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A CHARACTERIZATION OF SIGNED GRAPHS THAT ARE SWITCHING EQUIVALENT TO THEIR JUMP SIGNED GRAPHS

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Abstract

In this paper, we define the jump signed graph of a given signed graph and offer a structural characterization of signed graphs that are switching equivalent to their jump signed graphs.

For graph theory terminology and notation in this note we follow **[1]**. Additional terms and notation are introduced as and when necessary.

The *jump graph* $J(G)$ of a graph G is defined in **[2]** and shown to be the *complement* $\overline{L(G)}$ of the standard *line graph* $L(G)$ of G (see, for example, **[3]**).

The purpose of this note is to extend the concept of jump graphs to the class of *signed graphs* (or *sigraphs* in short **[4]**) since this appears to have interesting connections with certain long-standing questions in social psychology (see, for example, **[5]**).

By definition (see **[1][4]**) a *sigraph* is an ordered pair $S = (S^u, s)$, where $S^u = (V, E)$ is a graph called the *underlying graph* of S and $s: E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$ called a *signing* of the graph S^u. Let $E^+(S)$ denote the set of all edges of S^u that are mapped by s to the element "+" and let $E^-(S) = E - E^+(S)$. The elements of $E^+(S)$ are called *positive edges* of S and those of $E^-(S)$ are called *negative edges* of S.

The *line sigraph* $L(S)$ of a given sigraph S is defined in **[4]**. Since $J(G) \approx \overline{L(G)}$, as noted above, one would naturally like to extend the notion of jump graphs to the realm of sigraphs and seek an analogue of this relationship (see **[6]**).

We define the *jump sigraph* $J(S)$ of a sigraph S to be a sigraph such that $(J(S))^u \approx J(S^u)$, where two vertices of $J(S)$ (the edges of S) are connected by a negative edge if and only if the corresponding edges in S have opposite signs.

The *sign of a cycle* in a sigraph $S = (S^u, s)$ is defined as the product of the signs of its edges. S is then said to be *balanced* if every cycle in S is positive (see, for example **[1][7]**). A *marking* of S is a function $\mu: V(S) \to \{+, -\}$; S together with a particular marking μ is denoted by S_μ. A simple condition that characterizes balance in S is that it is possible to find a marking μ of S such that $s(uv) = \mu(u)\mu(v)$ for every edge uv of S **[8]**. We now give a straightforward, yet interesting, property of jump sigraphs.

Lemma 1: For any sigraph S, its jump sigraph $J(S)$ is balanced.

Proof: Let σ denote the signing of $J(S)$ and let the signing s of S be treated as a marking of the vertices of $J(S)$ (that is, of the edges of S). Then, by the definition of $J(S)$ we see that $\sigma(ee') = s(e)s(e')$ for every edge ee' of $J(S)$ and, hence, by the characterization of balance mentioned above, the result follows. ■

Given a marking μ of S, by *switching S with respect to* μ we mean reversing the sign of every edge of S whenever the end vertices have opposite signs in S_μ **[8]**. We denote the sigraph obtained in this way by $S_\mu(S)$ and this sigraph is called the *μ-switched sigraph* or just *switched sigraph*. A sigraph S_1 *switches* to a sigraph S_2 (that is, they are *switching equivalent* to each other), written $S_1 \sim S_2$, whenever there exists a marking μ such that $S_\mu(S_1) \approx S_2$.

We now characterize those sigraphs that are switching equivalent to their jump sigraphs.

In the case of graphs the following result is due to Simić [9] (see also [2]) where $H \circ K$ denotes the *corona* of the graphs H and K [3].

> **Lemma 2:** The jump graph $J(G)$ of a graph G is isomorphic with G if and only if G is either C_5 or $K_3 \circ K_1$. ■

> **THEOREM 1:** A sigraph S satisfies $S \sim J(S)$ if and only if S is a balanced sigraph on either C_5 or $K_3 \circ K_1$.

Proof: The Sufficiency part of the proof is straightforward, thus we prove only the necessity. Since $S^u \approx (J(S))^u$, S must be isomorphic to either C_5 or $K_3 \circ K_1$ by Lemma 2. Because $J(S)$ is balanced, by Lemma 1, S must be balanced. ■

Remark: Theorem 1 leaves very few cases for which $S \approx J(S)$; in fact, these cases form a subset of the set of solutions of $S \sim J(S)$. It would be of interest to know exactly what these cases are.

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