Graph Theory Notes of New York XLIV, 30–34 (2003)
New York Academy of Sciences.

[GTN XLIV:6]

A CHARACTERIZATION OF SIGRAPHS WHOSE LINE SIGRAPHS AND JUMP SIGRAPHS ARE SWITCHING EQUIVALENT

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Abstract

In this paper, we characterize signed graphs whose line sigraphs and jump sigraphs are switching equivalent.

1. Introduction

For standard graph theory terminology and notation used in the paper the reader is referred to Bollobás [1]. All graphs considered here are finite and simple. A graph in which every edge is designated positive or negative is called a *signed graph* (or *sigraph* in short; see e.g., [2]).

Let $G = (V, E)$ be a graph, the *jump graph* of G, $J(G)$ was defined by Chartrand *et al.*, [3] as follows: The set of vertices of $J(G)$ is the set E; two distinct vertices $e, e' \in E$ are defined to be adjacent in $J(G)$ whenever $e \cap e' = \varnothing$. As noted in [3], it is obvious from this definition that $J(G)$ is the complement $\overline{L(G)}$ of the standard *line graph* $L(G)$ of G.

Very recently, the notion of the jump graph was extended to the class of sigraphs [4] as follows: The *jump sigraph* $J(S)$ of a sigraph S has $J(S^u)$ as its underlying graph; an edge ee' of $J(S^u)$ is defined to be negative if and only if the corresponding edges e and e' in S have opposite signs.

Recall the definition of the *line sigraph* $L(S)$ of a sigraph S introduced by Behzad and Chartrand [5]: The underlying graph of $L(S)$ is $L(S^u)$; an edge ee' of $L(S^u)$ is defined to be negative if and only if both e and e' are negative edges in S. One is naturally lead to look for an analog of the graph equation

(1) $\quad J(G) \approx \overline{L(G)}$

for the case of jump sigraphs. For this to be meaningful, however, one first needs a proper definition for the complement of a given sigraph. It is a long-standing open problem to construct such a definition that is in consonance with *consistency theory* in *social psychology* [6]–[8]. One possible approach toward this end would be to start by looking at possible relationships between the structures of $J(S)$ and $L(S)$. Hence, we feel it worthwhile to first examine the relationship

(2) $\quad J(S) \sim L(S)$,

where ~ denotes the binary relation *switching equivalent* between two sigraphs as defined in the next section. In fact, in this paper, we solve the sigraph equivalence (2) in the sense that we determine the structure of all sigraphs satisfying (2). We also obtain conditions for which $J(S) \approx \overline{L(S)}$ (see Theorems 5 and 6).

2. Preliminaries

A *marking* of a sigraph S is an assignment of positive and negative signs to the vertices of S. That is, a marking is function from the vertex set of S to the set $\{\pm 1, 1\}$. Given a marking μ of S, a *switching* S with respect to μ is defined as the sigraph, called the *switched sigraph* and denoted by $S_\mu(S)$, obtained from S by complementing the sign of every edge whose end vertices are oppositely signed under μ. A sigraph S_1, *switches* to another sigraph S_2, written $S_1 \sim S_2$, if there exists a marking μ of S_1, such that $S_\mu(S_1) \approx S_2$. It may be easily verified that the binary relation ~ is in fact an equivalence relation on the class of all sigraphs; two sigraphs in the same equivalence class are therefore said to be *switching equivalent* to each other. This notion is illustrated in Figure 1, where positive and negative edges are depicted as solid and broken line segments, respectively.

By a *positive* (*negative*) section in a sigraph S we mean a maximal edge-induced subgaph consisting of only the positive (negative) edges of S. A cycle in S is said to be *positive* if it contains an even number of negative edges; or, equivalently, if the product of the signs of its edges is positive. A cycle that is not positive is said to be *negative*. A signed graph S is *balanced* if every cycle in S is positive **[2][9] –[13]**. It may be noted that balance is a property that is invariant under switching (see e.g., **[12][13]**). In fact, much more can be said. A bijection f between the vertex sets V_1 and V_2 of two sigraphs S_1 and S_2, respectively, is called a *cycle isomorphism* (or *weak isomorphism*, as in **[13]**) between S_1 and S_2 if f preserves both vertex adjacencies and the cycle signs of S_1 and S_2.

