$$\frac{L(a+1,c)\, f(z)}{L(a,c)\, f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Proof. Define the function $p(z)$ by (2.2). In view of (2.2) and (2.4), we get

$$\alpha\frac{L(a,c)\,f(z)}{L(a+1,c)\,f(z)} + \frac{L(a+1,c)\,f(z)}{L(a,c)\,f(z)}\left(\beta\frac{L(a+2,c)\,f(z)}{L(a+1,c)\,f(z)} + \gamma\frac{L(a+1,c)\,f(z)}{L(a,c)\,f(z)}\right)$$

$$= \frac{1}{a+1}\left\{\beta p(z) + [a\beta + \gamma(a+1)]\, p(z)^2 + \frac{\alpha(a+1)}{p(z)} + \beta z p'(z)\right\},$$

a hence the subordination (2.6) becomes

$$\beta p(z) + [a\beta + \gamma(a+1)]\, p(z)^2 + \frac{\alpha(a+1)}{p(z)} + \beta z p'(z)$$

(2.7)

$$\prec \beta q(z) + [a\beta + \gamma(a+1)]\, q(z)^2 + \frac{\alpha(a+1)}{q(z)} + \beta z q'(z).$$

By defining the functions ϑ and φ by

$$\vartheta(w) := \beta w + [a\beta + \gamma(a+1)]\, w^2 + \frac{\alpha(a+1)}{w} \quad \text{and} \quad \varphi(w) := \beta,$$

we see that the subordination (2.7) is same as (1.1). Note that $\varphi(w) \neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $\mathbb{C} - \{0\}$. The functions $Q(z)$ and $h(z)$ be defined by

$$Q(z) := zq'(z)\,\varphi(q(z)) = \beta z q'(z),$$

$$h(z) := \vartheta(q(z)) + Q(z) = \beta q(z) + [a\beta + \gamma(a+1)]\, q(z)^2 + \frac{\alpha(a+1)}{q(z)} + \beta z q'(z).$$

Clearly $Q(z)$ is starlike and

$$\Re\,\frac{zh'(z)}{Q(z)} = \Re\left\{1 + \frac{2[a\beta + \gamma(a+1)]}{\beta}\, q(z) - \frac{\gamma(a+1)}{\beta q(z)^2} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0.$$

Our Theorem 2.4 now follows by an application of Theorem 1.1. ■

By taking $a = n+1$ and $c = 1$ in Theorem 2.4, we have the following result:

Corollary 2.5. *Let α, β and γ be complex numbers, $\beta \neq 0$. Let $0 \neq q(z) \in \mathcal{A}$ be convex univalent in Δ and*

$$\Re\left\{1 + \frac{2[(n+1)\,\beta + \gamma(n+2)]}{\beta}\, q(z) - \frac{\gamma(n+2)}{\beta q(z)^2} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\alpha\,\frac{D^n f(z)}{D^{n+1} f(z)} + \frac{D^{n+1} f(z)}{D^n f(z)}\left(\beta\,\frac{D^{n+2} f(z)}{D^{n+1} f(z)} + \gamma\,\frac{D^{n+1} f(z)}{D^n f(z)}\right)$$

$$\prec \frac{1}{n+2}\left\{\beta q(z) + [(n+1)\,\beta + (n+2)\,\gamma]\, q(z)^2 + \alpha\,\frac{(n+2)}{q(z)} + \beta z q'(z)\right\},$$

then

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

By taking $a = c = 1$ and $2\alpha = \lambda$, $2\gamma = \delta$ in Theorem 2.4, we have the following result:

Corollary 2.6. *Let α, β and γ be complex numbers, $\beta \neq 0$. Let $0 \neq q(z) \in \mathcal{A}$ be convex univalent in Δ and*

$$\Re\left\{1 + \frac{2(\beta+\delta)}{\beta} q(z) - \frac{\delta}{\beta q(z)^2} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\lambda \frac{f(z)}{zf'(z)} + \frac{zf'(z)}{f(z)}\left[\beta + (\beta+\delta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]$$
$$\prec \beta q(z) + (\beta+\delta)\, q(z)^2 + \frac{\lambda}{q(z)} + \beta z q'(z)\,,$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Theorem 2.7. *Let α and δ be complex numbers, $\delta \neq 0$. Let $q(z) \in \mathcal{A}$ be univalent in Δ and $Q(z) = \delta z q(z) q'(z)$ be starlike in Δ. Further assume that*

$$\Re\left\{\frac{2\alpha}{\delta} + \frac{zQ'(z)}{Q(z)}\right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\alpha\left(\frac{L(a+1,\,c)f(z)}{L(a,\,c)f(z)}\right)^2 + \frac{\delta z}{2}\left[\left(\frac{L(a+1,\,c)f(z)}{L(a,\,c)f(z)}\right)^2\right]' \prec \alpha q(z)^2 + \delta z q(z)\, q'(z)\,, \tag{2.8}$$

then

$$\frac{L(a+1,\, c)f(z)}{L(a,\, c)f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by (2.2) and the functions ϑ and φ by

$$\vartheta(w) := \alpha w^2 \quad \text{and} \quad \varphi := \delta w\,.$$

Then the subordination (2.8) becomes (1.1) and our Theorem 2.7 follows by an application of Theorem 1.1. ■

By taking $a = n+1$ and $c = 1$ in Theorem 2.7, we have the following result:

Corollary 2.8. *Let α and δ be complex numbers, $\delta \neq 0$. Let $q(z) \in \mathcal{A}$ be univalent and $Q(z) := \delta z q(z)\, q'(z)$ be starlike in Δ. Further assume that*

$$\Re\left\{ \frac{2\alpha}{\delta} + \frac{zQ'(z)}{Q(z)} \right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\alpha\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2 + \frac{\delta z}{2}\left[\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2\right]' \prec \alpha q(z)^2 + \delta z q(z)\, q'(z)\,,$$

then

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

By taking $a = c = 1$ in Theorem 2.7, we have the following result:

Corollary 2.9. *Let α and δ be complex numbers, $\delta \neq 0$. Let $q(z) \in \mathcal{A}$ be univalent and $Q(z) := \delta z q(z)\, q'(z)$ be starlike in Δ. Further assume that*

$$\Re\left\{ \frac{2\alpha}{\delta} + \frac{zQ'(z)}{Q(z)} \right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ *satisfies*

$$\alpha\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{\delta z}{2}\left[\left(\frac{zf'(z)}{f(z)}\right)^2\right]' \prec \alpha q(z)^2 + \delta z q(z)\, q'(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Theorem 2.10. *Let* α, β, γ *and* δ *be complex numbers,* $\delta \neq 0$. *Let* $0 \neq q(z) \in \mathcal{A}$ *be univalent in* Δ *and* $Q(z) := \delta z q'(z)/q(z)$ *be starlike in* Δ. *Further assume that*

$$\Re\left\{\frac{\alpha(a+1)+a\delta}{\delta}q(z) + \frac{2\beta(a+1)}{\delta}q^2(z) - \frac{\gamma(a+1)}{\delta q(z)} + \frac{zQ'(z)}{Q(z)}\right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ *satisfies*

$$\begin{aligned}(2.9)\qquad &\alpha\frac{L(a+1,c)f(z)}{L(a,c)f(z)} + \beta\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^2 + \frac{\gamma L(a,c)f(z)}{L(a+1,c)f(z)} + \delta\left(\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right) \\ &\prec \frac{1}{a+1}\left\{[\alpha(a+1)+a\delta]\,q(z) + \beta(a+1)\,q(z)^2 + \gamma\frac{(a+1)}{q(z)} + \delta\frac{zq'(z)}{q(z)} + \delta\right\}\end{aligned}$$

then

$$\frac{L(a+1,\,c)f(z)}{L(a,\,c)f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Proof. Define the function $p(z)$ by (2.2) and the functions ϑ and φ by

$$\vartheta(w) := [\alpha(a+1)+a\delta]\,w + \beta(a+1)\,w^2 + \gamma\frac{(a+1)}{w} \quad\text{and}\quad \varphi := \frac{\delta}{w}\,.$$

Then the subordination (2.9) becomes (1.1) and our Theorem 2.10 follows by an application of Theorem 1.1. ■

