Characterizations of line sigraphs

Mukti Acharya[1] AND Deepa Sinha[2]

Department of Applied Mathematics, Delhi College of Engineering, Bawana Road, Delhi-110042, India.

[1]*E-mail: mukti1948@yahoo.com*

[2]*E-mail: deepa_sinha2001@yahoo.com*

Present address:

[2]*AIM & ACT, Banasthali Vidyapeeth, Banasthali P.O., Rajasthan – 304 022.*

Received June 14, 2004; Accepted July 20, 2004

Abstract

Given a *signed graph* (or, in short, *sigraph*) $S = (V, E; \sigma)$ on the graph $G = (V, E)$, its *line sigraph* $L(S)$ is a sigraph defined on the line graph $L(G)$ of G by defining an edge ef in it *negative* if and only if both e and f are adjacent negative edges in S. In this communication, we define a given sigraph S to be a *line sigraph* if there exists a sigraph H such that $L(H) \cong S$ (read as "$L(H)$ is isomorphic to S"). We then give structural characterizations of line sigraphs, extending the well-known characterization of line graphs due to L.W. Beineke.

(Keywords : sigraph/ line sigraph/ positive section)

Introduction

For standard terminology and notation in graph theory, not specifically defined in the communication, the reader is referred to Bollobás[1]. All graphs considered in this communication are finite, simple, and without self-loops and multiple edges.

A *signed graph*, or *sigraph* in short[2,3], is an ordered pair $S = (S^u, s)$, where S^u is a graph $G = (V, E)$ called the *underlying graph* of S, and $s : E(S^u) \rightarrow \{+, -\}$ is a function defined on the edge set $E(S^u) = E$ into the set $\{+, -\}$ of qualitative values '+' ('*positive*') and '-' ('*negative*') called *signs*. We let $E^+(S) = \{e \in E(G): s(e) = +\}$ and $E^-(S) = \{e \in E(G): s(e) = -\}$. Then the set $E(S) = E^+(S) \cup E^-(S)$ is called the *edge set* of S; the elements of $E^+(S)$ (($E^-(S)$)) are called *positive* (*negative*) *edges* in S. Two vertices $u, v \in V(S) = V(S^u) = V$ are said to be *adjacent* in S whenever they are adjacent in S^u (*i.e.* whenever $uv \in E(S^u)$). Thus, graphs may be regarded as sigraphs in which all the edges are positive; hence, we regard graphs as *all-positive* sigraphs (*all-negative* sigraphs are defined similarly). A sigraph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

Sigraphs are much studied in the literature because of their extensive use in modeling a variety of social processes[3,4,5] and also because of their strong connection with many classical mathematical systems[6]. It is also due to the fact that the class of graphs is contained in the class of sigraphs, whence the results proved for sigraphs must hold for graphs as well thereby providing generalizations of results from graph theory to the class of sigraphs. The purpose of this communication is to provide such a generalization, namely that of the well-known characterization of *line graphs* due to Beineke (*cf.*: Harary[7]).

Line Sigraphs

Behzad and Chartrand[2] have given a definition of the *line sigraph* $L(S)$ of a given sigraph S as follows: the vertices of $L(S)$ correspond one-to-one with the edges of S, $e_ie_j \in E(L(S)) \Leftrightarrow$ the edges of S corresponding to the vertices e_i and e_j of $L(S)$ have a vertex in common in S, and for any $e_ie_j \in E(L(S))$ one has $e_ie_j \in E^-(L(S)) \Leftrightarrow$ the adjacent edges of S corresponding to e_i and e_j are both negative in S.

We define a given sigraph S to be a *line sigraph* if and only if it is isomorphic to the line sigraph $L(T)$ of some sigraph T. The following theorem is the well-known characterization of a *line graph* given in most of the standard textbooks on graph theory (*e.g.*, *see* Harary[7]).

Theorem 1 : *The following statements are equivalent:*

(a) $G = (V, E)$ is a line graph.

(b) The edges of G can be partitioned into some of its complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.

The Main Result

We shall now embark on obtaining characterizations of line sigraphs. Towards this end, we need to define a nonstandard term introduced by the first author[8,9,10,11] as we will find its use in stating the main result of this communication: A *positive* (*negative*) *section* in a sigraph S is a maximal edge-induced subsigraph consisting of only the positive (negative) edges of S which turns out to be simply a path when S is a cycle or a path.

Hence, we give structural characterizations of a line sigraph, which is the main result of this communication.