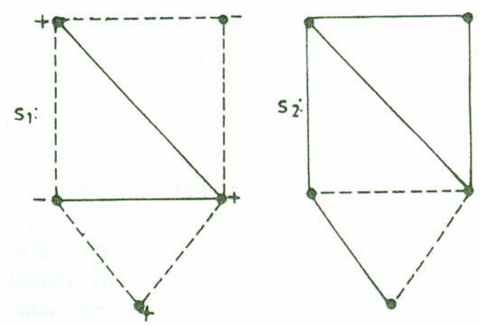

Figure 1: $S_1 \sim S_2$.

Theorem 1 (Zaslavsky [13]): Two sigraphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic. ■

Corollary 1.1: A sigraph S is balanced if and only if S is switching equivalent (cycle isomorphic) to its underlying graph S^u. ■

Theorem 2 (Acharaya and Sinha [4]): For any sigraph S, its jump sigraph $J(S)$ is balanced. ■

Theorem 3 (Wu and Guo [14]): The graph equation $J(G) \approx L(G)$ has only six solutions; namely, $K_2, P_5, C_5, P_5^+, K_{3,3} - e$, and $K_{3,3}$. ■

3. Main Results

To progress toward finding the solutions to graph Equation **(2)**, we first observe that if S is a solution to **(2)**, then in view of Theorem 2, $L(S)$ must be balanced. Thus, a sigraph S is a solution to **(2)** if and only if the underlying graph S^u is one of the six sigraphs given by Theorem 3 and $L(S)$ is balanced. We characterize sigraphs with balanced line sigraphs in the following result.

Theorem 4: For a sigraph S, $L(S)$ is balanced if and only if the following conditions hold:

(1) for any cycle Z in S

(a) if Z is all-negative, then Z has even length;

(b) if Z is heterogeneous, then Z has an even number of negative sections with even length;

(2) for $v \in V(S)$, if $\deg(v) > 2$, then there is at most one negative edge incident at v in S.

Proof:

Necessity

Suppose $L(S)$ is balanced. Then, by definition, every cycle Z' in $L(S)$ contains an even number of negative edges. The vertices of Z' correspond to the edges of a cycle Z in S. Since the length of any negative section in Z is reduced by one when transformation L is applied once to S, then (1) is immediate because of positivity of Z'.

On the other hand, suppose not all the vertices of Z' correspond to edges of a cycle in S. Then there is a vertex $e' \in Z'$ that corresponds to an edge in S incident at a vertex v with degree at least three. Suppose there are $r > 1$ negative edges incident at v. Clearly, if $r > 3$ then some three of these negative edges create a negative triangle in $L(S)$ contradicting the hypothesis that $L(S)$ is balanced. Therefore, $r \leq 2$. If $r = 2$, then since v has degree at least 3, the two negative edges together with one positive edge incident at v create a negative triangle in $L(S)$, again contradicting the hypothesis that $L(S)$ is balanced. Thus, (2) must hold.

Sufficiency

Suppose conditions (1) and (2) hold for a sigraph S. We show that $L(S)$ is balanced. If S is all-positive then, by definition, $L(S)$ is also all-positive and, hence, it is trivially balanced. Thus, suppose that S is a heterogeneous sigraph. We need to consider two cases.

Case 1: S has no cycle.

In this case, every cycle in $L(S)$ is due to edges in S that are incident at a vertex of degree greater than or equal to three. Hence, consider any vertex v whose total degree is at least three. By condition (2), there is at most one negative edge incident at any such vertex. Thus, either all edges incident at v are positive or there is exactly one negative edge incident at v. In either case, all the cycles in $L(S)$ contained in the clique formed by the edges incident at v are positive. Since there is no cycle in $L(S)$ created by any cycle in S (S is acyclic) it follows that every cycle in $L(S)$ is positive. Thus, $L(S)$ is balanced.

Case 2: S has at least one cycle.