By taking $a = n+1$ and $c = 1$ in Theorem 2.10, we have the following result:

Corollary 2.11. *Let α, β, γ and δ be complex numbers and $\delta \neq 0$. Let $0 \neq q(z) \in \mathcal{A}$ be univalent and $Q(z) := \delta z q'(z)/q(z)$ be starlike in Δ. Further assume that*

$$\Re\left\{ \frac{\alpha(n+2)+(n+1)\,\delta}{\delta}\, q(z) + \frac{2\beta(n+2)}{\delta}\, q^2(z) - \frac{\gamma(n+2)}{\delta q(z)} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\alpha \frac{D^{n+1}f(z)}{D^n f(z)} + \beta\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2 + \gamma \frac{D^n f(z)}{D^{n+1}f(z)} + \delta\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right)$$

$$\prec \frac{1}{n+2}\left\{[\alpha(n+2)+(n+1)\delta]\, q(z) + \beta(n+2)\, q(z)^2 + \gamma\frac{(n+2)}{q(z)} + \delta\frac{zq'(z)}{q(z)} + \delta\right\}$$

then

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

By taking $a = c = 1$ in Theorem 2.10, we have the following result:

Corollary 2.12. *Let α, β, γ and δ be complex numbers, $\delta \neq 0$. Let $0 \neq q(z) \in \mathcal{A}$ be univalent and $Q(z) := \delta z q'(z)/q(z)$ be starlike in Δ. Further assume that*

$$\Re\left\{ \frac{2\alpha+\delta}{\delta}\, q(z) + \frac{4\beta}{\delta}\, q^2(z) - \frac{2\gamma}{\delta q(z)} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\alpha \frac{zf'(z)}{f(z)} + \beta\left(\frac{zf'(z)}{f(z)}\right)^2 + \gamma \frac{f(z)}{zf'(z)} + \delta\left(1 + \frac{1}{2}\,\frac{zf''(z)}{f'(z)}\right)$$

$$\prec \left(\alpha + \frac{\delta}{2}\right) q(z) + \beta q(z)^2 + \frac{\gamma}{q(z)} + \frac{\delta}{2}\,\frac{zq'(z)}{q(z)} + \frac{\delta}{2}\,,$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Theorem 2.13. *Let* α, β *and* δ *be complex numbers,* $\delta \neq 0$. *Let* $q(z) \in \mathcal{A}$ *be univalent in* Δ *and* $Q(z) := \delta z q'(z)/q^2(z)$ *be starlike in* Δ. *Further assume that*

$$\Re\left\{ \frac{\alpha}{\delta}(a+1)\, q^2(z) + \frac{2\beta}{\delta}(a+1)\, q^3(z) - \frac{[\gamma(a+1)+\delta]}{\delta} + \frac{zQ'(z)}{Q(z)} \right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ *satisfies*

$$\alpha \frac{L(a+1,\, c)f(z)}{L(a,\, c)f(z)} + \beta\left(\frac{L(a+1,\, c)f(z)}{L(a,\, c)f(z)} \right)^2 + \gamma \frac{L(a,\, c)f(z)}{L(a+1,\, c)f(z)}$$

$$+\delta \frac{L(a+2,\, c)f(z)L(a,\, c)f(z)}{L(a+1,\, c)f(z)^2} \prec \alpha q(z) + \beta q(z)^2$$

$$+\left(\gamma + \frac{\delta}{a+1}\right)\frac{1}{q(z)} + \frac{\delta}{a+1}\,\frac{zq'(z)}{q^2(z)} + \frac{a\delta}{a+1}\,,$$

then

$$\frac{L(a+1,\, c)\, f(z)}{L(a,\, c)\, f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

The proof is similar to that of Theorem 2.10 and therefore omitted. By taking $a = n+1$ and $c = 1$ in Theorem 2.13, we have the following result:

Corollary 2.14. *Let* α, β *and* δ *be complex numbers,* $\delta \neq 0$. *Let* $q(z) \in \mathcal{A}$ *be univalent and* $Q(z) := \delta z q'(z)/q^2(z)$ *be starlike in* Δ. *Further assume that*

$$\Re\left\{ \frac{\alpha}{\delta}(n+2)\, q^2(z) + \frac{2\beta}{\delta}(n+2)\, q^3(z) - \frac{[\gamma(n+2)+\delta]}{\delta} + \frac{zQ'(z)}{Q(z)} \right\} > 0\,.$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\alpha \frac{D^{n+1}f(z)}{D^n f(z)} + \beta \left(\frac{D^{n+1}f(z)}{D^n f(z)} \right)^2 + \gamma \frac{D^n f(z)}{D^{n+1}f(z)} + \delta \frac{D^{n+2}f(z)\, D^n f(z)}{D^{n+1}f(z)^2}$$

$$\prec \alpha q(z) + \beta q(z)^2 + \left(\gamma + \frac{\delta}{n+2} \right) \frac{1}{q(z)} + \left(\frac{\delta}{n+2} \right) \frac{zq'(z)}{q^2(z)} + \delta \frac{n+1}{n+2},$$

then

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

By taking $a = c = 1$ in Theorem 2.13, we have the following result:

Corollary 2.15. *Let α, β and δ be complex numbers, $\delta \neq 0$. Let $q(z) \in \mathcal{A}$ be univalent and $Q(z) := \delta z q'(z)/q^2(z)$ be starlike in Δ. Further assume that*

$$\Re \left\{ \frac{2\alpha}{\delta} q^2(z) + \frac{4\beta}{\delta} q^3(z) - \frac{[2\gamma + \delta]}{\delta} + \frac{zQ'(z)}{Q(z)} \right\} > 0 .$$

If $f(z) \in \mathcal{A}_0$ satisfies

$$\alpha \frac{zf'(z)}{f(z)} + \beta \left(\frac{zf'(z)}{f(z)} \right)^2 + \gamma \frac{f(z)}{zf'(z)} + \delta \frac{1 + \frac{1}{2} \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}$$

$$\prec \alpha q(z) + \beta q(z)^2 + \left(\gamma + \frac{\delta}{2} \right) \frac{1}{q(z)} + \frac{\delta}{2} \frac{zq'(z)}{q^2(z)} + \frac{\delta}{2},$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

For $\alpha = \beta = 0$, $\gamma = -\delta/2$, $q(z) = \dfrac{1+z}{1-z}$, the Corollary 2.15 reduces to a recent result of Obradovič and Tuneski [4], (Theorem 3, p. 62).

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References

[1] B. C. CARLSON and D. B. SHAFFER, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), 737-745.

[2] J. -L. LIU and S. OWA, *On a class of multivalent functions invlolving certain linear operator*, Indian J. Pure Appl. Math. **33** (11) (2002), 1713-1722.

[3] S. S. MILLER and P. T. MOCANU, *Differential Subordinations. Theory and Applications*, Pure and Applied Mathematics, a series in monographs and textbooks; No. 225, Marcel Dekker, New York 2000.

[4] M. OBRADOVIČ and N. TUNESKI, *On the starlike criteria defined by Silverman*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. **181** (24) (2000), 59-64.

[5] K. S. PADMANABHAN, *On sufficient condition for starlikeness*, Indian J. Pure Appl. Math. **32** (4) (2001), 543-550.

[6] J. PATEL and P. SAHOO, *Properties of a class of analytic functions involving a linear operator*, Demonstratio Math. **35** (3) (2002), 495-507.

[7] V. SINGH, *On some criteria for univalence and starlikeness*, Indian J. Pure Appl. Math. **34** (4) (2003), 569-577.

Abstract

In the present investigation, we obtain some sufficient conditions involving $\frac{L(a+1,\,c)f(z)}{L(a,\,c)f(z)}$ *and* $\frac{L(a+2,\,c)f(z)}{L(a+1,\,c)f(z)}$ *for certain analytic function* $f(z)$ *defined on the open unit disk in the complex plane to satisfy the subordination* $\frac{L(a+1,\,c)f(z)}{L(a,\,c)f(z)} \prec q(z)$, *where* $L(a,\,c)$ *is the familiar Carlson-Shaffer linear operator.*