Theorem 2 : The following statements are equivalent for any sigraph S :

(i) S is a line sigraph ;

(ii) S^u is a line graph and for any two vertices u, $v \in V(S)$ and for any u-v path in S either $u = v$ and the cycle so created is not a triangle with exactly two negative edges or the u-v path contains no positive section of length one unless one of its ends is u or v;

(iii) S does not contain an induced subsigraph isomorphic to either of the two sigraphs, S_1 formed by taking the path $P_4 = (x, u, v, y)$ with both the edges (x,u) and (v,y) negative and the edge (u,v) positive and S_2 formed by taking S_1 and identifying the vertices x and y, or to any sigraph on Beineke's nine forbidden subgraphs for a graph to be a line graph;

(iv) S^u is a line graph and for any positive edge uv of S either there is no negative edge at u or there is no negative edge at v;

(v) S^u is a line graph and vertices of S can be assigned signs '+' or '-' such that both the ends of every negative edge receive '-' sign and the same is not true for any positive edge;

(vi) S^u is a line graph and the vertex set $V(S)$ of S can be partitioned into two subsets V_1 and V_2, one of them possibly empty, such that all the negative edges of S join vertices of just one of the subsets.

Proof: (i) $\Rightarrow$ (ii). Suppose S is a line sigraph. Then there exists a sigraph T such that $S^u \cong L(T^u)$ so that S^u is a line graph.

To prove the other part of (ii), we first note that if S is homogeneous, then there is nothing to prove. Hence, suppose that S is a heterogeneous sigraph. Let, on the contrary, there exist a pair of distinct vertices u and v and a u-v path having a positive section of length one such that it does not contain u or v. This implies that the edges of the u-v path incident to the end vertices of the positive section are both negative. Let (u, e_1, e_2, v) be a u-v path in S such that $e_1e_2 \in E^+(S)$ and ue_1

and e_2v are negative edges incident to e_1 and e_2 respectively. Now, by the definition of the sign of an edge in the line sigraph, the pair u and e_1 as also the pair e_2 and v must be adjacent negative edges in T, whence in particular e_1 and e_2 would then both be negative, a contradiction to our assumption that e_1e_2 is a positive edge in S. Thus, the proof follows by contraposition in this case.

(ii) $\Rightarrow$ (i). Suppose S is the sigraph satisfying (ii). We shall show that S is the line sigraph, i.e., there exists a sigraph T such that $S \cong L(T)$.

Now, S^u being a line graph its lines can be partitioned into a family $\Im$ of complete subgraphs of S^u such that no vertex of S^u lies in more than two of these subgraphs. Then take S^u to be the *intersection graph* $\Omega(F)$ of the family $F = \Im \cup V'(S)$ where $V'(S)$ denotes the family of all one-point subsets of $V(S)$ each of which belongs to exactly one of the complete subgraphs in $\Im$. Then, by the construction described in the characterisation of line graphs, $S \cong L(\Omega(F))$ where if $F = (L_1, L_2, L_3,\ldots\ldots, L_q)$ then there exists an isomorphism $\Psi : V(S^u) \rightarrow V(L(\Omega(F)))$ such that $\Psi(v) = L_l L_j$ if and only if $L_i \cap L_j = \{v\}$. We shall show that $S \cong L(T)$ where $T^u \cong \Omega(F)$ and s_l: $E(\Omega(F)) \in \{+, -\}$ is such that

$$s_1(L_iL_j) = - \Leftrightarrow d^-(v) \neq 0 \text{ and } L_i \cap L_j = \{v\}.$$

Let L_iL_j and L_jL_k be adjacent in $L(\Omega(F))$. Then there exist $u, v \in V(S)$ such that $L_i \cap L_j = \{u\}$ and $L_j \cap L_k = \{v\}$. Hence, for the sign of the edge (L_iL_j, L_jL_k) in $L(T)$ we have $s_t(L_iL_j, L_jL_k) = -$ if and only if $s_1(L_iL_j) = s_l(L_jL_k) = -$ so that $d^-(u) \neq 0$ and $d^-(v) \neq 0$. This implies that there exist $x, y \in V(S)$ such that $ux, vy \in E^-(S)$. Since $\Psi(u) = L_iL_j$, $\Psi(v) = L_jL_k$ and Ψ is an isomorphism, $\Psi^{-1}(L_iL_j)\Psi^{-1}(L_jL_k) = uv \in E^-(S)$.

Suppose, $uv \in E^+(S)$. Then $x = y$, for otherwise in the x-y path (x, u, v, y), we see that (u, v) is the positive section of length one, a contradiction to the second condition of the hypothesis. But then S would contain a triangle with exactly two negative edges contradicting the second condition of the hypothesis again. Thus, for each pair of distinct vertices u and v, neither any u-v path contains a positive section of length one unless u or v is an end vertex of it nor a triangle with exactly two negative edges.

Now, suppose $s_t(L_iL_j, L_jL_k) = +$. This implies that $s_l(L_iL_j) \neq s_l(L_jL_k)$ so that either $d^-(u) \neq 0$ and $d^-(v) = 0$ or $d^-(u) = 0$ and $d^-(v) \neq 0$. Now $uv \in E(S)$ because of the isomorphism Ψ. In the first case, since $d^-(u) \neq 0$ and $d^-(v) = 0$, $uv \in E^+(S)$. Similarly, in the second case, since $d^-(u) = 0$ and $d^-(v) \neq 0$ the same conclusion can be arrived at.