Suppose S has at least one cycle and conditions (1) and (2) hold. If there is no vertex of degree greater than or equal to three, then every component of S is either a path or a cycle; furthermore, one of these components must be a cycle. By condition (1a), if a cycle in S is all-negative then it must have even length and every such cycles generates an all-negative cycle of even length in $L(S)$. Now, if a cycle in S is heterogeneous then, by condition (1b), it has an even number of negative sections of even length. Thus, by definition, the length of each of these negative sections is reduced by one in $L(S)$. Consequently, the number of negative sections of odd length in $L(S)$ is even. It follows that each such cycle in S goes to a cycle in $L(S)$ with an even number of negative edges. Hence, $L(S)$ is balanced.

Next, suppose that S has at least one vertex with degree at least three. For simplicity, we may assume that S is connected. Then, clearly, every cycle in S must contain a vertex of degree at least three and at any such vertex v_i on a cycle Z of S, two of the edges e_{i-1} and e_{i+1} incident at v_i must belong to Z.

Now, suppose that $L(S)$ is not balanced. Then $L(S)$ must contain a negative cycle

$$Z'_k = (e_1, e_2, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_k, e_1)$$

of length $k \geq 3$. Without loss of generality, we may assume that k is the least possible length for any such cycle. By conditions (1) and (2), not all the vertices of Z'_k correspond to edges that are incident at a single vertex in S, nor do they form a cycle in S. Thus, some vertices of Z'_k correspond to edges of a cycle Z_k in S and some correspond to edges incident at a vertex of degree greater than equal to three. Since v_i is any such vertex and e_{i-1}, e_i, e_{i+1} are edges incident at v_i in Z_k, where e_{i-1} and e_{i+1} are edges of the cycle Z_k in S, it follows from the definition of $L(S)$, that the negativity of Z'_k is not is not changed as a result of the positive clique formed by the edges constituting a star at v_i. However, the length of the cycle in $L(S)$ is reduced by one, contradicting minimality of the length of Z'_k in $L(S)$. The contradiction shows that $L(S)$ must be balanced. ∎

By Theorem 3, solutions to Equation **(2)** are the sigraphs S that satisfy the conditions of Theorem 4 and have underlying graphs that are any of the graphs given by Theorem 3. Thus,

Theorem 5: A sigraph S satisfies $J(S) \sim L(S)$ if and only if
(1) S^u is isomorphic to any of the graphs $K_2, P_5, C_5, P_5^+, K_{3,3} - e$, or $K_{3,3}$, and
(2) S satisfies the conditions of Theorem 4. ∎

Specifically, Theorem 5 yields the following solutions to Equation **(2)**.

Solution set 1: All sigraphs on the acyclic underlying graphs K_2 and P_5.

Solution set 2: The sigraphs with underlying graph C_5 that are shown in Figure 2.

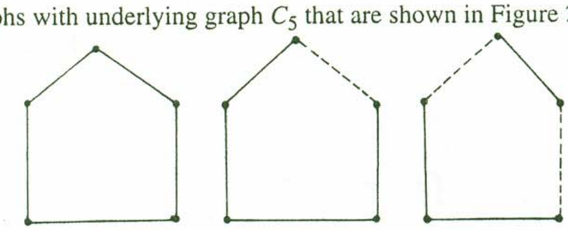

Figure 2:

Solution set 3: The sigraphs with underlying graph P_5^+ that are shown in Figure 3.

Solution set 4: The sigraphs with underlying graph $K_{3,3} - e$ that are shown in Figure 4.

Solution set 5: The sigraphs with underlying graph $K_{3,3}$ that are shown in Figure 5.

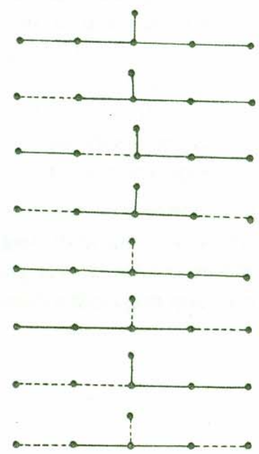

Figure 3:

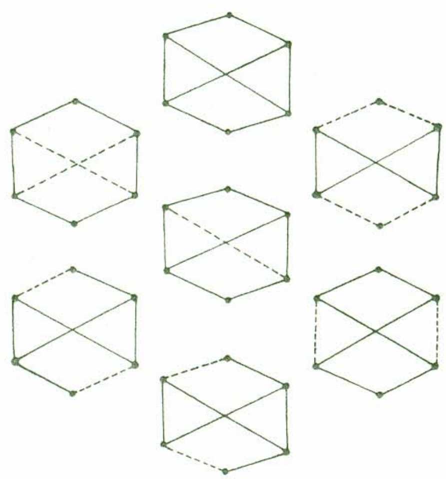

Figure 4:

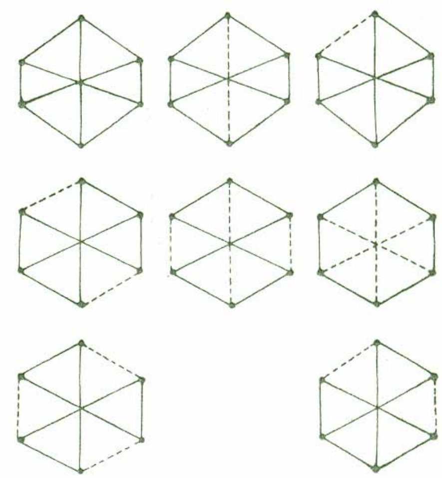

Figure 5:

We now consider the question of defining in a reasonable way the complement of a given sigraph S. To this end we propose the following *unary operation* on S: Let σ denote a signing of S and let μ be the marking of S obtained by defining $\mu(u)$, for each vertex u, as the product $\prod_{x \in E_u} \sigma(x)$ of the signs $\sigma(x)$ of the edges x incident at u.

We define an operator c that transforms S into a sigraph $\bar{S}$ with the same vertex set as that of S and two vertices are adjacent whenever the vertices are not adjacent in S^u and with each edge uv signed $\mu(u)\mu(v)$. Clearly, $\bar{S}$ as defined here is a sigraph whose underlying graph is the usual graph complement of S^u and it is also balanced. Consequently, the following result, which is not difficult to verify, shows that an analog of (1) (namely Equation (4)) is highly restrictive; however, the relation of switching equivalence is more suitable for this comparison, as can be seen from the universality of Equation (3).

Theorem 6: For any sigraph S,

(3) $J(S) \sim \overline{L(S)}$, and

(4) $J(S) \approx \overline{L(S)}$,

if and only if S satisfies the two conditions of Theorem 5. ∎

4. Concluding Remarks

As mentioned, one of the observations made in **[3]** is that $J(G)$ is isomorphic to the complement $\overline{L(G)}$ of $L(G)$ for every graph G. However, this leads to the question of whether such a relation holds for $J(S)$ and $L(S)$, both of which are now well defined. Toward answering this question, we found a unary operator c that transforms S into a balanced sigraph $\bar{S}$ defined on the usual complement of the underlying graph S^u of S. As a result we discovered that an analogue of Equation **(1)** for a sigraph is highly restrictive if one uses isomorphism as the binary relation, but there does exist a *universal* analogue of **(1)** if we use switching equivalence as the binary relation to compare $J(S)$ and $\overline{L(S)}$. However, there still appear to be two major shortcomings: (1) whether S is balanced or not, $J(S)$ and $\overline{L(S)}$ are balanced; and (2) the unary operator c is not involutory in the sense that $\bar{\bar{S}}$ is not isomorphic to S as required for a unary operator to function as a complementation.

Thus, our suggestion for a reasonable definition of the *complement* of a sigraph remains to be refined. Since it is a long-cherished dream of social psychologists to propose an acceptable definition of the complement of a sigraph **[6]**, it would perhaps be worth studying several related problems, such as Equation **(2)** in a comparable manner since the effort may eventually resolve this open question.

Acknowledgements

The authors wish to thank Dr. B.D. Acharya for generously providing us with useful and rigorous discussion on the subject of this research note.

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Received: October 10, 2001
Revised: January 16, 2002