(ii) $\Rightarrow$ (iii). S^u being a line graph, it is known that none of the nine Beineke's forbidden subgraphs[7] can be induced subgraphs of S^u, and hence no sigraph on any one of them can be a line sigraph. Also, from the second condition of (ii) none of the two sigraphs S_1 and S_2 can be an induced subsigraph of the line sigraph.

(iii) $\Rightarrow$ (iv). Since none of the sigraphs on any of the nine Beineke's forbidden subgraphs is an induced subsigraph of S none of those nine graphs in particular is an induced subgraph of S^u whence by Beineke's theorem mentioned above it follows that S^u is a line graph. Also, the two sigraphs S_l and S_2 are forbidden subsigraphs of S ; this condition implies the second part of (iv) as follows:

Suppose uv is an edge of S and there exist negative edges ue_l and ve_2 of S at u and v respectively. Then the pair u and e_l and also the pair v and e_2 must be adjacent negative edges in T, so uv is a negative edge in S. Hence, this part of the proof is seen to be complete by contraposition.

(iv) $\Rightarrow$ (i). Suppose (iv) holds for a given sigraph S. Then $S^u = L(H)$ for some graph H. Let $T = (V(H), E(H), \sigma)$ be the sigraph with $\sigma(e) = -$ if and only if there is at least one negative edge at the vertex e of S. Then, an edge uv in $E(L(T))$ is negative if and only if $\sigma(u) = -$ and $\sigma(v) = -$ in T

which is true if and only if there exist negative edges at u and at v in S if and only if uv is negative in S.

(iv) $\Rightarrow$ (v). Assuming the truth of (iv) implies that there does not exist any positive edge $uv \in S$ such that both u and v are incident to negative edges, so the end vertices of the negative edge can be assigned `-' sign so that the vertices of the positive section, which is of length at least two, can both be assigned '+ ', or '+ ' to one of them and ' - ' sign to its other end; this is not possible if uv is a positive edge and adjacent to negative edges at both u and v.

(v) $\Rightarrow$ (vi). Let $V = V_1 \cup V_2$ be such that V_1 is the set of vertices which have received ' + ' sign; hence, $V_2 = V - V_1$ would be the set of vertices that have received the ' – ' sign. Thus, obviously, every edge connecting two vertices u, v of V_2 is negative while all the edges across V_1 and V_2 are positive and those joining vertices within V_1 are also positive. Hence, $<V_2>$ contains negative edges only.

(vi) $\Rightarrow$ (v). Given the partition of $V(S)$ into the subsets V_1 and V_2 as described in (vi), assign ' - ' sign to all the vertices in V_2 and ' + ' sign to all the vertices in V_1.

(vi) $\Rightarrow$ (ii). That S^u is a line graph is given in (vi). So, if (ii) is false then there exist vertices u, v $\in V(S)$ such that for any u-v path in S there exists a positive section of length one such that it does not contain u or v. Hence, let (u, e_1, e_2, v) be a u-v path in S such that $e_1e_2 \in E^+(S)$ and ue_1 and e_2v are negative edges incident to e_1 and e_2 respectively. But then, (v) is violated. The conclusion implies violation of (vi). Thus, the proof in this case follows by contraposition.

Acknowledgement

We are pleased to thank Dr. B.D. Acharya for pointing out the problem solved in this communication.

References

1. Bollobás, B. (1998) *Modern Graph Theory*, Graduate Texts in Mathematics #184, Springer-Verlag, New York.
2. Behzad M. & Chartrand, G.T. (1969) *Elemen: der Mathematik, Band* **24**(3): 52.
3. Chartrand G.T. (1977) Graphs as *Mathematical Models*, Prindle, Weber and Schmidt, Inc., Boston, Massachusetts.
4. Harary, F., Norman R.Z. & Cartwright, D. (1965) *Structural Models: An Introduction to the Theory of Directed Graphs*, Wiley, New York.
5. F.S. Roberts (1978) *Graph Theory and its Applications to Problems of Society*, CBMS-NSF Regional Conference Series in Appl. Math., 29 SIAM Publications, Philadelphia, PA, USA.
6. Zaslavsky, T. (2001) *Electronic J. Combinatorics*, **8**(1) : 124.
7. Harary, F. (1969) *Graph Theory*, Addison-Wesley Publ. Comp., Reading, Massachusetts.
8. Acharya, M. (1986) *Proc. Symp. on Optimization, Design of Experiments and Graph Theory*, (IIT, Bombay : Dec. 15-17, 1986), *Eds.*: G.A. Patwardhan and H. Narayanan, p. 342. Indian Institute of Technology, Bombay, 1988.
9. Gill, M.K. & Patwardhan, G.A. (1981) *J. Math. & Phys. Sci.* **15**(6) : 567.
10. Gill, M.K. & Patwardhan, G.A. (1983) *J. Combin. Inf. & Syst. Sci.*, **8** : 287.
11. Gill, M.K. & Patwardhan, G.A. (1986) *Discrete Math.* **61** : 189.